The Twin Paradox

In thinking about the twin paradox you have a big advantage over the presentation in the book because you covered the special section on the Lorentz Transformations. In this exercise, those equations will be used to try to shed light on the paradox.

Dick and Jane are twins that have volunteered for this experiment. Dick will stay home and Jane will travel to a nearby star, Delta Pavonis, that is 20 light-years from earth. The star is stationary in the earth frame which is also Dick's frame.

The picture you will draw is based on Dick's perspective and will include the outgoing rocket that takes Jane from earth to Delta Pavonis. When she arrives at Delta Pavonis, she immediately jumps onto an incoming ship that is passing Delta Pavonis at the same instant that the outgoing ship passes the star.

Consequently, Dick has to draw a space time graph that includes a reference frame for the out-going rocket and another for the incoming rocket. Dick understands that to successfully use the Lorentz Transformations he has to set up those two reference frames so that their origins are coincident with his when all three clocks read zero.

Let x and t represent Dick's coordinates, x' and t' the coordinates of the outgoing rocket, and x" and t" the coordinates in the incoming rocket's frame. From Dick's perspective, the outgoing rocket is moving with a velocity of +4/5 that of light while the incoming rocket is moving with a velocity of -4/5.

1. The first step is to write down the Lorentz Transformation equations connecting x and t to x' and t' and the inverse equations connecting x' and t' to x and t.

2. The next step is to write down the Lorentz Transformation equations connecting x and t to x" and t" and the inverse equations connecting x" and t" to x and t.

Jane starts her trip to Delta Pavonis, DP for short, at $t = t' = t'' = 0$.

3. At that instant of time, where is the incoming ship located according to Dick. Remember that ship is on track to rendezvous with Jane at DP.

4. At the space time point in 3, the incoming ship is located at what space point in the incoming frame and the clock onboard the incoming ship reads what time, $x'' = ?$ And $t'' = ?$

5. According to Dick, how long will the trip to DP take?

Use the information in 3 and 5 to draw a graph that includes Dick's worldline on earth, the path of the outgoing ship carrying Jane to DP, and the path of the incoming ship that will return Jane to earth.

6. Use the Lorentz equations to find the time on the outgoing ship's clock, which is the same as the time on Jane's personal chronometer, as the ship reaches DP.

7. Again use the Lorentz equations to find the time on the incoming ship's clock as it passes DP.

Note that at the instant of arrival at DP Jane's chronometer reads _____ years. When she looks at the clock of the earth frame person stationed on DP, she sees that clock reading _____ years. And of course the earth person sees the same two readings when looking at his clock and Jane's chronometer. Jane quickly teleports onto the incoming ship. When she arrives, she notices that the clock on the incoming ship reads _____ years. Again, there is absolutely no disagreement among the three sets of observers about what any of the clocks read.

8. Now use the equation for t' in terms of x and t to draw the line of constant t' that corresponds to the time on the outgoing ship as it passes DP. Extend this line until it reaches Dick patiently waiting on earth. What is the time on Dick's chronometer at that instant.

Note that the intersection of those two lines corresponds to an observer in the

outgoing frame whose clock is synchronized with Jane's passing Dick patiently waiting at x = 0. That observer would surmise that Dick has been moving toward her at a constant speed of 4/5 and consequently would expect to see that Dick's chronometer has been running slow during the time the rocket has been traveling to DP.

9. With this in mind, are the times on the two passing clocks consistent?

10. Now use the equation for t" in terms of x and t to draw the line of constant t" that corresponds to the time on the incoming ship as it passes DP. Extend this line until it reaches Dick patiently waiting on earth. What is the time on Dick's chronometer at that instant.

Note that Jane's chronometer and the incoming ship's clock do not read the same times. This is a result of Jane having jumped from one reference frame to another. But during the trip back to earth, Jane's chronometer and the ship's clock will tick off the same amount of time.

11. When Jane reaches earth, her chronometer reads _____ years while the ship's clock reads _____ years and Dick's chronometer reads _____ years.

Another way of picturing this trip is to have Dick and Jane, as per an agreement made before the trip began, send a message to the other person each time their chronometer ticks off 5 years. Those messages travel at light speed and can be added to space time drawing. The messages get Doppler shifted on both legs of the journey but note that Jane receives exactly the right number of messages corresponding to the time ticked off on Dick's chronometer while Dick also receives the correct number of messages corresponding to the time ticked off of Jane's chronometer. Consequently, neither Dick nor Jane is surprised by the aging difference between them that accrued during her trip to Delta Pavonis and back.