The Path of an Electron Orbiting an Accelerating Nucleus

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Abstract

The orbit of an electron in an atom that accelerates from rest to a constant relativistic velocity in the laboratory is found using Maxwell's equations and Newton's second law. The shape of the orbit changes from a circle to an ellipse consistent with the predictions of special relativity.

I. Introduction

The rationale for calculating the orbit of an electron around an accelerating nucleus was motivated by the essay "How to teach special relativity" in the book **Speakable and Unspeakable in Quantum Mechanics**[1] by J. S. Bell. The essay outlines an alternative path for deducing the Lorentz transformation equations[2] of special relativity. This article shows how technologically advanced scientists living in an imaginary world *before the discovery of special relativity* could have done a series of experiments, Sections II and IV, supported by calculations of the path of an electron around an accelerating nucleus, Sections III and the Appendix, that would have led to the Lorentz transformations.

The pedagogical value of this alternative approach to special relativity is that it highlights the connection between the microscopic force laws governing the motion of an electron with the macroscopic properties of rods and clocks in a way that is completely consistent with the more usual approach to special relativity.

II. ***Gedanke*n Experiment: Round One**

Two skilled experimentalists living in the imaginary world described in Section I decide to measure the length of a meter stick as it moves through their laboratory at a high rate of speed. They select two identical meter sticks, one will ride on a rocket and the other will remain in the laboratory. Their plan for measuring the length of the moving meter stick is to position themselves a distance D apart with synchronized watches. To synchronize their watches, the two scientists will meet at point A, the middle of the marked off distance D, simultaneously start their watches, and then walk with equal speeds to their designated spots. Each scientist will record the time when the ends of the moving meter stick pass. After collecting these data, the scientists will walk at equal speeds back to point A to confirm that their watches are still synchronized. Using the time it took the front or back ends of the meter stick to travel the distance D, they can determine the speed of the meter stick. Knowing the speed and the time it took for the meter stick to pass by; they can calculate the length of the moving meter stick.

With this protocol established, the scientists do this experiment several times with the rocket passing through the laboratory at different constant speeds. After all the runs were finished, the rocket returned to the laboratory. The meter stick aboard the rocket was re-examined[3] confirming that it was still identical to the meter stick that remained in the laboratory. When the results were collated for various rocket speeds, v, they discover that the moving meter stick was contracted by $\sqrt{1-\frac{v^2}{c^2}}$, where c is the speed of light.

The two scientists ponder this surprising result. First they agree that the "length" of the meter stick, moving or not, is a consequence of the interactions between the atoms making up the meter stick. They realize that the forces acting in the meter stick are electromagnetic in nature, and that the electric and magnetic fields created by charges depend on their state of motion.[4] They hypothesize that the fields in the moving meter stick change by just the correct amount to cause the moving meter stick to be shorter by

$\sqrt{1-\frac{v^2}{c^2}}$. They also recognize that the interactions inside a real meter stick are much too complicated for them to have any hope of calculating, from first principles, its length, whether moving or not.

Instead of trying to solve the problem outlined above, the scientists decide to tackle the simpler problem of a single electron moving under the influence of the electric and magnetic fields, **E** and **B**,[5] produced by an accelerating nucleus. They will use those fields to calculate in detail the path of the electron as the nucleus accelerates to a high rate of speed. Their goal is to see if changes in the electron's orbit are correlated with the changes they observed in the moving meter stick.

III. Orbit of an Electron about an Accelerating Nucleus

The problem the scientists solved is summarized by equation 1,

$$\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) = \frac{d}{dt}\left[\frac{m_e \boldsymbol{V}}{\sqrt{1-\frac{V^2}{c^2}}}\right] \qquad (1)$$

where e, **V**, and m_e are the charge, velocity, and mass of the electron. The scientists have experimentally discovered this modified version of

Newton's second law by doing experiments on objects accelerated to high speeds.[6] The numerical solution of equation 1 is outlined in the Appendix.

The scientists assumed that the nucleus moved in the Z-direction with the electron orbiting in the XZ-plane. The results for $X(t)$ and $Z(t)$, the laboratory coordinates of the electron, were found by solving equations A26 to A28 in the Appendix. $X(t)$ and $Z(t)$, the output of those calculations,[7] were tabulated and used to plot the path of the electron as the nucleus moved through the laboratory while accelerating to its terminal speeds, v_∞. Figure 1 shows the electron's path for the first 100 units of time, 10 times the electron's orbital period,[8] as the nucleus just begins to accelerate. Figure 2 shows the path at a later time when the nucleus is moving faster. The arrow shows the direction of motion of the electron. Both graphs were for the case $v_\infty = 0.8c$.

Figure 1

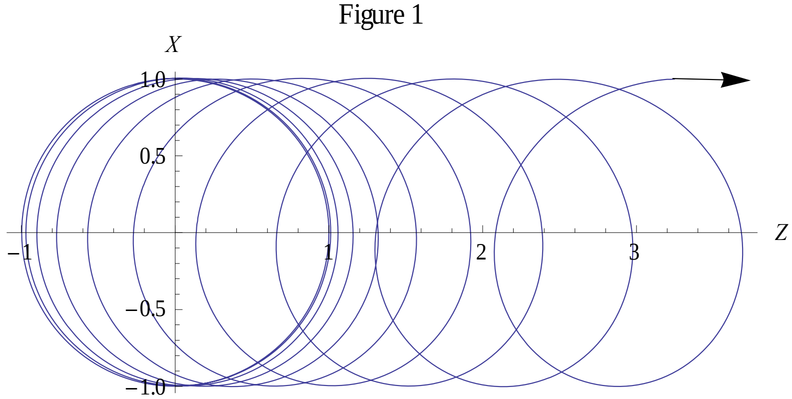
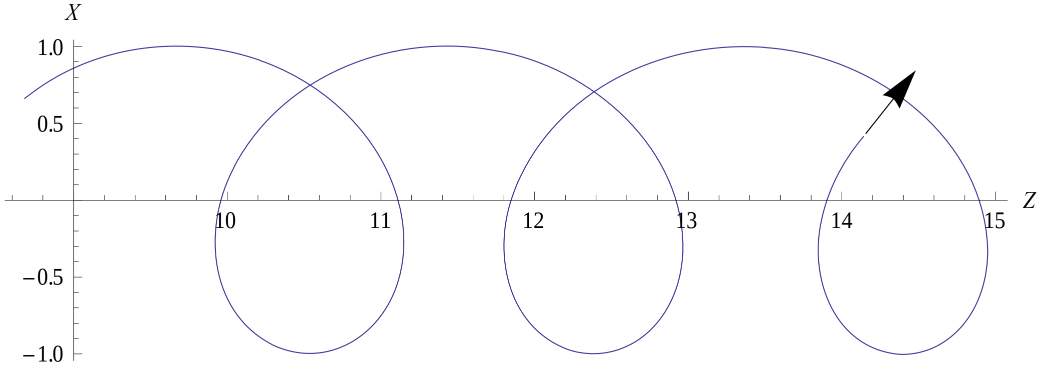


Figure 2

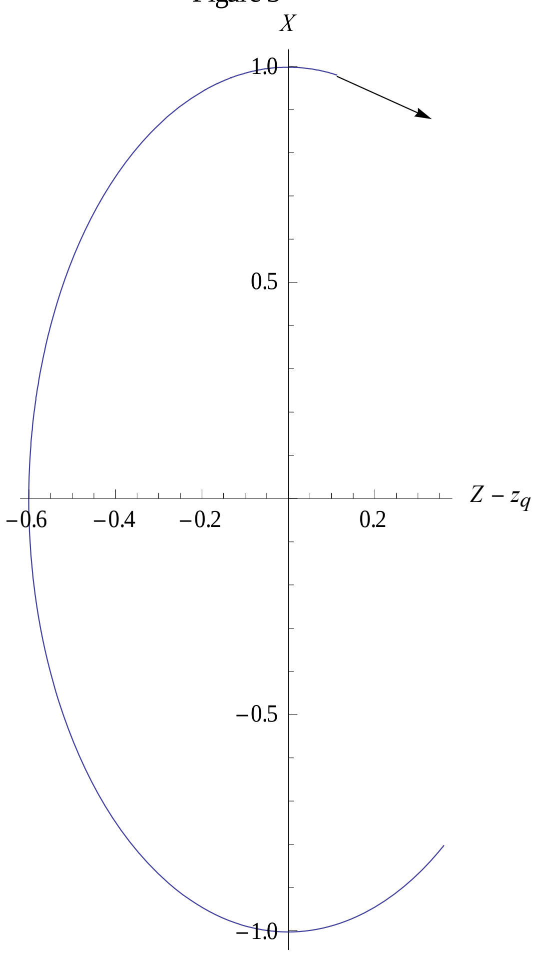


These plausible paths for the electron through the laboratory gave the scientists confidence that their solution for $X(t)$ and $Z(t)$ were correct.

In order to see the orbit of the electron in the atom as it moved through the lab, the scientists subtracted $z_q(t)$**,** the location of the nucleus at time t, from the location of the electron at that time. Figure 3 is one such graph of $Z(t) - z_q(t)\ vs\ X(t)$. The range of times, t = 3000 to 3010, were chosen to insure that the nucleus was moving at a constant speed, 0.8c, and included just enough time for the electron to complete a single orbit.

The original circular orbit in this case evolved during the acceleration into

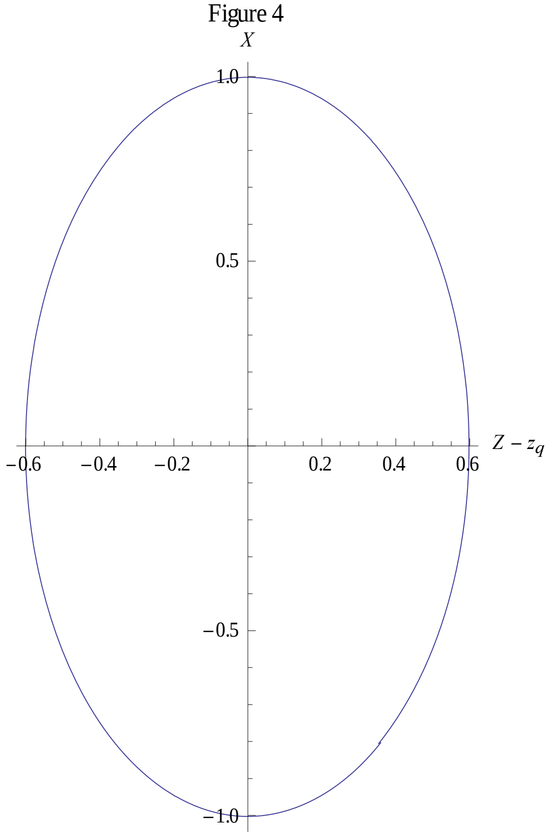
Figure 3



an ellipse with the minor axis contracted to 0.6 from 1.0. This agreed with the experiments described in Section II. Furthermore, for all tested values of v_∞, up to 0.995c, the orbits were contracted by $\sqrt{1-\frac{v_\infty^2}{c^2}}$. This supported their hypothesis that the observed contraction was due to the difference in forces between the charges in a moving meter stick compared to those in a stationary one. But they were puzzled by the fact that as v_∞ got larger and larger, the electron completed less and less of a complete orbit in 10 units of time. Using their results for $X(t)$ and $Z(t)$, they soon discovered that in each case the time required to ensure that the electron completed precisely one orbit was longer by the same factor by which length was contracted. Since the stationary electron had an orbital period of 10, the period grew to

$\frac{10}{\sqrt{1-\frac{v_\infty{}^2}{c^2}}}$ when the nucleus moved with speed v_∞. Figure 4 is a re-plot of figure 3 with the time interval stretched by $\sqrt{1-\frac{v_\infty{}^2}{c^2}}$, from 10 to 16.67 units of time. The modified time range, 3000 to 3016.67, allowed the electron to complete exactly one orbit.

Figure 4



The results were encouraging but raised new questions. The first question was whether moving clocks would mirror the behavior of their constituent atoms and run slow compared to stationary clocks. More troubling was the recognition that when the rocket reached its terminal velocity, v_∞, the Galilean transformation equations connecting the rocket coordinates to the laboratory coordinates,[9] $X'(t) = X(t)$, $Z'(t) = Z(t) - v_\infty t$, and $t' = t$, could be used to plot the shape of the orbit in the rocket frame. That graph of Z' vs X' would be identical to figure 4. This implied that observers in the rocket would see their meter stick contract! But scientists riding in the rocket at the constant speed v_∞ could rightly claim the atoms in their meter stick consisted of stationary nuclei with electrons in circular orbits caused by a simple electric field. If their claim was valid, it would contradict the orbital contraction seen in figure 4 and invalidate the almost sacrosanct Galilean transformation equations. Consequently the two scientists decide to do a second round of experiments.

IV. *Gedanken* Experiment: Round Two

For these experiments, the scientists enlisted two colleagues who agreed to ride in the rocket. The laboratory scientists and the rocket scientists would measure the length of the meter stick that was moving with respect to them using the procedures described in section II. Also, for this set of experiments, identical clocks would be placed in the middle of the lab and rocket meter sticks. The stationary observers would compare the time ticked off the moving clock as it traveled the distance D that separated their stationary and carefully synchronized watches. Before starting, the scientists in the rocket agreed to synchronize their watches after the rocket had reached its terminal speed. This last step guaranteed that the synchronization process would not be affected by the rocket's acceleration.

After the experiments were completed, the four scientists gathered to compare results. The rocket riding scientists measured the laboratory meter stick, the one moving through the rocket, to be contracted by $\sqrt{1-\frac{{v_\infty}^2}{c^2}}$ compared to their meter stick. Next, it was observed that the two moving clocks ran slow by $\sqrt{1-\frac{{v_\infty}^2}{c^2}}$. This matched the period of the electrons orbiting a nucleus moving with speed v_∞. There were no observational contradictions: the stationary observers, two in the lab and two riding on the rocket, saw identical changes when they compared the moving and stationary meter sticks and clocks.

The fact the rocket observers did not see their meter stick shrink nor their clock run slow implied that the orbit of the electron in the rocket frame, $Z'\ vs\ X'$, was circular with a period of 10 and a radius of 1.0, and not the ellipse seen in figure 4. Now they had experimental evidence that the Galilean transformation equations were incorrect.

The scientists recognized that the minor axis of the ellipse in figure 4 had contracted by $\sqrt{1-\frac{{v_\infty}^2}{c^2}}$. Since this contraction was in the Z-direction, they decided to leave the transformation in the X-direction unchanged and to undo the contraction by dividing the Galilean equation for Z' by $\sqrt{1-\frac{v_\infty{}^2}{c^2}}$.

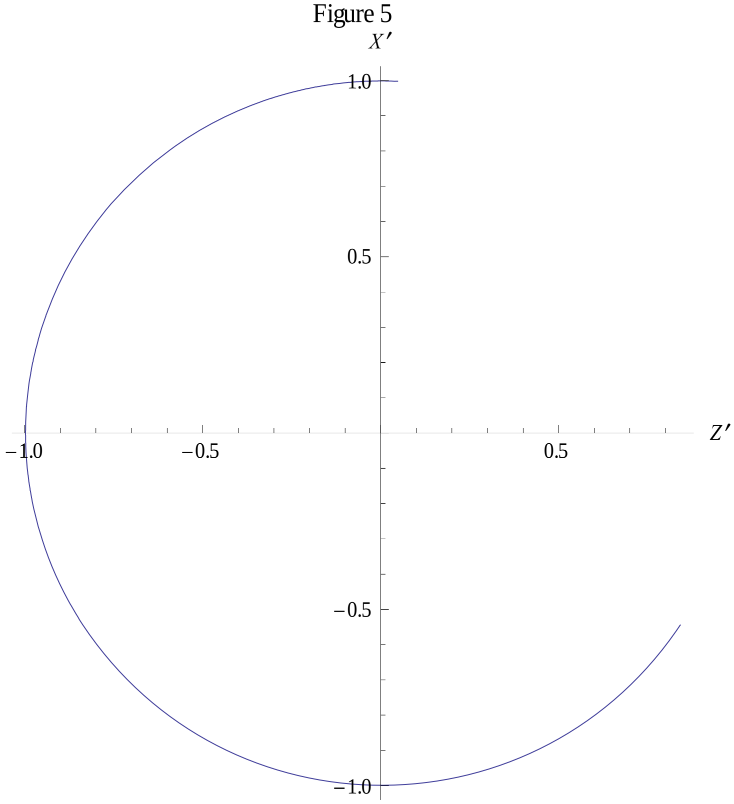
The result was a pair of modified Galilean equations for the space coordinates:[10]

$$X' = X \qquad (2)$$

$$Z' = \frac{Z - v_\infty t}{\sqrt{1-\frac{v_\infty{}^2}{c^2}}} \qquad (3)$$

The shape of the orbit seen by the rocket observers was re-plotted by inserting the numerical values for $X(t)$ and $Z(t)$ into equations 2 and 3. Again the values of t were chosen to insure that the speed of the nucleus was a constant 0.8c. The new orbit was the perfect circle with a radius of 1.0 shown in figure 5.

Figure 5



But before the scientists could feel too self-satisfied, they saw that the orbit took longer than the expected 10 units of time, the period of an electron orbiting a stationary nucleus: in fact the period was still 16.67.

The scientists realized that since moving clocks tick at a different rate than stationary clocks, the Galilean equivalence of rocket and laboratory time, $t' = t$, also had to be modified. They tried the same trick that worked before, multiplying t by $\sqrt{1-\frac{{v_\infty}^2}{c^2}}$ to undo the observed time dilation. This produced a new equation for rocket time in terms of laboratory time,

$$t' = t\sqrt{1-\frac{{v_\infty}^2}{c^2}} \tag{4}$$

When they used equation 4 to calculate the orbital period of the electron in the rocket frame, $t'(3016.67) - t'(3000)$, they got the desired result, 10. But when equation 4 was solved for lab time in terms of rocket time,

$$t = {t'}\Big/{\sqrt{1-\frac{{v_\infty}^2}{c^2}}} \tag{5}$$

they immediately saw a fatal inconsistency. The rocket scientists could rightly claim that an electron orbiting a nucleus in the lab had a period of

16.67 units of time. Using that value, equation 5 predicted that the scientists in the lab would see that same electron orbit with a period of 27.8 instead of 10 units of time! Clearly the equations connecting rocket and laboratory time in equations 4 and 5 were incorrect.

Then the scientists remembered that any set of internally consistent transformation equations connecting the two frames, whether Galilean or modified, had to have the property that the equations linking the coordinates of the lab to those in the rocket frame became those connecting the rocket coordinates to those in the lab when the sign of v_∞ was switched. Equations 4 and 5 also failed this simple test. The scientists applied this strategy to equation 3 to get an equation for Z in terms of Z' and t', equation 6.

$$Z = \frac{Z' + v_\infty t'}{\sqrt{1 - \frac{{v_\infty}^2}{c^2}}} \qquad (6)$$

Based on their earlier result shown in figure 5, they were confident that equation 6 would convert the ellipse seen by the rocket scientists for the electron's orbit in the lab back to a circle of radius 1.0, albeit with the incorrect period, namely 16.67 instead of 10.

But now they had two equations, 3 and 6, connecting lab and rocket coordinates. It was easy to eliminate Z' from those two equations and arrive at the following equation[11] for rocket time in terms of Z and t,

$$t' = \frac{t - \frac{v_\infty Z(t)}{c^2}}{\sqrt{1 - \frac{{v_\infty}^2}{c^2}}} \qquad (7)$$

When the scientists used equation 7 to calculate the period for the electron's orbit in terms of rocket time, $t'(3016.67) - t'(3000)$, they got 10. Moreover, equations 2, 3, and 7 transformed properly when the sign of v_∞ was switched. These new equations connecting the lab and rocket coordinates confirmed that the orbit of the model electron about a stationary nucleus was a circle of radius 1.0 with a period of 10 whether it is in the lab or riding on the rocket.

These modified equations connecting laboratory and rocket observers eliminated all the apparent contradictions that arose when laboratory and rocket observers each measured the “moving” meter sticks to be shorter and “moving” clocks to run slow.

V. Final Observations

The scientists who did the *gedanken* experiments outlined above used their understanding of Maxwell’s equations and **F** = d**p**/dt to formulate a set of transformations, equations 2, 3, and 7, connecting observers moving at a constant velocity with respect to one another. Those equations`, deduced without any prior knowledge of special relativity, are the well known Lorentz transformation. J. S. Bell, in the essay cited in reference 1, shows how Lorentz invariance connects this approach to special relativity with the more typical axiomatic approach based on the constancy of the speed of light and the principle of relativity. The book, **Theory of Relativity Based on Physical Reality**,[12] develops the alternative approach outlined here in greater depth. In a more recent article, D. J. Miller[13] also argues for more visibility for the constructive approach to special relativity.

The orbit of an electron around an accelerating nucleus morphs from a circle to an ellipse. This suggests that the contraction of the meter stick takes place during the acceleration phase of the motion when the forces acting on the electron are changing. Edwin F. Taylor and A. P. French,[14] starting from the Lorentz equations, show how the stress-free acceleration of a rocket causes it to contract in the frame in which the acceleration began. The stress-free acceleration of the rocket requires that each piece of the rocket move at the same speed when viewed from a local inertial reference frame co-moving with the rocket. These two apparently different "causes" of the contraction of moving objects are also connected by Lorentz invariance.

Lastly, note that the modified transformation equations 3 and 7 reduce to the Galilean equations as the speed of light approaches infinity.[15] In the same limit, the electric and magnetic fields produced by an accelerating charge, equations A7 to A8 in the Appendix, reduce to the fields produced by a stationary nucleus,

$$E_x = \frac{q}{4\pi\epsilon_0}\frac{x(t)}{R^3} \qquad (8a)$$

$$E_y = \frac{q}{4\pi\epsilon_0}\frac{y(t)}{R^3} \qquad (8b)$$

$$E_z = \frac{q}{4\pi\epsilon_0}\frac{z(t) - z_q(t)}{R^3} \qquad (8c)$$

with the two components of the magnetic field, equations A14 and A15, going to zero.[16] The fact that length contraction and time dilation depend on the finite speed of light can be demonstrated by looking directly at the Lorentz transformation equations or by showing that the forces acting on an electron orbiting an accelerating nucleus are identical to those acting on an electron in a stationary atom. Both views lead to the same conclusion: namely that length contraction and time dilation would not exist if light had an infinite speed. Consequently, the most insightful answer to the question of why special relativity may well be "because the speed of light is finite!"

Appendix: Derivation of the Equations of Motion

Equations A1 and A2 are the electric and magnetic fields produced by an accelerating charge.[17, 18]

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{(1-\frac{\boldsymbol{v}_q\cdot\boldsymbol{n}}{c})^3}\left[\frac{\left(1-\frac{{v_q}^2}{c^2}\right)\left(\boldsymbol{n}-\frac{\boldsymbol{v}_q}{c}\right)}{R^2}\right.$$

$$\left.+\frac{\boldsymbol{n}\frac{\boldsymbol{a}_q\cdot\boldsymbol{n}}{c^2}-\frac{\boldsymbol{a}_q}{c^2}-\boldsymbol{v}_q\frac{\boldsymbol{a}_q\cdot\boldsymbol{n}}{c^3}+\boldsymbol{a}_q\frac{\boldsymbol{v}_q\cdot\boldsymbol{n}}{c^3}}{R}\right] \qquad (A1)$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{q}{4\pi c\epsilon_0}\frac{1}{(1-\frac{\boldsymbol{v_q}\cdot\boldsymbol{n}}{c})^3}\left[\frac{\left(1-\frac{{v_q}^2}{c^2}\right)\left(\frac{\boldsymbol{v_q}\times\boldsymbol{n}}{c}\right)}{R^2}\right.$$

$$\left. -\frac{\boldsymbol{n}\times(\frac{\boldsymbol{a_q}}{c^2}-\boldsymbol{n}\times\frac{(\boldsymbol{a_q}\times\boldsymbol{v_q})}{c^3}}{R}\right] \qquad (A2)$$

$\boldsymbol{R}$ is the vector connecting the nucleus at the retarded time t_r to the field point $\boldsymbol{r}(t)$, and $\boldsymbol{n}$ is the unit vector in the $\boldsymbol{R}$ direction, $\boldsymbol{R}/R$.

$$\boldsymbol{R} = \boldsymbol{r}(t) - \boldsymbol{r_q}(t_r) \qquad (A3)$$

The retarded time is given by,

$$t_r = t - {}^{|\boldsymbol{R}|}/_{c} = t - {}^{R}/_{c} \qquad (A4)$$

$\boldsymbol{v_q}$ and $\boldsymbol{a_q}$ are the velocity and acceleration of the point charge at the retarded time, $\boldsymbol{v_q}(t_r)$ and $\boldsymbol{a_q}(t_r)$, respectively.

For the calculations used in the body of the manuscript, it was assumed that the point charge moved along the z-axis starting from rest at x = y = z = 0.

Under this assumption, the velocity and acceleration in equations A1 and A2 can be replaced by,

$$\boldsymbol{v_q}(t) = v_q(t)\boldsymbol{k} \tag{A5}$$

$$\boldsymbol{a_q}(t) = a_q(t)\boldsymbol{k} \tag{A6}$$

With this simplification the components of the electric field become the following:

$$E_x = \frac{q}{4\pi\epsilon_0} \frac{x(t)}{\left(1 - \frac{v_{\boldsymbol{q}}(t_r)(z(t) - z_q(t_r))}{c^2(t - t_r)}\right)^3} \left[\frac{1 - \frac{v_q(t_r)^2}{c^2}}{c^3(t - t_r)^3} + \frac{a_{\boldsymbol{q}}(t_r)(z - z_q(t_r))}{c^5(t - t_r)^3}\right] \tag{A7}$$

$$E_y = \frac{q}{4\pi\epsilon_0} \frac{y(t)}{\left(1 - \frac{v_{\boldsymbol{q}}(t_r)(z(t) - z_q(t_r))}{c^2(t - t_r)}\right)^3} \left[\frac{1 - \frac{v_q(t_r)^2}{c^2}}{c^3(t - t_r)^3} + \frac{a_{\boldsymbol{q}}(t_r)(z - z_q(t_r))}{c^5(t - t_r)^3}\right] \tag{A8}$$

$$E_z = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(1 - \frac{v_{\boldsymbol{q}}(t_r)\left(z(t) - z_q(t_r)\right)}{c^2(t-t_r)}\right)^3} \left[\frac{\left(z(t) - z_q(t_r) - v_q(t_r)(t-t_r)\right)\left(1 - \frac{v_q(t_r)^2}{c^2}\right)}{c^3(t-t_r)^3} + \frac{a_{\boldsymbol{q}}(t_r)\left(z(t) - z_q(t_r)\right)^2}{c^5(t-t_r)^3} - \frac{a_{\boldsymbol{q}}(t_r)}{c^3(t-t_r)} \right] \quad (A9)$$

The following relationships were used to simplify expressions A7 to A9.

$$\boldsymbol{R}(\boldsymbol{t}) = \boldsymbol{r}(\boldsymbol{t}) - z_q(t_r)\boldsymbol{k} \quad (A10)$$

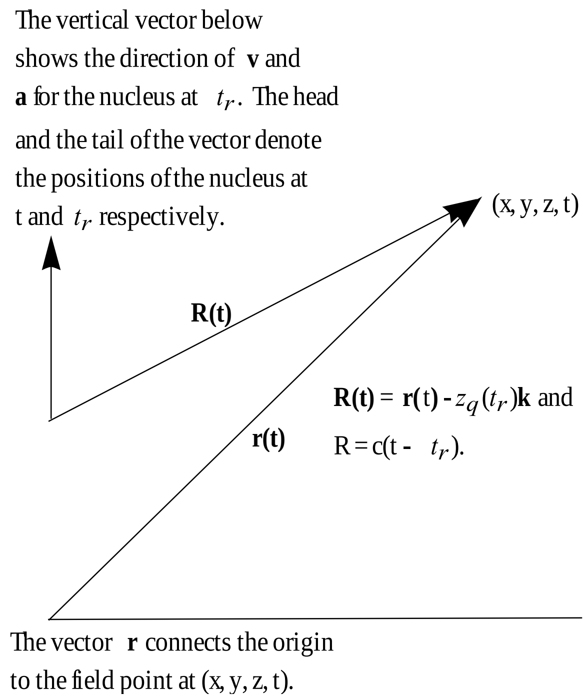
$$R = c(t - t_r) \quad (A11)$$

$$\boldsymbol{v_q} \cdot \boldsymbol{n} = \frac{v_{\boldsymbol{q}}(t_r)(z - z_q(t_r))}{\left|\boldsymbol{r}(t) - z_q(t_r)\boldsymbol{k}\right|} = \frac{v_{\boldsymbol{q}}(t_r)(z - z_q(t_r))}{c(t-t_r)} \quad (A12)$$

$$\boldsymbol{a_q} \cdot \boldsymbol{n} = \frac{a_{\boldsymbol{q}}(t_r)(z - z_q(t_r))}{\left|\boldsymbol{r}(t) - z_q(t_r)\boldsymbol{k}\right|} = \frac{a_{\boldsymbol{q}}(t_r)(z - z_q(t_r))}{c(t-t_r)} \quad (A13)$$

Figure A1 below shows the relationship between **R**(t) and **r**(t).

FigureA1: The relationship between **r**(t) and **R**(t)



The equations for the components of the magnetic field are,

$$B_x = -\frac{q}{4\pi c\epsilon_0}\frac{y(t)}{\left(1-\frac{v_{\boldsymbol{q}}(t_r)(z(t)-z_q(t_r))}{c^2(t-t_r)}\right)^3}\left[\frac{v_q(t_r)\left(1-\frac{v_q(t_r)^2}{c^2}\right)}{c^4(t-t_r)^3} + \frac{a_q(t_r)}{c^4(t-t_r)^2}\right] \qquad (A14)$$

$$B_y = \frac{q}{4\pi c\epsilon_0} \frac{x(t)}{\left(1 - \frac{v_{\boldsymbol{q}}(t_r)(z(t) - z_q(t_r))}{c^2(t - t_r)}\right)^3} \left[\frac{v_q(t_r)\left(1 - \frac{v_q(t_r)^2}{c^2}\right)}{c^4(t - t_r)^3} + \frac{a_q(t_r)}{c^4(t - t_r)^2} \right] \quad (A15)$$

$$B_z = 0 \quad (A16)$$

A17 and A18 were used in arriving at equations A14 and A15.

$$\boldsymbol{a_q} \times \boldsymbol{n} = -\frac{a_q(t_r)y}{c(t - t_r)}\boldsymbol{i} + \frac{a_q(t_r)x}{c(t - t_r)}\boldsymbol{j} \quad (A17)$$

$$\boldsymbol{v_q} \times \boldsymbol{n} = -\frac{v_q(t_r)y}{c(t - t_r)}\boldsymbol{i} + \frac{v_q(t_r)x}{c(t - t_r)}\boldsymbol{j} \quad (A18)$$

The Lorentz force acting on the electron is given by,

$$\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}) \quad (A19)$$

The velocity of the electron is represented by **V** to distinguish it from the velocity of the nucleus, $\boldsymbol{v_q}$, and the charge on the electron is e. Assume the electron orbits the stationary nucleus in the xz-plane, y = 0. When the electron's orbit crosses the x-axis its velocity is in z-direction. If the acceleration of the nucleus begins just as the electron crosses the x-axis, the forces on the electron are only due to E_x, E_z , and B_y and are in the x and z-directions keeping the orbit confined to the xz-plane. Using equation A19, the components of the force acting on the orbiting electron are,

$$F_x = e\left(E_z - V_z B_y\right) \qquad (A20)$$

$$F_y = 0 \qquad (A21)$$

$$F_z = e\left(E_z + V_x B_y\right) \qquad (A22)$$

These forces cause changes in the electron's relativistic momentum. The equations of motion for the electron are,

$$F_x = \frac{d}{dt}\left[\frac{m_e V_x}{\sqrt{1 - \frac{V^2}{c^2}}}\right] = e\left(E_x - V_z B_y\right) \qquad (A23)$$

$$F_z = \frac{d}{dt}\left[\frac{m_e V_z}{\sqrt{1 - \frac{V^2}{c^2}}}\right] = e\left(E_z + V_x B_y\right) \qquad (A24)$$

Equations A23 and A24 involve the coordinates of the orbiting electron in the laboratory, $X(t)$ and $Z(t)$, and the velocity and acceleration of the nucleus at t_r. The relationship between t and t_r is found explicitly by dotting equation A10 with itself and then differentiating it with respect to t. The result is,

$$\frac{dt}{dt_r} = \frac{c^2(t-t_r) - V_x(t)X(t) - V_z(t)(Z(t) - z_q(t_r))}{c^2(t-t_r) - v_q(t_r)(Z(t) - z_q(t_r))} \qquad (A25)$$

Before the three coupled second order differential equations for $X(t)$, $Z(t)$, and $t_r(t)$ were solved numerically, equation A24 was used to eliminate $Z"(t)$ from equation A23 and $X"(t)$ was removed from equation A24 in an analogous fashion. The primes in these equations denote derivatives with respect to t.

Next, the following simplifications were made, $\frac{q}{4\pi\epsilon_0}$ and c were set equal to 1 and e, the charge on the electron, was set equal to -1. This left the mass of the electron, m_e, as the one remaining adjustable variable. The new set of coupled differential equations are listed below in terms of $E_x(t)$, $E_z(t)$, and $B_y(t)$, equations A7, A9, and A15.

$$0 = X"(t)(1 - Z'(t)^2) - \frac{X'(t)Z'(t)}{1 - X'(t)^2}\left(X"(t)Z'(t)X'(t) + \frac{E_z(t) + X'(t)B_y(t)}{m_e}(1 - X'(t)^2 - Z'(t)^2)^{\frac{3}{2}}\right) + \frac{E_x(t) - Z'(t)B_y(t)}{m_e}(1 - X'(t)^2 - Z'(t)^2)^{\frac{3}{2}} \qquad (A26)$$

$$0 = Z''(t)(1 - X'(t)^2) - \frac{X'(t)Z'(t)}{1-Z'(t)^2}\Big(Z''(t)Z'(t)X'(t) + \frac{E_x(t) - Z'(t)B_y(t)}{m_e}(1 -$$

X't2−Z't232+Ezt+X'tBy(t)me1−X't2−Z't232 *(A27)*

$$0 = t_r'(t) - \frac{t - t_r(t) - X'(t)X(t) - Z'(t)\left(Z(t) - z_q\left(t_r(t)\right)\right)}{t - t_r(t) - v_q\left(t_r(t)\right)\left(Z(t) - z_q\left(t_r(t)\right)\right)} \qquad (A28)$$

Before these equations were solved using the NDSolve command in *Mathematica*, an appropriate set of initial conditions had to be established and an explicit form chosen for the acceleration of the nucleus. Since the orbital parameters are in arbitrary units, the following choices were made: the electron orbited the stationary nucleus with a radius of 1 and a period of 10 and the initial position of the electron was X = 1 and Z = 0. These choices made the initial velocity $\frac{\pi}{5}$, the electron's mass $\frac{5\sqrt{25-\pi^2}}{\pi^2}$, and $t - t_r = 1$. This led to the following initial conditions:

$$t_r(1) = 0 \qquad (A29)$$

$$X(1) = 1 \qquad (A30)$$

$$V_x(1) = 0 \qquad (A31)$$

$$Z(1) = 0 \tag{A32}$$

$$V_z(1) = \frac{\pi}{5} \tag{A33}$$

The acceleration of the nucleus was selected to be,

$$a_q(t) = \frac{v_\infty}{125} e^{-\frac{t}{250}} \left(1 - e^{-\frac{t}{250}}\right) \tag{A34}$$

The velocity of the nucleus reaches 0.999 of its asymptotic velocity, v_∞, when t ≥ 2000. The numerical solutions of these equations for $X(t)$ and $Z(t)$ at selected values of v_∞ were used to produce the graphs in sections III and IV.

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[1] J.S. Bell, *Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy*, (Cambridge University Press, Cambridge, 1987), pp. 67-80.

[2] A. Einstein, *Relativity, the Special and General Theory*, Authorized translation by Robert W. Lawson, (Three Rivers Press, New York, 1961), pp. 131-138.

[3] In order for the meter stick to be re-examined, it had to first be accelerated to the speed v and then decelerated back to rest. Consequently, the role of acceleration in the contraction of the meter stick is a germane topic of concern.

[4] The following quote comes from page 227 of N. David Mermin's book, *Space and Time in Special Relativity*, (McGraw Hill, New York, 1968), "It is remarkable that the electromagnetic forces between the particles of a moving stick differ from those between the particles of a stationary stick in just such a way as to cause the length of the moving stick to be smaller by a factor $\sqrt{1 - \frac{v^2}{c^2}}$." Mermin elaborates this idea in, *It's About Time: understanding Einstein's relativity*, (Princeton University Press, New Jersey, 2005), pp. 179-186.

[5] Note that the Liénard-Weichert potentials, used to find the fields due to a moving charge, were discovered before 1905, the year Einstein published his article on special relativity. Consequently, these scientists were able to derive the electric and magnetic fields due a moving charge without knowing about special relativity.

[6] The scientists discovered the modification of F = dp/dt by accelerating a charge of known mass in a large parallel plate capacitor and noting its position every small time increment, Δt. During each time increment the change in momentum was the same, Δp = eEΔt, but the velocity, Δx/Δt, changed by less and less and asymptotically approached the speed of light. By plotting p vs v the scientists were able to deduce the correct equation for momentum, $p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}$.

[7] http://www.physics.unlv.edu/~lenz/AJP.calc.nb, this *Mathematica* notebook has the solution of the equations of motion for X(t) and Z(t).

[8] The parameters used in the numerical solution were chosen so that the electron orbited the stationary nucleus with a radius of one and a period of ten.

[9] A. P. French, Special Relativity, (W. W Norton & Company, New York, 1968), pp. 66-68

[10] See equations 14 in reference 1, page 72.

[11] Ibid. Note that the equation for t′ erroneously has the x instead of the z coordinate.

[12] L. Janossy, *Theory of Relativity Based on Physical Reality*, (Academia Kiado, Budapest 1971).

[13] J. D. Miller, "A constructive approach to the special theory of relativity," Am. J. Phys, **78** (6), pp. 663-638.

[14] Edwin F. Taylor and A. P. French, "Limitation on Proper Length in Special Relativity," Am. J. Phys. **51** (10), pp. 889-893.

[15] Wolfgang Rindler, *Relativity: Special, General, and Cosmological*, 2nd Edition, (Oxford University Press, Oxford, 2006), page 57.

[16] In order for c to go to infinity, either μ_0 or ϵ_0 has to approach zero. If we keep the electric field finite, $\epsilon_0 \neq 0$, then $\mu_0 = 0$, and the magnetic field disappears!

[17] Wolfgang K. H. Panofsky and Melba Phillips, *Classical Electricity and Magnetism*, 2nd Edition, (Addison - Wesley Publishing, Massachusetts, 1962), pp. 354-357.

[18] John David Jackson, *Classical Electrodynamics*, 3rd Edition, (John Wiley & Sons, New York, 1999), pp. 661-665.