$$C_3^{(1)} = \frac{1}{\sqrt{\varepsilon^2 + (1+\varepsilon^2-\varepsilon^4)^2}} \begin{pmatrix} 0 \\ \varepsilon \\ 1+\varepsilon^2-\varepsilon^4 \end{pmatrix}$$

$$\cong \begin{pmatrix} 0 \\ \varepsilon \\ 1 \end{pmatrix}$$

$$C_2^{(1)\dagger} \cdot C_3^{(1)} = \varepsilon - \varepsilon = 0$$

So orthogonal to 1st order

$$\left. \begin{aligned} C_2^{(1)\dagger} \cdot C_2^{(1)} &= 1+\varepsilon^2 \cong 1 \\ C_3^{(1)\dagger} \cdot C_3^{(1)} &= \varepsilon^2 + 1 \cong 1 \end{aligned} \right\} \text{Normalized to 1st order}$$

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9) Degenerate Perturbation Theory

Well we have all the ingredients.

Recall from p. 6-33 for 1st order perturbation derivation we had an equation for $i \neq j$

$$\langle \psi_{0j} | H_1 | \psi_{0i} \rangle = (E_{0i} - E_{0j}) \langle \psi_{0j} | \psi_{1i} \rangle$$

We we trying to get this coefficient

a) if $\langle \psi_{0j} | H_1 | \psi_{0i} \rangle \neq 0$ & $E_{0i} - E_{0j} \neq 0$

b) No problem and no problem if $\langle \psi_{0j} | H_1 | \psi_{0i} \rangle = 0$ & $E_{0i} - E_{0j} \neq 0$

c) If $\langle \chi_{0i} | H_1 | \chi_{0j} \rangle \neq 0$

and $E_{0i} - E_{0j} = 0$,

then $\langle \chi_{0j} | \psi_{1i} \rangle$ diverges

and degenerate
perturbation theory
fails.

d) But if $\langle \chi_{0j} | H_1 | \chi_{0j} \rangle = 0$

and $E_{0i} - E_{0j} = 0$

$\langle \chi_{0j} | \psi_{1i} \rangle$ is
indeterminate

Any value will satisfy
the 1st order perturbation
equation.

So why not zero.

6-114)

— all higher orders depend on 1st order, and so this choice will affect them.

But they are higher order and so our choice seems the good choice for fast convergence.

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$$\delta_{ij} H_{ij} = E_{0i} \delta_{ij} + \lambda \langle \psi_{0j}' | H_1 | \psi_{0i}' \rangle$$

$\therefore$ if $i \neq j$ $\langle \psi_{0j}' | H_1 | \psi_{0i}' \rangle = 0$

One assumes that someone has proven divergences don't occur in higher order perturbation.

The off-diagonal elements of the perturbation matrix are zero for $\{ | \psi_{0i}' \rangle \}_{subset}$.

If $i = j$

$$H_{ij} = E_{0i} + \underbrace{\lambda \langle \psi_{0i}' | H_1 | \psi_{0i}' \rangle}_{\text{Just the 1st order perturbation correction.}}$$

But what do we do about case (c)? Make it case (d)

We diagonalize the subset of degenerate states $\{|\psi_{0i}\rangle\}_{subset}$

to $\{|\psi'_{0i}\rangle\}_{subset}$ for $H = H_0 + \lambda H_1$

These are linear combination of $\{|\psi_{0i}\rangle\}_{subset}$

and so are eigenstates of both

truncated H and exact H_0

---

since diagonal

$\cancel{H^{mat}_{subset}}$

Now by diagonalization

$$H_{ij}\delta_{ij} = \langle \psi'_{0j} | H | \psi'_{0i} \rangle$$

$$= \langle \psi'_{0j} | H_0 | \psi'_{0i} \rangle + \lambda \langle \psi'_{0j} | H_1 | \psi'_{0i} \rangle$$

So now we can just do ordinary perturbation theory with $\{|\psi'_{0i}\rangle\}_{\text{subset}}$ & $\{|\psi_{0i}\rangle\}_{\text{non-subset}}$

The divergences vanish

$$|\psi_{1i}\rangle = \sum_{j \neq \{i\}} \frac{\langle \psi_{0j} | H_1 | \psi'_{0i}\rangle}{E_{0i} - E_{0j}} |\psi_{0j}\rangle$$

in subset

any i in subset

1st order correction state

Similarly

$$E_{2i} = \sum_{j \neq \{i\}} \frac{|\langle \psi_{0j} | H_1 | \psi'_{0i}\rangle|^2}{E_{0i} - E_{0j}}$$

in subset

any in subset

The 1st order energy corrections are $E_{1i} = \langle \psi_{0i}' | H_1 | \psi_{0i}' \rangle$ for the degenerate subset.

Just the values that diagonalization of the subset case gave us.

Double Degeneracy

Well we already have the exact 2×2 solution

See p. 6-90

$$E_{\pm} = \frac{1}{2}\left[(H_{11} + H_{22}) \pm \sqrt{(H_{11} - H_{22}) + 4|H_{12}|^2} \right]$$

We see perturbation always lifts the degeneracy unless $H_{11} = H_{22}$ and $H_{12} = 0$.

This effect is called the "repulsion of the energy levels".

But nothing mysterious I think.

— If states are degenerate but you perturb them, ~~with~~ then it is unlikely that they will both perturb ~~to~~ such that their energies stay equal.

the perturbation isn't likely to affect them equally.

Good ~~L~~inear Combinations

The subset $\{|\chi_{0i}'\rangle\}$ obtained from diagonalization in the jargon (although maybe not such common jargon) are called the "good" linear combinations.

6-120

Is ~~there~~ some way to know them with diagonizing?

Say A is an observable ~~that~~ commutes with H_0 and H_1:

$$[A, H_0] = 0$$
$$[A, H_1] = 0$$

Then $[A, H] = [A, H_0] + [A, H_1]$
$= 0 + 0$
$= 0$.

So A commutes with H.

Since A & H_0 commute, there exist a common eigenset $\{|\Psi_{0i}\rangle\}$

and say $A|\Psi_{0i}\rangle = a_i|\Psi_{0i}\rangle$

and the set of a_i

are NOT degenerate.

Now $0 = \langle \psi_{0j} | [A, H_1] | \psi_{0i} \rangle$

since $[A, H]$

$= \langle \psi_{0j} | A H_1 | \psi_{0i} \rangle - \langle \psi_{0j} | H_1 A | \psi_{0i} \rangle$

$= a_j \langle \psi_{0j} | H_1 | \psi_{0i} \rangle - a_i \langle \psi_{0j} | H_1 | \psi_{0i} \rangle$

$= (a_j - a_i) \langle \psi_{0j} | H_1 | \psi_{0i} \rangle$,

but all a_i's are NOT degenerate

$\langle \psi_{0j} | H_1 | \psi_{0i} \rangle = 0$

So if you could find A and the common eigenset $\{ | \psi_{0i} \rangle \}$ with H_0 & H_1, you don't need to diagonalize. This may seem a rare case, but in fact, it's common (or so Griffiths p. 260 opines).

At this point, we could go on to more realistic examples of perturbation theory — e.g., fine structure of H-atom, Zeeman effect, hyperfine splitting.

— But they are kind of involved and somewhat unmemorable.

Probably best to look at time-dependent perturbation ~~theory~~ and radiative transitions, however briefly.

Formally nearly degenerate state elude me, but some book must do it.

Nearly Degenerate states in Perturbation theory

This is a bit of a formalism puzzle.

In this case one could still find "good" linear combination that made $\langle \psi_{0j}' | H_1 | \psi_{0i}' \rangle$ vanish, but $\{ |\psi_{0j}'\rangle \}_{\text{subset}}$ would NOT be ~~eigenvalues~~ eigenstates of H_0.

A formally good procedure eludes me.

Just using non-degenerate perturbation theory has no infinities, but the ~~ser~~ expansion

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may not converge

or NOT converge quickly enough to be useful.

There must be a way.