Cosmology & Galaxies NAME:

Homework 24: Cosmic-Present Star-Forming Galaxies (SFGs)

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024 qmult 00180 1 1 3 easy memory: Milk Way stellar mass and virial mass

1. In solar mass units, the Milky Way stellar mass (i.e., the mass in stars M_*) is $\sim$ __________ $M_\odot$ and its virial mass ($M_{\rm vir}$: i.e., fiducial total mass which is mostly dark matter) is $\sim$ __________ $M_\odot$. At least these values were standard circa 2023. However, a downward revision may have become accepted just about that year.

   a) 10^{12}; 10^{10} b) 5×10^{10}; 5×10^{10} c) 5×10^{10}; 10^{12} d) 10^9; 5×10^{10}
   e) 10^9; 5×10^8

   SUGGESTED ANSWER: (c) See Ci-55. See Ou et al. (2023, arXiv:2303.12838) for a downward revision of the virial mass to $\sim 2 \times 10^{11}\,M_\odot$

   Wrong answers:
   a) As Lurch would say AAAarrgh.

   Redaction: Jeffery, 2008jan01

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024 qmult 00230 1 4 2 easy deducto-memory: exponential profile for face-on spiral galaxies

2. "Let's play *Jeopardy*! For \$100, the answer is:

$$I_\lambda = I_{\lambda,0} e^{-(R/R_{\rm d})} ,$$

   where I_λ is the surface brightness, $I_{\lambda,0}$ is the central surface brightness, R is the radius coordinate, and $R_{\rm d}$ disk scale length (and not the effective or half-light radius)."

   What is the standard __________ surface brightness profile, Alex?

   a) edge-on spiral disc b) face-on spiral disc c) elliptical d) dwarf irregular
   e) general Sérsic

   SUGGESTED ANSWER: (b)

   Wrong answers:
   a) As Lurch would say AAAARGH.

   Redaction: Jeffery, 2008jan01

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024 qmult 00600 1 1 4 easy memory: two main classes of galaxy bulges

3. There are two main classes of galaxy bulges:

   a) classical bulges and non-classical bulges b) big bulges and disc-like bulges
   c) little bulges and disc-like bulges d) classical bulges and disc-like bulges
   e) little bulges and big bulges

   SUGGESTED ANSWER: (d)

   Wrong answers:
   a) Seems reasonable.

   Redaction: Jeffery, 2008jan01

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024 qmult 00620 1 4 2 easy deducto-memory: Schmidt-Kennicutt law

4. "Let's play *Jeopardy*! For \$100, the answer is:

$$\Sigma_{\rm SFR} = B \left(\frac{\Sigma_{\rm gas}}{1\,M_\odot/{\rm pc}^2} \right)^\alpha M_\odot/{\rm yr}/{\rm kpc}^2 ,$$

   where SFR means star formation rate, $\Sigma_{\rm SFR}$ is surface star formation rate in units of $M_\odot/{\rm yr}/{\rm kpc}^2$, $\Sigma_{\rm gas}$ is gas surface density in units $M_\odot/{\rm pc}^2$ (the denominator below it makes the overall factor dimensionless), $B \approx 10^{-4}$ is an empirical constant, and $\alpha = 1.40(15)$ is another empirical constant with some theoretical understanding.

What is the ____________, Alex?

a) Press-Kennicutt law b) Schmidt-Kennicutt law c) Press-Schecter law
d) Martin-Schmidt e) Martin-Schmidt-Kennicutt law

SUGGESTED ANSWER: (b)

Wrong answers:

c) As Lurch would say AAAARGH.

Redaction: Jeffery, 2008jan01

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024 qfull 00100 1 3 0 easy math: inclined circle analyzed: On exams, do all parts.

5. An inclined circle (ideal disc galaxy is) is seen in projection as an ellipse.

NOTE: There are parts a,b. On exams, do all parts.

a) The equation for a circle is written elaborately is

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2=1\ ,$$

where a is the radius. Find the explicit formula for y. The circle is rotated on its x-axis to inclination angle i where inclination angle is measured from the direction to the observer to a normal to the circle. What is the projected height of every y point (i.e., what is the inclined y_i)? Prove the inclined circle (i.e., projected circle) is an ellipse and find its semi-minor axis b.

b) The area of an ellipse is $A=\pi ab$ and the circularized radius an ellipse created by inclination is defined by

$$R_i=\sqrt{ab}=a\sqrt{\cos(i)}\ .$$

Prove that the differential area of an inclined circle is

$$dA=2\pi R_i\,dR_i\ .$$

SUGGESTED ANSWER:

a) Behold:

$$y=\pm a\sqrt{1-\left(\frac{x}{a}\right)^2}$$

By inspection of a diagram you will have to imagine, every inclined y_i is given by $y_i=y\cos(i)$. If you multiple the circle formula for y by $\cos(i)$, you get

$$y_i=\pm a\cos(i)\sqrt{1-\left(\frac{x}{a}\right)^2}=\pm b\sqrt{1-\left(\frac{x}{a}\right)^2}\ ,$$

where we have defined $b=a_i\cos(i)$. The new formula identified by inspection as that of an ellipse where y_i is the y coordinate and b is the semi-minor axis. Thus, the inclined circle is an ellipse with semi-minor axis b

b) Behold:

$$dA=\pi d(ab)=d\left[a^2\cos(i)\right]=2\pi a\,da\cos i=2\pi a\sqrt{\cos(i)}\,da\sqrt{\cos(i)}=2\pi R_i\,dR_i\qquad\text{QED.}$$

Redaction: Jeffery, 2018jan01

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028 qfull 00350 1 3 0 easy math: free-fall time and collapse to star time: On exams, only do parts a,b,c,d.

6. The free-fall time for a straight line fall of a particle of mass m starting from rest to a point source or spherically symmetric source of mass M (always interior to the infalling particle) is

$$t_{\text{ff}}=\frac{t_{\text{orbit}}}{2}=\frac{\pi}{\sqrt{G(M+m)}}\left(\frac{r}{2}\right)^{3/2}\quad ,$$

where t_{orbit} is the orbital period predicted by the Newtonian physics version of Kepler's 3rd law and r is the initial distance from the particle to the source center and is twice the relative semi-major axis of an elliptical orbit of the particle to the source (Wikipedia: Free-fall time; Wikipedia: Kepler's laws of planetary motion Third law; Ci-246). The Kepler's 3rd law orbital period is independent of eccentricity $e \leq 1$, and so half of it is the free-fall time.

NOTE: There are parts a,b,c,d,e,f. On exams, only do parts a,b,c,d.

a) What is the free-fall time for test particle (i.e., one of negligible mass)?

b) What is the free-fall time as a function of r for a spherical mass distribution with initially constant density ρ and outer radius $r \leq R$. The matter is initiall all at rest and there is zero pressure at all times. Assume the (infinitely thin) shells of matter in the distribution at all the r values never cross during free fall which is true and plausible, but seems tricky to prove. Describe the order of arrival of the shells at the center?

HINT: Remember the shell theorem

$$\vec{g} = -\frac{GM(r)}{r^2}\hat{r}$$

where the mass distribution is sperically symmetric and $M(r)$ is the interior mass to radius r. Note $M(r)$ must increase monotonically since there is no negative mass, but it can be zero out some radius r.

NOTE: For all subsequent parts, we assume a spherically symmetric mass distribution at all times with initial outer radius R and there is zero pressure at all times.

c) Say that the interior mass $M(r)$ to radius r obeys a power law $M(r) = M_0(r/r_0)^\alpha$ where $\alpha \leq 3$. When does the mass all collapse to the center assuming that it magically all stops there on arrival and the shells of matter at all the r values never cross during free fall which is true and plausible, but seems tricky to prove.

d) Do infalling shells ever cross for any possible mass distribution?

HINT: First, show that an infinitely thin shell of nonzero mass should self-gravitate in the most simple physically realistic limit. To be convincing this proof is a bit trickier than it seems. Consider a finite thin shell of constant mass m, constant density ρ (which we consider leads to the the most simple physically realistic limit), inner radius r_{a}, outer radius $r_{\text{a}} + \Delta r$, let $\delta r = r - r_{\text{a}}$, let $m(\delta r)$ be the mass from r_{a} to $r_{\text{a}} + \delta r$, and work to 1st order in Δr and δr. Second, consider a mass distribution that consists of just two infinitely thin shells (that follow from limit with constant density) initially at rest. Let the inner shell have mass m_1 and radius r_1 and the outer shell have mass m_2 and radius r_2. Third, show that that the masses can be chosen such that the outer shell has shorter free-fall time than the inner shell. Remember, the shell theorem. This is a very idealized setup, but it will lead to the answer.

e) For star formation, we want to relate density ρ to the particle density n which can be measured more directly. The relating formula is

$$n = \rho \left(\sum_i \frac{X_i}{A_i m_{\text{p}}} \right) ,$$

where X_i is the mass fraction of species i (which could be any atom or a molecule including those that are distinct due to their isotopic nature), A_i is the atomic mass number (which could be a molecular mass number), and $m_{\text{p}} = 1.67262192369(51) \times 10^{-24}\,\text{g}$ is the proton mass. Note this special case atomic mass number is in units of proton masses, not daltons (symbol u or Da and AKA atomic mass units). The fact is most of the universe is made of hydrogen (which made of protons) and not made of daltonium (which is made of daltons). Worrying about corrections due to electron masses, binding energies, and isotopes abundances (which aside from hydrogen and helium are rather uncertain) is below the level of accuracy of this problem. The mean atomic mass is defined by

$$\mu^{-1} = \sum_i \frac{X_i}{A_i}$$

which gives

$$n = \frac{\rho}{\mu m_{\text{p}}} \qquad \text{or} \qquad \rho = n\mu m_{\text{p}} \ .$$

Fiducial cosmic values for X_i are: $X = 0.73$ for H, $Y = 0.25$ for He, and $Z = 0.02$ for metals. Two fiducial mean atomic masses are given by

$$\mu_{\mathrm{H}_1,\mathrm{dominated}} = \left(\frac{X}{1} + \frac{Y}{2} + \frac{Z}{30}\right)^{-1} \qquad \text{and} \qquad \mu_{\mathrm{H}_2,\mathrm{dominated}} = \left(\frac{X}{2} + \frac{Y}{2} + \frac{Z}{30}\right)^{-1} ,$$

where the atomic mass for Z is a rough fiducial average based on the fiducial atomic masses of very abundant metals: i.e., $A_{\mathrm{C},6} = 12$, $A_{\mathrm{O},8} = 16$, $A_{\mathrm{Si},14} = 28$, and $A_{\mathrm{Fe},28} = 56$. Compute the $\mu_{\mathrm{H}_1,\mathrm{dominated}}$ and $\mu_{\mathrm{H}_2,\mathrm{dominated}}$ values to 3-digit precision which probably 1 more digit than is significant, but it is useful to know insignificant digits sometimes to check for consistency between different calculations.

HINT: Write a small computer program to do the calculation.

f) The part (b) answer gives a fiducial lower limit for the formation time for a star. It is just a fiducial lower limit since real initial clouds of molecular gas do not have uniform density, are not spherically symmetric, and do not have zero pressure and zero initial kinetic energy. It is just a lower limit since the pressure force and kinetic energy in the molecular cloud resist collapse during the collapse process and delay collapse to a star sized object. However, it is useful to rewrite the part (b) answer in terms fiducial values: particle density $10^3\,\mathrm{cm}^{-3}$, $\mu_{\mathrm{H}_2,\mathrm{dominated}}$ from part (e), and Julian years (i.e., 365.25 days). Do the rewrite.

HINT: Write a small computer program to do the calculation.

SUGGESTED ANSWER:

a) Behold:

$$t_{\mathrm{ff}} = \frac{\pi}{\sqrt{GM}}\left(\frac{r}{2}\right)^{3/2} .$$

b) Note a shell always has the same mass interior to it during free fall and that mass always acts as a point mass at the center by the shell theorem. The exterior shells have no affect by the shell theorem and since they never cross first mentioned shell. Thus, the free-fall time for all shells is

$$t_{\mathrm{ff}}(r) = \frac{\pi}{\sqrt{G(4\pi/3)\rho r^3}}\left(\frac{r}{2}\right)^{3/2} = \sqrt{\frac{3\pi}{32G\rho}} .$$

Since t_{ff} is, in fact, independent of r, the shells all arrive simultaneously at the center.

c) In this case,

$$t_{\mathrm{ff}} = \frac{\pi}{\sqrt{GM(r)}}\left(\frac{r}{2}\right)^{3/2} = \frac{\pi}{\sqrt{GM(r)}}\left(\frac{r_0}{2}\right)^{3/2}\left(\frac{r}{r_0}\right)^{3/2} = \frac{\pi}{\sqrt{GM_0}}\left(\frac{r_0}{2}\right)^{3/2}\left(\frac{r}{r_0}\right)^{(3-\alpha)/2} ,$$

where $(3-\alpha) \geq 0$. Assuming the shells never cross, the time when all test particles are at the center is where $r = R$, its maximum value. Thus,

$$t_{\mathrm{ff}} = \frac{\pi}{\sqrt{GM_0}}\left(\frac{r_0}{2}\right)^{3/2}\left(\frac{R}{r_0}\right)^{(3-\alpha)/2} .$$

d) To 1st order in δr, we have

$$m(\delta r) = 4\pi\rho r_{\mathrm{a}}^2\delta r ,$$

and thus

$$m = 4\pi\rho r_{\mathrm{a}}^2\Delta r \qquad \text{and constant density} \qquad \rho = \frac{m}{4\pi r_{\mathrm{a}}^2\Delta r} .$$

The inward force on a differential solid angle $d\Omega$ of the finite thin shell to 1st order in δr and Δr is

$$F = \int_0^{\Delta r}\frac{Gm(\delta r)}{r_{\mathrm{a}}^2}\rho r_{\mathrm{a}}^2\,d\delta r\,d\Omega = G(4\pi\rho)(\rho r_{\mathrm{a}}^2)\frac{\Delta r^2}{2}\,d\Omega$$

$$F = \frac{Gm^2}{r_{\mathrm{a}}^2}\left(\frac{1}{2}\right)\frac{d\Omega}{4\pi} ,$$

where we have used the shell theorem. In the last result holds in the limit that $\Delta r \to 0$, and thus the infinitely thin shell does self-gravitate in the physically realistic limit. Note the 1/2 factor somewhat magically gives the average between gravitational field $g = 0$ just inside the thin shell and and gravitational $g = Gm/r_{\rm a}^2$ just outside the thin shell. It is a self-gravitating correction.

Now consider the two free-fall times for the two thin shells:

$$t_1 = \frac{\pi}{\sqrt{G(m_1/2)}}\left(\frac{r_1}{2}\right)^{3/2} \qquad \text{and} \qquad t_2 = \frac{\pi}{\sqrt{G[m_1+(m_2/2)]}}\left(\frac{r_2}{2}\right)^{3/2} ,$$

where the 1/2 factors are the self-gravitating corrections and recall the infalling matter is regarded as test particles of negligible mass. If $t_1 > t_2$, the shells must cross. Now $t_1 > t_2$ implies

$$\sqrt{\frac{[m_1+(m_2/2)]}{m_1/2}}\left(\frac{r_1}{r_2}\right)^{3/2} > 1 \ .$$

Clearly, the inequality holds if m_2 is made large enough. Thus, there are mass distributions where infalling shells must cross. But what happens then they cross requires further analysis.

Note, I think my self-gravitating correction is right at least at this moment. If you imagine a single thin finite shell of constant density, the inner edge feels no self-gravity by the shell theorem and the outer edge feels all the self-gravity of the shell. So the overall self-gravity on the shell is the average force on the shell which leads to as self-gravity due to one half of the mass of the shell and this one half mass acts as a point mass at the center of the shell by the shell theorem. If the shell magically stays constant in density as it falls, then this self-gravity stays the same. So the infinitely thin shell (with density constant as the limit is taken) should behave the same way. The case of the infinitely thin shells is highly idealized, but the point is that there is a physically realistic limit case where infalling shells cross, and so there must be crossing in some physically real cases too.

e) The two fiducial mean atomic masses are

$$\mu_{\rm H_1,dominated} = \left(\frac{X}{1}+\frac{Y}{2}+\frac{Z}{30}\right)^{-1} = 1.26 \quad \text{and} \quad \mu_{\rm H_2,dominated} = \left(\frac{X}{2}+\frac{Y}{2}+\frac{Z}{30}\right)^{-1} = 2.34 \ .$$

f) Behold:

$$t_{\rm ff}(r) = \sqrt{\frac{3\pi}{32G\rho}} = (1.07\times10^6\,{\rm Jyr})\left(\frac{\mu_{\rm H_2,dominated}}{2.34}\right)^{-1/2}\left(\frac{n_{H_2}}{10^3\,{\rm cm}^{-3}}\right)^{-1/2} \ .$$

Fortran-95 Code

```
         print*
         print*,'Fiducial mean atomic masses and fiducial free-fall time'
         x=0.73_np
         xm=1.0_np
         xm2=2.0_np
         y=0.25_np
         ym=4.0_np
         z=0.02_np
         zm=30.0_np
         xmu=1.0_np/(x/xm+y/ym+z/zm)
         xmu2=1.0_np/(x/xm2+y/ym+z/zm)
         print*,'xmu,xmu2'
         print*,xmu,xmu2
    !    1.2607690691321706241         2.3355391202802646944
         pi=acos(-1.0_np)
         pi=3.14159265358979323846264338327950288419716939937510_np
    !
!!23456789a123456789b123456789c123456789d123456789e123456789f123456789g12
```

```
    !           ! https://en.wikipedia.org/wiki/Pi#Approximate_value_and_digits 51
digits
          gcon=6.67430e-11_np  ! (15)
https://physics.nist.gov/cuu/Constants/Table/allascii.txt mks
          xmp=1.67262192369e-27_np  ! (51)
https://physics.nist.gov/cuu/Constants/Table/allascii.txt mks
          xn=1.e+3_np*(1.e+6_np)   ! from cm**(-3) to m**(-3)
          den=xmu2*xmp*xn
          xjy=365.25_np
          daysec=86400.0_np
          con=1.0_np/(xjy*daysec)
          tfid=sqrt(3.0_np*pi/(32.0_np*gcon*den))*con
          print*,'tfid'
          print*,tfid  ! 1065028.6238804413767
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Redaction: Jeffery, 2018jan01

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024 qfull 01030 1 3 0 easy math: galaxy potential energy and escape velocity: On exams, only do parts a,b,c,d.

7. In this question, we consider escape velocities from galaxies. The path is long if one does not gloss over tricky points like Ci-86–87.

   NOTE: There are parts a,b,c,d,e,f. On exams, only do parts a,b,c,d.

   a) From introductory physics, the change mechanical energy of particle is

   $$\Delta E = \Delta KE + \Delta PE = W_{\rm noncon} \ ,$$

   where KE is kinetic energy, PE is potential energy, and $W_{\rm noncon}$ work done by nonconservative forces. If there are no nonconservative forces, mechanical energy is conserved and

   $$1) \quad \Delta E = 0 \qquad 2) \quad \Delta KE = -\Delta PE \qquad 3) \quad E = KE + PE \quad \text{is constant.}$$

   The escape velocity from some point (with no nonconservative forces) can be found from some point noting that $KE = 0$ at infinity where the graviational potential Φ (which is potential energy PE per unit mass) is defined to be zero. Find the general formula for escape velocity $v_{\rm esc}$ given that kinetic energy is initially KE and gravitational potential is initially Φ.

   b) Assume a spherically symmetric mass distribution for a galaxy which seems to be often approximately true since dark matter halos are often quite spherically symmetric it seems though not always. Let the density profile be a power law

   $$\rho = \rho_{\rm s}\left(\frac{r}{r_{\rm s}}\right)^{-\alpha} = \rho_{\rm s}x^{-\alpha} \ ,$$

   where $\rho_{\rm s}$ is a scale density, $r_{\rm s}$ is a scale radius, $x = r/r_{\rm s}$ is a dimensionless radius, and α is the power. Determine the formula for interior mass $M(r)$ (i.e., mass interior to radius r) in terms of a scale $M_{\rm s}$ and x assuming $\alpha < 3$.

   c) Why can't a galaxy have pure power law density profile from $r = 0$ to $r = \infty$, in fact? **HINT:** Consider the divergence behavior of the interior mass formula.

   d) There is a tricky point in considering potential change. When integrating up the potential energy of a gravitating sphere, we use

   $$PE(r) = \int_0^r \left[\frac{-GM(r)}{r}\right] 4\pi r^2 \rho \, dr \ ,$$

   where $M(r)$ is the interior mass and $\Phi = -GM(r)/r$ is the gravitational potential r. This is the right thing to do, but $-GM(r)/r$ is not the potential at r in the fully assembled gravitating sphere. Why not? Show what the potential at r is (relative to infinity which is zero) for a gravitating sphere of total radius R. **HINT:** Getting the signs right for potential is tricky. You have to do the sign on every step right—or chance of being right is only 50 %.

e) Making use of the part (b) and the part (d) answers find the potential from $x \leq X$ for $\alpha < 3$. Show explicitly the cases for 1) $\alpha \neq 2$, 2) $\alpha \in (2,3)$ and $x << X$, 3) $\alpha < 2$ and nd $x << X$, and 4) $\alpha = 2$.

f) From the part (e) answer from the escape velocity formula for the case of $\alpha \in (2,3)$ and $x << X$ in terms of the circular velocity for scaled radius $x = 1$. What is the escape velocity if circular velocity is 200 km/s and $\alpha = 9/8$? Why are galactic outflows hard to understand if α gets very close to 2? Having α close to 2 is what is implied by the flat velocity curve ranges of observed disc galaxies.

SUGGESTED ANSWER:

a) Given $\Delta E = 0$, $\Delta KE = -KE$, and $\Delta\Phi = -\Phi$, we find

$$1)\quad 0 = \Delta KE + m\Delta\Phi = -KE - m\Phi \qquad 2)\quad KE = \frac{1}{2}mv_{\rm esc}^2 = -m\Phi \qquad 3)\quad v_{\rm esc} = \sqrt{2(-\Phi)} = \sqrt{2|\Phi|}\ .$$

b) Behold:

$$M(r) = \int_0^r 4\pi r'^2 \rho\, dr' = 4\pi r_{\rm s}^3 \rho_s \int_0^x x'^{2-\alpha}\, dx' = 4\pi r_{\rm s}^3 \rho_s \left(\frac{x^{3-\alpha}}{3-\alpha}\right) = M_{\rm s} x^{3-\alpha}$$

where we have assumed that $\alpha < 3$.

c) If $\alpha < 3$, the integral diverges for $x \to \infty$. If $\alpha = 3$, the integral is a logarithm and diverges for both $x \to 0$ and $x \to \infty$. If $\alpha > 3$, the integral diverges for $x \to 0$. The upshot is that no real galaxy can have a pure power law density profile.

d) The integral is a process of adding mass to the growing sphere of radius r and mass $M(r)$ with nothing above r. The gravitational potential $\Phi = -GM(r)/r$ is exactly right for the surface of the growing sphere relative to infinity. You bring differential mass $4\pi r^2 \rho\, dr$ from infinity and add it the growing sphere with the correct contribution to potential energy. But after you have added a finite amount of mass above r to radius R, the potential at r is given by

$$\begin{aligned}
\Phi &= -\int_\infty^r \vec{F}\cdot d\vec{r'} = \int_r^\infty \vec{F}\cdot d\vec{r'} = -\int_r^\infty \frac{GM(r')}{r'^2}\, dr' \\
&= -\int_r^R \frac{GM(r')}{r'^2}\, dr' - \int_R^\infty \frac{GM(R)}{r'^2}\, dr' = -\int_r^R \frac{GM(r')}{r'^2}\, dr' - GM(R)\left(-\frac{1}{r}\right)\Big|_R^\infty \\
&= -\int_r^R \frac{GM(r')}{r'^2}\, dr' - \frac{GM(R)}{R}\ .
\end{aligned}$$

Note that signs are infernally tricky with graviational potential. You have do every step exactly right or you will make random sign errors and only be right 50 % of the time.

e) Recalling that we are requiring $\alpha < 3$, we find

$$\begin{aligned}
\Phi &= -\int_r^R \frac{GM(r')}{r'^2}\, dr' - \frac{GM(R)}{R} \\
\Phi &= -\frac{GM_{\rm s}}{r_{\rm s}} \int_x^X x'^{1-\alpha}\, dx' - \frac{GM_{\rm s}}{R} X^{3-\alpha} \\
\Phi &= \begin{cases}
-\dfrac{GM_{\rm s}}{r_{\rm s}} \left(\dfrac{X^{2-\alpha} - x^{2-\alpha}}{2-\alpha}\right) - \dfrac{GM_{\rm s}}{R} X^{3-\alpha} & \text{for } \alpha \neq 2; \\
-\dfrac{GM_{\rm s}}{r_{\rm s}} \left[-\left(\dfrac{x^{2-\alpha}}{2-\alpha}\right)\right] = -\dfrac{GM_{\rm s}}{r_{\rm s}} \left(\dfrac{x^{2-\alpha}}{\alpha-2}\right) & \text{for } \alpha \in (2,3) \text{ and } x << X; \\
-\dfrac{GM_{\rm s}}{r_{\rm s}} \left(\dfrac{X^{2-\alpha}}{2-\alpha}\right) - \dfrac{GM_{\rm s}}{R} X^{3-\alpha} & \text{for } \alpha < 2 \text{ and } x << X; \\
-\dfrac{GM_{\rm s}}{r_{\rm s}} \ln\left(\dfrac{X}{x}\right) - \dfrac{GM_{\rm s}}{R} X^{3-\alpha} & \text{for } \alpha = 2
\end{cases}
\end{aligned}$$

We see that for $\alpha \in (2,3)$ and $x << X$, the result is independent of X. This is an easy result to guess, but takes some care to prove.

f) For $\alpha \in (2,3)$ and $x << X$, we find the escape velocity formula to be

$$v_{\rm esc} = \begin{cases} \sqrt{\dfrac{2GM_{\rm s}}{r_{\rm s}}\left(\dfrac{x^{2-\alpha}}{\alpha-2}\right)} = v_{\rm cir}\sqrt{\dfrac{2x^{2-\alpha}}{\alpha-2}} & \text{in general;} \\ (200\,{\rm km/s}) \times \sqrt{\dfrac{2}{\alpha-2}} & \text{for } x=1 \text{ and } v_{\rm cir}=200\,{\rm km/s}; \\ 800\,{\rm km/s} & \text{for } x=1,\ v_{\rm cir}=200\,{\rm km/s}, \text{ and } \alpha=9/8. \end{cases}$$

Since disc galaxies often have large density profile ranges where $\alpha \approx 2$, it is clear the escape velocities can be very high. For example the escape speed from the center of the Milky Way is $\sim$ 800 km/s (Ci-87). Such high escape velocities makes understanding galactic outflows challenging since stellar winds and supernovae do not typically reach such velocities for the bulk of their material.

Redaction: Jeffery, 2018jan01

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024 qfull 01050 1 3 0 easy math: metallicity saturation in galaxies: On exams, omit part e.

8. The metallicity of galaxies does not generally increase with cosmic time, but reaches an (approximate) plateau due to gas inflow from the intergalactic/circumgalactic medium (which if intergalactic is of nearly primordial gas: primordial cosmic gas fiducial mass fractions $X = 0.75$ H, $Y = 0.25$ He, $Z = 0.001$ metallicity which is overwhelmingly deuterium counted as a metal: Wikipedia: Big Bang: Abundance of primordial elements) and the outflow of metal enriched gas from stellar evolution (i.e., stellar winds and supernovae) back to the intergalactic/circumgalactic medium or into compact astro-bodies (compact remnants, long-lived small mass stars, brown dwarts, planets, and smaller astro-bodies). The plateau phase will probably not last forever since cosmological constant acceleration isolates all bound systems not participating in the mean expansion of the universe from fresh primordial gas. So a slow metallicity increase should occur despite gas inflow/outflows as the the overall isolated bound system gas gradually enriches. However, this enrichment seems very slow since cosmic time $\sim$ 5 Gyr after the Big Bang (Weinberg 2016, arXiv:1604.07434) and will gradually turn off with the end of the stelliferous era (theoretically cosmic time $\sim$ 0.15–10^5 Gyr: Wikipedia: Graphical timeline of the Stelliferous Era; Wikipedia: Future of an expanding universe: The Stelliferous Era). In this question, will do a simple modeling of the plateauing of galaxy metallicity.

   NOTE: There are parts a,b,c,d,e. On exams, omit part e.

   a) Write a (1st order ordinary autonomous) differential equation for galaxy gas density ρ (assumed to be uniform) in terms of a constant inflow rate of gas $F = (d\rho/dt)_{\rm inflow}$ (not necessarily primordial gas) and an outflow rate $-\kappa\rho = -\rho/\tau$, where κ is the rate constant and $\tau = 1/\kappa$ is the time constant. The outflow rate includes both outflow of gas back to the intergalactic/circumgalactic medium and into compact objects.

   b) Using an integrating factor solve the differential equation of part (a) with ρ_0 as the initial density at time zero (i.e., $t = 0$). Give 1st-order-in-small-t solution and the asymptotic solution as $t \to \infty$ (which is also the constant solution Give the asymptotic solution as $t \to \infty$ (which is also the constant solution of the differential equation). What name can be the time constant τ? Why do we have the asymptotic solution we have?

   c) Write a (1st order ordinary autonomous) differential equation for galaxy gas metal density $Z\rho$ (assumed to be uniform) in terms of a constant inflow rate of metal-only gas $Z_{\rm in}F = Z_{\rm in}(d\rho/dt)_{\rm inflow}$, where $Z_{\rm in} \in [0,1]$. Let the outflow rate be the same as in part (b): i.e., $-\kappa\rho = -\rho/\tau$, where κ is the rate constant and $\tau = 1/\kappa$ is the time constant. There is also a rate constant γ for the creation metal-only gas in the galaxy from zero-metallicity gas with density $(1 - Z)\rho$.

   d) The differential equation in part (c) can be solved for Z for general time t using the solution of part (b), but it seems a bit tedious to get this solution. However, finding the asymptotic solution $Z_{\rm asy}$ as $t \to \infty$ is easy. Find it. Check that $Z_{\rm asy}$ is dimensionally correct and show that it satisfies $Z_{\rm asy} \in [0,1]$.

   e) We can make a crude estimate of current cosmic $Z_{\rm asy}$. First, let

$$\kappa = \frac{(d\rho/dt)_{\rm outflow}}{\rho} = \frac{3\,{\rm M}_\odot/{\rm yr}}{\rho} \ ,$$

where $3\,\mathrm{M}_\odot/\mathrm{yr}$ is roughly the rate of star formation for a galaxy like the Milky Way (Ci-383) and we assume this is of order the overall gas loss rate due gas outflow back to the intergalactic/circumgalactic medium and locking up of gas in compact astro-bodies. Second, let

$$\gamma = \frac{[d(Z\rho)/dt]_{\text{metal creation}}}{\rho} = \frac{[5\,\text{SNe}/(100\,\text{yr})] \times (1\,\mathrm{M}_\odot\,\text{metals/per SNe})}{\rho} \ ,$$

where $5\,\text{SNe}/(100\,\text{yr})$ is roughly the rate of supernovae for a galaxy like the Milky Way (Wikipedia: Supernova: Milky Way candidates) and we assume that this is of order the metal creation given that each supernovae yields of order $1\,\mathrm{M}_\odot$ of metals. Let $Z_{\text{in}} = 0.001$ the fiducial primordial cosmic metallicity. Calculate Z_{asy} with these values and discuss whether the result is reasonable or not.

SUGGESTED ANSWER:

a) Behold:

$$\frac{d\rho}{dt} = F - \kappa\rho \ .$$

b) Behold:

$$1)\quad \frac{d\rho}{dt} = F - \kappa\rho \qquad 2)\quad \frac{d\rho}{dt} + \kappa\rho = F \qquad 3)\quad e^{\kappa t}\frac{d\rho}{dt} + e^{\kappa t}\kappa\rho = Fe^{\kappa t}$$
$$4)\quad e^{\kappa t}\rho|_0^t = F\tau(e^{\kappa t} - 1) \qquad 5)\quad \rho = F\tau(1 - e^{-\kappa t}) + \rho_0 e^{-\kappa t} \ .$$

Alternatively since F does not depend on ρ, we could solve as follows:

$$1)\quad \frac{d\rho}{dt} = F - \kappa\rho \qquad 2)\quad \frac{d\rho}{\rho - F\tau} = -\frac{dt}{\tau} \qquad 3)\quad \ln|\rho' - F\tau|\Big|_{\rho_0}^{\rho} = -\frac{t}{\tau}$$
$$4)\quad \ln\left|\frac{\rho - F\tau}{\rho_0 - F\tau}\right| = -\frac{t}{\tau} \qquad 5)\quad \rho - F\tau = (\rho_0 - F\tau)e^{-t/\tau} \qquad 6)\quad \rho = F\tau(1 - e^{-t/\tau}) + \rho_0 e^{-t/\tau} \ .$$

Thus, we have

$$\rho = \begin{cases} F\tau(1 - e^{-\kappa t}) + \rho_0 e^{-\kappa t} & \\ \qquad = F\tau(1 - e^{-t/\tau}) + \rho_0 e^{-t/\tau} & \text{in general;} \\ Ft + \rho_0(1 - \kappa t) = Ft + \rho_0\left(1 - \dfrac{t}{\tau}\right) & \text{for small } t; \\ F\tau & \text{for the asymptotic solution given for } t \to \infty. \end{cases}$$

The time constant τ is the e-folding time. We get the solution we get since the outflow rate increases/decreases until it matches the inflow rate which formally only happens as $t \to \infty$, but practically will be nearly true after a few e-folding times and the initial density just gives a transient solution (which unless it is large compared to the asymptotic solution) will be relatively small in a few e-folding times.

c) Behold:

$$\frac{d(Z\rho)}{dt} = Z_{\text{in}}F - \kappa Z\rho + \gamma(1 - Z)\rho = Z_{\text{in}}F - (\kappa + \gamma)Z\rho + \gamma\rho \ .$$

d) The asymptotic solution is found by setting $d(Z\rho)/dt = 0$. We get

$$Z_{\text{asy}} = \frac{Z_{\text{in}}F + \gamma\rho_{\text{asy}}}{(\kappa + \gamma)\rho_{\text{asy}}} = \frac{Z_{\text{in}}F + \gamma F\tau}{(\kappa + \gamma)F\tau} = \frac{Z_{\text{in}} + \gamma\tau}{(\kappa + \gamma)\tau} = \frac{Z_{\text{in}} + \gamma\tau}{1 + \gamma\tau} \ ,$$

where we have used the fact that $\kappa\tau = 1$. Note the formula is dimensionally correct since the terms in the numerator and denominator are all dimensionless and yield a dimensionless Z_{asy} as they should. Note a quantity being dimensionless does not mean not having a physical nature. It just means the quantity is written in terms of natural units. Since $Z_{\rm in} \in [0,1]$, we have $Z_{\rm asy} \in [0,1]$ by inspection. Note

$$Z_{\rm asy} = \begin{cases} \dfrac{Z_{\rm in} + \gamma\tau}{1+\gamma\tau} & \text{in general;} \\ Z_{\rm in} & \text{for } \gamma\tau = 0; \\ 1 & \text{for } \gamma\tau \to \infty. \\ \dfrac{\gamma\tau}{1+\gamma\tau} & \text{for } Z_{\rm in} = 0; \\ 1 & \text{for } Z_{\rm in} = 1; \end{cases}$$

Note if $\tau = 0$, the gas inflowed is instantly outflowed and there is no way its metallicity can be increased above $Z_{\rm in}$. On the other hand, if $\tau = \infty$, the density goes to infinity asymptotically and so does the metallicity.

e) First, with the given values

$$\gamma\tau = \frac{\gamma}{\kappa} = \frac{[5\,\text{SNe}/(100\,\text{yr})] \times (1\,\text{M}_\odot\,\text{metals/per SNe})}{3\,\text{M}_\odot/\text{yr}} \approx 0.017\ .$$

Now

$$Z_{\rm asy} = \frac{Z_{\rm in} + \gamma\tau}{1+\gamma\tau} \approx \frac{0.001 + 0.017}{1 + 0.017} \approx 0.017\ .$$

The solar (surface) metallicity is fiducially 0.02 though one precise determination puts it at 0.0134 (Wikipedia: Metallicity: Mass fraction). The solar value is thought to be of order of the cosmic present metallicity, and so our calculated value is of order correct. The fact that is so close to 0.02 must be considered an accident since our input values have an estimated error of a factor of 2 at least.

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