6) Permutation Operator and Particle ~~[illegible]~~ Exchange

This is a ~~topic~~ which is, in my opinion, poorly explained in many textbooks.

I've read through the relevant ~~the~~ sections in CT - 1378

Baym - 389

Merzbacher - 423

But I've ~~thought~~ that before

and believe I finally "get it".

Probably if one was really smart one could "get it" from any of them, but I'm not so smart.

→ It's a case of each one ~~omitting a clarifying detail or~~ ~~[illegible]~~ omitting a different set of clarifying remarks or details or examples.

Let's see if I can do any better.

5-138)

First let's say permutation and particle exchange {which is what permutation does when explained. are a formalism for studying the symmetry properties of states (AKA wave functions when that term is appropriate) and operators.

The permutation operator "effects" a particle exchange.

But this is NOT a physical process — it's a way of mathematically constructing a new state (but with symmetrized states, it doesn't construct a new state)

Let's start (and only look in much detail) at the case of 2 particles.

In this section, we'll assume they are distinct.

— So they can be told apart by some interaction.

But we will assume they have the same spin S.
(Usually, we are thinking of spin $\frac{1}{2}$ particles).

Each particle has its own ~~spin~~ state space
(this is the formalism that works)

~~But~~ And assume that we have a complete set of commuting observables (C.S.C.O.) for each particle (CT-143)

synonym: or compatible CT-231

5-140

What is a C.S.C.O.?

A C.S.C.O

is enough observables

so that their

eigenvalues

or

quantum numbers

complotoly specify

the state of a particle

quantum number are dimensionless indexes for eigenvalues. For example a spin 1/2 particle has spin quantum number 1/2

The observables have to commute

e.g., $[A, B] = 0$.

When they do ~~the~~ a particle can be in a definite ~~state~~ eigenstate of both observables. (CT-140)

But this is kluty to remember and say

The actual spin ang. mom. squared eigenvalue is $s(s+1)\hbar^2 = \frac{3}{4}\hbar^2$

If $[A, B] \neq 0$,

then ~~the on~~ a particle can't be in an eigenstate of both simultaneously and their is an uncertainty relation for the results of "measurements" of the observable.

Really standard deviations in the eigenvalues of superimposed states.

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2}\langle i[A,B]\rangle\right)^2$$

(Gr-110)

$$\left(i[A,B]\right)^\dagger$$

$$= (-i)\left[(AB)^\dagger - (BA)^\dagger\right]$$

$$= -i\left[B^\dagger A^\dagger - A^\dagger B^\dagger\right]$$

$$= -i\left[BA - AB\right]$$

$$= i[A,B]$$

and so is itself Hermitian.

~~Now if you have~~

~~The~~

Particles do have alternative ~~sets~~ ~~of~~ C.S.C.O.'s

~~Of~~ The most basic ~~basis~~ C.S.C.O, I think,

5-142

in the space - spin C.S.C.O.
(or position)
There are only 2 observables
in the C.S.C.O

$\underset{\sim}{r}_{op}$ and $s_{z\,op}$

(well 3 if you count
S^2_{op} but it has only
one ~~eigenvalue s~~ quantum number s
For a spin 1/2 particle
$s = \frac{1}{2}$)

The eigenvalues of $\underset{\sim}{r}_{op}$
are all points in space
— a continuous spectrum
of them.
— the eigenfunctions in the
space representation are
Dirac delta functions
$\delta(\underset{\sim}{r} - \underset{\sim}{r}')$.

Well the wavenumber (k-space, AKA momentum space) — spin C.S.C.O may be equally basic depending on your point of view.

Maybe Not though

Particles at some point in space are part of the same system, But particles Not entangled but remote with same k are not part of same system. But maybe particles at same point in alternate world space in alternate world are equally disentangled?

But not quite symmetric
— Can't particles have same $\underset{\sim}{r}$ & $\underset{\sim}{k}$ and be disentangled — and unsymmetrical if identical? Many worlds says yes.

in bra-ket notation
the spatial eigenstate
is the abstract ket

$$|\underset{\sim}{r}\rangle$$

(I always picture these as attached to every point in space)

Expansion coefficient of $|\underset{\sim}{r}\rangle$

in space

a sort of normalization condition for a continuous spectrum of eigenstates

$$\langle \underset{\sim}{r}' | \underset{\sim}{r} \rangle = \delta(\underset{\sim}{r} - \underset{\sim}{r}')$$

(CT-145)

The quantum numbers for s_{zop}
are $m = -s, -s+1, \ldots, s-1, s$

there are always $2s+1$ of these.

So the space-spin eigenket
of a particle is

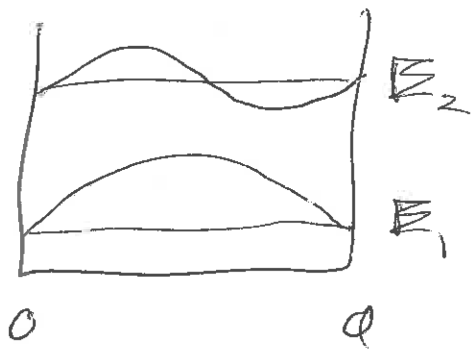
$$|\underset{\sim}{r}\, m\rangle$$

Another C.S.C.O would
be one ~~including~~ specifying energy
eigenstates (stationary states)

5-144

e.g., say we had ~~a~~ 1-d infinite square well.

The stationary states are $|\psi_n\rangle$



where $E_n = \left(\frac{\hbar^2}{2m}\right)\left(\frac{\pi}{a}\right)^2 n^2$ (Gr-32)

In space representation, the coefficient is what we ordinarily call a wave function:

$$\langle x | \psi_n \rangle = \sqrt{\frac{2}{a}} \sin(kx)$$

$ka = n\pi$
$n = 1, 2, 3, \ldots$

n is the quantum number indexing energy.

If the particle has spin, then we need 2 quantum numbers for an eigenket $|\psi_{n,m}\rangle$

which you could just write $|n\, m\rangle$ if you like.

~~That n~~ If you specify n, m you have uniquely specified the state.

— in general more than one particle can be in the state.

So having a C.S.C.O. allows you to specify the full state of a particle — but not in general specify a particle uniquely.

That's enough on C.S.C.O.'s for now.

Return to our two particles — they have the same spin recall, but are distinct.

They each have their own state space and their own individual observables (It's just the formalism that works.)

We demand or assume further that they have isomorphic C.S.C.O.'s

ones with the same eigenstates and eigenvalues.

For example, all particles of the same spin have isomorphic space-spin C.S.C.O.

And k-space-spin

wavenumber or momentum C.S.C.O. too.

Particle 1

r_{op1}, s_{zop1} for 1

r_{op2}, s_{zop2} for 2

Other C.S.C.O.'s
can be isomorphic
for the two particles
depending on cases.

e.g., two distinct, but
non-interacting particles
in an infinite-square well.

as far as the system
behaves, the ~~particle~~
single-particle states could be fully
specified by n & m.

(Actually we could tell
they were distinct particles
(if they are distinct) since
the overall 2-particle state
doesn't have to be
(though it could be) symmetrized

we'll get to
this soon)

5-148

Anyway we have our two ~~particles~~ distinct particles with the same spin and same isomorphic C.S.C.O's for them.

Their joint eigenstates we further assume can be formed by a tensor product

and remember -spin space

I think this is always possible for the space-spin C.S.C.O. — CT-1378 grows coy on the subject of whether this can generally be done.

~~There are~~ I think not.

~~Now~~ For example we can't form a tensor product for the ~~energy~~ stationary states positronium

In the ideal case of neglecting spin and weak interaction the positronium stationary states are hydrogenic.

see p. 5-95 and 5-86

and we can't factor these energy eigenstates into products.

Ideal Positronium is actually an interacting system as we'll see.

The particles are NOT identical but the Hamiltonian is symmetric — and eigenstates are symmetrized, but NOT mixed states in general

but include both symmetric and antisymmetric states.

5-150)

u_i and u_j ~~state~~ stand for ~~the~~ complete set of eigenvalues of C.S.C.O that uniquely specify states i, j

$$|1\,u_i\rangle|2\,u_j\rangle$$

just formally ~~taking~~ ~~the~~ tensor product.

$$= |1\,u_i,\ 2\,u_j\rangle$$

$$= |2\,u_j,\ 1\,u_i\rangle$$

Tensor product commutes.

The permutation operator P_{21} is a linear operator ~~that~~ transforms the state so

$$P_{21}|1\,u_i,\ 2\,u_j\rangle$$

$$= |2\,u_i,\ 1\,u_j\rangle$$

$$= |1\,u_j,\ 2\,u_i\rangle$$

same thing since tensor product commutes.

5-55

So initially particle 1 has set u_i and 2 has set u_j

and the permutation has given us a different state.

Since the particles are <u>distinct</u>

$|1u_i, 2u_j>$ and $|1u_j, 2u_i>$

particles have been exchanged.

are different states and would <u>evolve differently</u> under an ~~perturbation Hamiltonian~~ interaction that distinguished the particles.

For example.

$\chi_a(1)\chi_b(2)$

$\chi_a(\underset{\sim}{r})$

function of r

So $\chi_a(1)\chi_b(2)$ and $\chi_a(2)\chi_a(1)$ are different state

There's nothing <u>physical</u> doing the <u>exchange</u>

We are just saying physically

5-152

that the two distinct states are possible

So much for eigenstates.

<u>What</u> of <u>general states</u>

~~that~~ are formed by a linear combination of the eigenstates — which recall form a <u>complete set</u> (a basis

We look at two cases in <u>parallel</u>.

— A case of expansion in a discrete but otherwise general basis.

— A case of expansion in the space-spin basis

The space basis is a continuum basis.

(Wavenumber - spin basis would be analogous) general discrete basis

$$|\Psi 1 2\rangle = \sum_{ij} c_{ij} |1u_i, 2u_j\rangle$$

$|\Psi 2 1\rangle$

$$\cancel{|\Psi' 1 2\rangle} = P_{21} |\Psi 1 2\rangle$$

$$= \sum_{ij} c_{ij} P_{21} |1u_i, 2u_j\rangle$$

$$= \sum_{ij} c_{ij} |1u_j, 2u_i\rangle$$

$$= \sum_{ij} c_{ji} |1u_i, 2u_j\rangle$$

relabeling the dummy indices.

$i \to j$

$j \to i$

$$= \sum_{ij} c'_{ij} |1u_i, 2u_j\rangle$$

$$\therefore\ c'_{ij} = c_{ji}$$

The coefficients can be regarded as functions of indices which are "variables" or "coordinates".

space-spin basis (5-153

$$|\Psi 1 2\rangle = \sum_{mm'} \int \Psi(\underset{\sim}{r}m, \underset{\sim}{r}'m') |1\underset{\sim}{r}m, 2\underset{\sim}{r}'m'\rangle\, d\underset{\sim}{r}\, d\underset{\sim}{r}'$$

$$|\Psi 2 1\rangle = P_{21} |\Psi 1 2\rangle$$

$$= \sum_{mm'} \int \Psi(\underset{\sim}{r}m, \underset{\sim}{r}'m') |1\underset{\sim}{r}'m', 2\underset{\sim}{r}m\rangle\, d\underset{\sim}{r}\, d\underset{\sim}{r}'$$

Interchanging the dummy labels ~~gets~~

$\underset{\sim}{r} \longleftrightarrow \underset{\sim}{r}'$

$m \longleftrightarrow m'$ gives

$$= \sum_{mm'} \int \Psi(\underset{\sim}{r}'m', \underset{\sim}{r}m) |1\underset{\sim}{r}m, 2\underset{\sim}{r}'m'\rangle\, d\underset{\sim}{r}\, d\underset{\sim}{r}'$$

$|\Psi 1 2\rangle$ has spatial-spin representation $\Psi(\underset{\sim}{r}m, \underset{\sim}{r}'m')$

prime associated with particle 2

$|\Psi 2 1\rangle$ has spatial spin representation $\Psi(\underset{\sim}{r}'m', \underset{\sim}{r}m)$

5-154

$c'_{ij} = c_{ji}$

has a different dependence on ij

than c_{ij}

unless there is some symmetry:

$c_{ji} = e^{i\phi} c_{ij}$

where $e^{i\phi}$ is ~~a phase~~ a global phase factor of the state.

$\Psi(\underset{\sim}{r}' m', \underset{\sim}{r} m)$

is distinct from

$\Psi(\underset{\sim}{r} m, \underset{\sim}{r}' m')$

because the particles distinct

unless we have some symmetry:

$\Psi(\underset{\sim}{r}' m', \underset{\sim}{r} m)$

$= e^{i\phi} \Psi(\underset{\sim}{r} m, \underset{\sim}{r}' m')$

In space-spin representation

$P_{21} \Psi(\underset{\sim}{r} m, r' m')$

$= \Psi(\underset{\sim}{r}' m', \underset{\sim}{r} m)$

~~Often for~~ Or for notational simplicity.

$P_{21} \Psi(x_1 m_1, x_2 m_2) = \Psi(\underset{\sim}{r}_2 m_2, \underset{\sim}{r}_1 m_1)$

The $r_1 m_1$ and $r_2 m_2$ are like indices.

Permutation operator is formally a QM observable

~~Or~~ one can regard it that way.

Note $\mathcal{P}_{21}^2 = 1$ the unit operator

Say we have $|1 u_i, 2 u_j\rangle$ as an eigenstate of $\mathcal{P}_{21}$

Then $\mathcal{P}_{21} |1 u_i, 2 u_j\rangle = \underbrace{\lambda}_{\text{eigenvalue.}} |1 u_i, 2 u_j\rangle$

Now $\mathcal{P}_{21}^2 |1 u_i, 2 u_j\rangle = |1 u_i, 2 u_j\rangle = \lambda^2 |1 u_i, 2 u_j\rangle$

Anyons have $e^{i\phi}$ as eigenvalue somehow, Some weird formalism. But they are just quasi-particles (Wik)

$\therefore \lambda^2 = 1$ Note NOT $|\lambda|^2 = 1$.

$\lambda = \pm 1$

The only two ~~possible~~ eigenvalues ~~The eigen~~ are pure real just as for an observable.

$(e^{i\phi})^2 = e^{i(2\phi)} = \cos(2\phi) + i \sin(2\phi) = 1$

5-15E

only for $\phi = 0$ and π
for $\phi \in [0, 2\pi)$

These eigenvectors do
form a complete set.

Proof Say $|1u_i, 2u_j\rangle$
is not an eigenket,
then neither is $|1u_j, 2u_i\rangle$

But one can then construct eigenvectors

$$|1u_i, 2u_j\rangle_S = \frac{1}{\sqrt{2}}\Big(|1u_i, 2u_j\rangle + |1u_j, 2u_i\rangle\Big)$$

and

$$|1u_i; 2u_j\rangle_A = \frac{1}{\sqrt{2}}\Big(|1u_i, 2u_j\rangle - |1u_j, 2u_i\rangle\Big)$$

are symmetric and antisymmetric
eigenket of P_{21}

Since you can reconstruct

$|1u_i, 2u_j\rangle$

& $|1u_j, 2u_i\rangle$

from $|1u_i, 2u_j\rangle_S$ & $|1u_i, 2u_j\rangle_A$,

it is clear that a complete basis of eigenkets of P_{21} exists.

Since the eigenvalues are real and a complete basis exists P_{21} should be an observable,

One can prove it is formally Hermitian too.

Recall the Hermitian conjugate is defined by

$$\langle \alpha | Q | \beta \rangle = \langle \beta | Q^{+} | \alpha \rangle^{*}$$

and Q is Hermitian if

$$Q = Q^{+}$$

~~F~~ 15 8

Consider

$a = \langle 1u_{i'}, 2u_{j'} | P_{21} | 1u_i, 2u_j \rangle$

$= \langle 1u_{i'}, 2u_{j'} | 1u_j, 2u_i \rangle$

$= \delta_{i'j} \delta_{j'i}$

We've assumed the single-particle state basis is orthonormal which we always can.

and

$b = \langle 1u_i, 2u_j | P_{21}^{\dagger} | 1u_{i'}, 2u_{j'} \rangle^*$

$a = b$ by definition of Hermitian conjugate

$c = \langle 1u_i, 2u_j | P_{21} | 1u_{i'}, 2u_{j'} \rangle^*$

$= \langle 1u_i, 2u_j | 1u_{j'}, 2u_{i'} \rangle^*$

$= \langle 1u_{j'}, 2u_{i'} | 1u_i\ 2u_j \rangle$

$= \delta_{j'i} \delta_{i'j}$

$= \delta_{i'j} \delta_{j'i} = a = b$

$\therefore\ P_{21} = P_{21}^{\dagger}$ for the basis and so for any state expanded in the basis

Q Say $\{|\phi_i\rangle\}$ is a basis but not of Q and Q is Hermitian for all basis states. ~~but not of Q in general.~~

$$\langle\alpha|Q|\beta\rangle$$

$$= \sum_{ij} a_i^* b_j \langle\phi_i|Q|\phi_j\rangle$$

$$= \sum_{ij} a_i^* b_j \langle\phi_j|Q^+|\phi_i\rangle^*$$

using Hermitian conjugate defn.

$$= \sum_{ij} a_i^* b_j \langle\phi_j|Q|\phi_i\rangle^*$$

Using our assumption Q is Hermitian for eigenstates

$$= \left(\sum_{ij} a_i b_j^* \langle\phi_j|Q|\phi_i\rangle\right)^*$$

$$= \langle\beta|Q|\alpha\rangle^*$$

and so $Q = Q^+$ for all states.

~~The permutation operator and other operators. We will just look at the position representation~~

A general rule deduced from p. 5-153 is that $\mathcal{P}_{21}$ just exchanges the particle indexes. The things that get summed on,

Say $A(1)$ is an operator for particle 1

$$\mathcal{P}_{21} A(1) \Psi(1,2)$$

$$= A(2) \Psi(2,1)$$

$$= A(2) \mathcal{P}_{21} \Psi(1,2)$$

$$\mathcal{P}_{21} A(1) = A(2) \mathcal{P}_{21}$$

Since $A(1) \neq A(2)$ in general $A(1)$ and P_{21} don't commute in general.

Let's look at spatial representation for concreteness.

$$\mathcal{P}_{21} \left(V(\underset{\sim}{r}_1)\, \psi(\underset{\sim}{r}_1, \underset{\sim}{r}_2) \right)$$

$$= V(r_2)\, \psi(\underset{\sim}{r}_2, \underset{\sim}{r}_1)$$

$$= V(\underset{\sim}{r}_2)\, \mathcal{P}_{21}\, \psi(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$$

Potential that acted on particle 1 now acts on particle 2

What about a differentiating operator like

$$T_1 = -\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_1^2}$$

for particle 1

$$\mathcal{P}_{21}\, T_1\, \psi(x_1, x_2)$$

$$= \mathcal{P}_{21} \left(-\frac{\hbar^2}{2m_1}\right) \frac{\partial^2}{\partial x_1^2} \psi(x_1, x_2)$$

$$= -\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_2^2} \psi(x_2, x_1)$$

$$= -\frac{\hbar^2}{2m_1} \frac{\partial^2}{\partial x_2^2} \mathcal{P}_{21}\, \psi(x_1, x_2)$$

Particle 2 now mysteriously has the mass of particle 1 after the permutation operation.
— a confusing notation, but common.

5-162a

Consider now a <u>Hamiltonian</u> for ~~identical~~ two particles

$$H_{12} = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V(x_1, x_2)$$

$$P_{21} H_{12} = \left(-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_2^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_1^2} + V(x_2, x_1)\right) P_{21}$$

$$= H_{21} P_{21}$$

Not the original Hamiltonian

<u>unless</u> the particles were <u>identical</u>

$m_1 = m_2$

and $V(x_2, x_1) = V(x_1, x_2)$

necessarily

For example positronium neglecting weak interaction (maybe?)

<u>or</u> the particles were distinct but had the same mass and in this system had the same potential.

If $H_{21} = H_{12}$

then $[P_{21}, H_{12}] = 0$

they commute and ~~have~~ can have a common basis set.

CT-140

5-162 b

Actually, you might ~~wha~~ wonder if there is something futile about ~~and operating~~ ~~about~~

$$\mathcal{P}_{21} H_{12} \Psi(12)$$

$$= H_{21} \mathcal{P}_{21} \Psi(12)$$

$$= H_{21} \Psi(2,1)$$

$\mathcal{P}_{21}$ on $\Psi(12)$ exchanges the particles and creates a <u>different state</u> if the ~~the~~ particles ~~are identical~~ ~~(or if~~ are NOT identical

(unless the state is only changed by a global phase factor $e^{i\phi}$ which could be 1 or ~~o~~ -1)

~~Both~~ $\mathcal{P}_{21} H_{12} \Psi(12) = H_{21} \Psi(2,1)$

You've got particle 1 having

t - 162c

~~the Hamiltonian~~

Hamiltonian attributes particle 2 used to have

and particle 2 having the ~~particle~~ Hamiltonian attributes particle 1 used to have.

<u>But</u> knowing ~~how~~ the permutation operator acts on operators is useful in study the time evolution of a system

Recall general time evolution equation from Gr - 115

$$\frac{d\langle Q\rangle}{dt} = \frac{1}{\hbar}\langle i[H, Q]\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle$$

where Q is a general observable.

(actually Q just has to be an operator, not an observable Gr - p.1 (0))

Hamiltonian stays symmetric See p5-179

So if $[H, \mathcal{P}_{21}] = 0$

then $\frac{d\langle \mathcal{P}_{21}\rangle}{dt} = 0$ or $\langle \mathcal{P}_{21}\rangle$ = Constant,

~~If you are in a~~ If a state ~~has a particular symmetry (± 1), then one~~ ~~which~~ is an eigenstate of $\mathcal{P}_{21}$, then it stays an eigenstate as long as the

5-16-22 b

Actually, you might ~~wha~~ wonder if there is something futile about ~~ab operating~~ ~~about~~

$$\mathcal{P}_{21} H_{12} \Psi(12)$$

$$= H_{21} \mathcal{P}_{21} \Psi(12)$$

$$= H_{21} \Psi(2,1)$$

$\mathcal{P}_{21}$ on $\Psi(12)$ exchanges the particles and creates a different state if the ~~th~~ particles ~~are identical~~ ~~(or if~~ are NOT identical (unless the state is only changed by a global phase factor $e^{i\phi}$ which could be 1 or ~~o~~ -1)

~~Both~~ $\mathcal{P}_{21} H_{12} \Psi(12) = H_{21} \Psi(2,1)$

You've got particle 1 having

5-162c

~~the Hamiltonian~~

Hamiltonian attributes particle 2 used to have

and particle 2 having the ~~particle~~ Hamiltonian attributes particle 1 used to have.

<u>But</u> knowing how the permutation operator acts on operators is useful in study the time evolution of a system.

Recall general time evolution equation from Gr-115

$$\frac{d\langle Q\rangle}{dt} = \frac{1}{\hbar}\langle i[H, Q]\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle$$

where Q is a general observable.

(actually Q just has to be an operator, not an observable Gr-p.(0))

Hamiltonian stays symmetric See p5-179

So if $[H, P_{21}] = 0$

then $\frac{d\langle P_{21}\rangle}{dt} = 0$ or $\langle P_{21}\rangle =$ Constant.

~~If you are in a~~ If a state ~~has a particular symmetry (±1), then~~ or ule is an eigenstate of P_{21}, then it stays an eigenstate as long as the

Identical Particles & Symmetrization Principle

The symmetrization principle is a postulate.

It cannot be derived although the world be ~~paradoxical~~ weirder than it is if it didn't exist

If you exchange ~~particles~~ any two identical particles of multi-particle state the state must be unchanged (i.e., be symmetric)

or

be changed only by an overall factor of -1 (i.e., be antisymmetric)

In other words the states must be eigenstates of ~~[illegible]~~ any two particle permutation operator.

Let's just consider our 2 particle basis system again

$$\mathcal{P}_{21} | 1u_i , 2u_j \rangle = | 1u_j , 2u_i \rangle$$

To satisfy the symmetrization principle

$$| 1u_j , 2u_i \rangle = \pm | 1u_i , 2u_j \rangle$$

~~It's actually a second postulate [illegible] boson states~~

What of general states given a symmetrized basis?

$$|\psi 21\rangle = \sum_{ij} c_{ji} | 1u_i , 2u_j \rangle$$

(see p. 5-153)

$P|\psi 12\rangle$

$$= \sum_{ij} c_{ji} (\pm 1) | 1u_j , 2u_i \rangle$$

$$= (\pm 1) \sum_{ij} c_{ij} | 1u_i , 2u_j \rangle$$

relabeling

$= \pm |\psi 1 2\rangle$

and similarly for the particle-spin basis.

$$P_{21}\psi(\underset{\sim}{r}_1 m_1, \underset{\sim}{r}_2 m_2) = \psi(\underset{\sim}{r}_2 m_2, \underset{\sim}{r}_1 m_1)$$

$$= \pm \psi(\underset{\sim}{r}_1 m_1, r_2 m_2)$$

For N particles ~~Particles~~ $P_{ji}\Psi(\ldots r_i m_i \ldots \underset{\sim}{r}_j m_j \ldots) = \pm\Psi(\ldots r_j m_j \ldots r_i m_i \ldots)$

Identicle particles with only symmetric states are called bosons

Identicle particles with only antisymmetric states are called fermions.

It is actually a 2nd postulate

that integer spin particles are bosons (i.e., have symmetric states)

and half-integer spin particles are fermions (i.e., have antisymmetric states)

5-166

~~By the way you might say~~

~~$\Psi(\nu_1 m_1, \nu_2 m_2)$~~

~~and $\Psi(\nu_2 m_2, \nu_1 m_2)$~~

~~are always the *same state* because the particles are *identical*.~~

~~*True* in a *sense*, but the state is not symmetrized in respect to its *coordinate* sets~~

~~(i.e., eigenvalues $\nu_1 m_1$, ν_2, m_2)~~

Exchange Degeneracy

The symmetrization principle is an axiom. It cannot be derived.

But ~~there~~ is an argument ~~that suggests that the~~ world would be weirder than it is without it

Say you had two single particle $^{\text{eigen}}$states

~~they~~ they are orthonormal

$\Psi_a(\underset{\sim}{r})$ and $\Psi_b(\underset{\sim}{r})$

— We consider first two distinct particles with isomorphic $^{\text{energy}}$eigenstates ~~for~~ ~~same~~ the system (particles 1 and 2) for which $\Psi_a(\underset{\sim}{r})$ + $\Psi_b(\underset{\sim}{r})$ are eigenstates.

5-168)

They have the same spin **and** are in same spin state, so we don't need to fuss with spin coordinates.

The 2-particle eigenstates we can construct ~~any~~

$\cancel{c[\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2)] + c_b}$

by simple products are

$\psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2)$ and $\psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)$

There are distinct states even if indistinct in the system at hand

You or nature can always impose a perturbation to detect which particle is in which state

— since they are distinct.

So it seems these are two good 2-particle eigenstates.

One can form an infinite continuum of mixed states.

$$\Psi_M = c_i \psi_a(\mathbf{r}_1)\psi_b(\mathbf{r}_2) + c_j \psi_a(\mathbf{r}_2)\psi_b(\mathbf{r}_1)$$

$$\int |\Psi_M|^2 d\mathbf{r}_1 d\mathbf{r}_2 = |c_i|^2 + |c_j|^2 = 1$$

So we require $|c_i|^2 + |c_j|^2 = 1$ for normalization but otherwise c_i and c_j are free ~~to~~ Set by all past history (i.e., initial conditions)

Note $H = H_1 + H_2$ has eigenvalue $E_a + E_b$ no matter what c_i and c_j are. They are energy degenerate.

5-170

~~You can m~~

In principle some measurement can collapse the mixed state

to $\Psi_a(\underset{\sim}{r}_1)\Psi_b(\underset{\sim}{r}_2)$

or $\Psi_a(\underset{\sim}{r}_2)\Psi_b(\underset{\sim}{r}_1)$

A measurement of particle 1 and particle 2 energies ~~Ex~~ xample ~~1~~ has E_a and 2 has E_b or vice versa.

and you could measure $|c_i|^2$ and $|c_j|$.

Harder to get ~~their phase the phases of~~ c_i and c_j ~~themselves~~ but somehow it could be done

— There are no paradoxes. The ~~states ary~~ eigenstates are distinct since the particles are and some past history could arrange either of these states.

Now we go to identical particles:

$$\psi_a(\underset{\sim}{r}_1)\,\psi_b(\underset{\sim}{r}_2)$$

and $\psi_a(\underset{\sim}{r}_2)\,\psi_b(\underset{\sim}{r}_1)$

are now the same state

Formalism gives the particles different labels, but they are the same state since the particles are identical

~~Now~~ But

$$\psi_M = c_i\,\psi_a(\underset{\sim}{r}_1)\psi_b(\underset{\sim}{r}_2) + c_j\,\psi_a(\underset{\sim}{r}_2)\psi_b(\underset{\sim}{r}_1)$$

exchange degeneracy.

are all distinct states

— a continuum infinity of ~~energy~~ degenerate states

— only condition $|c_i|^2 + |c_j|^2 = 1$

5-172

Now there is always a continuum infinity for mixed states.
— So that in itself is not a problem.

But there is a bit of a odd ~~paradox~~ that you can create a continuum infinity from one eigenstate

But there is a lot of odd thinge in QM.

Also odd that any collapse of the wave function on measurement can only be a collapse to one state

$\psi_a(r_1)\psi_b(r_2)$

intenchange and it's the same

And how would it been by nature?

You'd measure E_a for a particle and E_b for another but can't distinguish which is which.

— So has there even been a collapse? Maybe. (counterfactual situation are hard to get a right answer for)

ψ_M is certainly a different state from $\psi_a(\underset{\sim}{r}_1)\psi_b(\underset{\sim}{r}_2)$

Ex

$$f(x,y) = x^2 y$$
$$f(y,x) = y^2 x^2$$

Not real wave function, but they illustrate the point.

$$f_M = c_i x^2 y + c_j y^2 x^2$$

different functional behavior.

You could measure the particle position somehow.

5-174

[Note $|a+b|^2 = (a^*+b^*)(a+b)$
$= |a|^2 + |b|^2 + a^*b + b^*a$
$= |a|^2 + |b|^2 + 2\,\mathrm{Re}(ab^*)$]

$$\rho(\underset{\sim}{r}_1, \underset{\sim}{r}_2) = |c_i|^2 |\psi_a(\underset{\sim}{r}_1)|^2 |\psi_b(\underset{\sim}{r}_2)|^2$$
$$+ |c_j|^2 |\psi_a(\underset{\sim}{r}_2)|^2 |\psi_b(\underset{\sim}{r}_1)|^2$$
$$+ 2\,\mathrm{Re}\left[c_i c_j^* \psi_a(\underset{\sim}{r}_1)\psi_b(\underset{\sim}{r}_2)\psi_a(r_2)^*\psi_b(r_1)^*\right]$$

$$\mathrm{Re}[z_1 z_2] = x_1 x_2 - y_1 y_2$$
$$\mathrm{Re}[z_1]\mathrm{Re}[z_2] = x_1 x_2$$

Not equal unless $y_1 y_2 = 0$

Actually stationary states can always be chosen pure real since $H\psi = E\psi$ is pure real.

so $\mathrm{Re}[c_i c_j^* \underbrace{\psi_a \ldots}_{z_2 \text{ with } y_2 = 0}] = \mathrm{Re}[c_i c_j^*]\,\psi_a \ldots$

~~So~~ $\text{Re}[c_i c_j^*]$

$$= \text{Re}\left[|c_i||c_j| e^{i\phi_i} e^{-i\phi_j}\right]$$

$$= |c_i||c_j| \cos(\phi_i - \phi_j)$$

Recall $e^{ix} = \cos x + i \sin x$ (Wiki: Euler's Formula)

So "3 measurements" of $\rho(\underset{\sim}{r}_1, \underset{\sim}{r}_2) d\underset{\sim}{r}_1 d\underset{\sim}{r}_2$ can allow you to solve for $|c_i|$, $|c_j|$ and $\phi_i - \phi_j$

One only needs relative phase since the global phase of wave function is arbitrary.

Of course, ~~3 mea~~ just 3 measurements is ~~too~~ difficult since ρ ~~is~~ is just a probability density and says nothing about any actual measurement — So one needs to do many and build up all of $\rho(\underset{\sim}{r}_1, r_2)$.

But it can be done in principle.

So one can determine c_i and c_j fully in principle.

But what if $c_i = 1 - \varepsilon$ and $c_j = \varepsilon$ where ε is very small.

Then $\psi_a(\underset{\sim}{v}_1)\psi_b(v_2)$ is strongly favored over $\psi_a(\underset{\sim}{v}_2)\psi(\underset{\sim}{v}_1)$.

<u>But</u> the two states are same, since the particles are identical.

What favors one over the other?

Are the particles distinguished after all.

So the neutral choice
is supported where the

$$|c_i| = |c_j| = \frac{1}{\sqrt{2}}$$

so both forms are
equally weighted

So maybe all eigenstates
must be symmetrized
for identical particles

~~Now if~~ → symmetric or
antisymmetric

Now if all eigenstates for
a kind of particle
have the same symmetrization,
then all states for those particles
are symmetrized. See p. 5-164.

There are other ~~tr~~ arguments why
the exchange degeneracy suggests

5-178

symmetrization
↳ but they don't communicate much to me (see CT-1375-1377)

Hamiltonian for Identical Particles

— It must be symmetrical

$$\cancel{H = -\sum_i \frac{1}{m_i}\frac{\partial^2}{\partial x_i^2} + V(}$$

$$H = \sum_k \frac{-\hbar^2}{2m_k} \nabla_k^2 + V(r_1, m_1, \ldots, r_i, m_i, \ldots, r_j, m_j, \ldots)$$

all masses the same
— so exchanging any indexes changes nothing.

interchange any particular values and nothing can change since the particles are identical.

So $\mathcal{P}_{ji} H(\ldots i \ldots j \ldots)$

$= H(\ldots j \ldots i \ldots) \mathcal{P}_{ji}$ (see p. 5-160)

by symmetry $\hookrightarrow$ $= H(\ldots i \ldots j \ldots) \mathcal{P}_{ji}$

$$\therefore \quad [H, \mathcal{P}_{ji}] = 0$$

~~comm~~ commute.

Recall the general time evolution equation for an ~~observable~~ expectation value

$$\frac{d\langle Q\rangle}{dt} = \frac{1}{\hbar}\langle i[H,Q]\rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle$$

(Gr-115)

explicit time dependence of observable.

Now $\mathcal{P}_{ji}$ has no explicit time dependence.

$$\frac{d\langle \mathcal{P}_{ji}\rangle}{dt} = 0$$

5–180

So if a state is an eigenstate of P_{ji}

~~a th a~~ it has

$$\langle P_{ji} \rangle = \pm 1, \left\{ \begin{array}{l} \text{boson} \\ \text{fermion} \end{array} \right.$$

~~then~~ this value is constant.

So a symmetrized state stays symmetrized no matter what.

Maybe ~~so integer spin particles~~ integer/half integer spin particles were bosons/fermion at the beginning and that initial condition means they stay that way.

On the other hand, our argument relies on Sch. equ. evolution

— if wave function collapse violates that (and it does NOT in pure decoherence ~~theory~~), then our argument has a flaw — takes another {argument / axiom} to say symmetrization is conserved in wave function collapse.

So for this reason, one can't say $[H, P_{ji}] = 0$ proves symmetrization.

5-182)

another reason for NOT believing that $[H, P_{ji}] = 0$ proves symmetrization is that non-identical particles can have symmetric Hamiltonians at least it seems that way

Consider an ideal isolated positronium — neglecting any weak force effects (of which I know nothing), the Hamiltonian is symmetric

$$H = -\frac{\hbar^2}{2m_e}\nabla_1^2 - \frac{\hbar^2}{2m_e}\nabla_2 - \frac{e^2}{4\pi\varepsilon|r_2 - r_1|}$$

equal masses

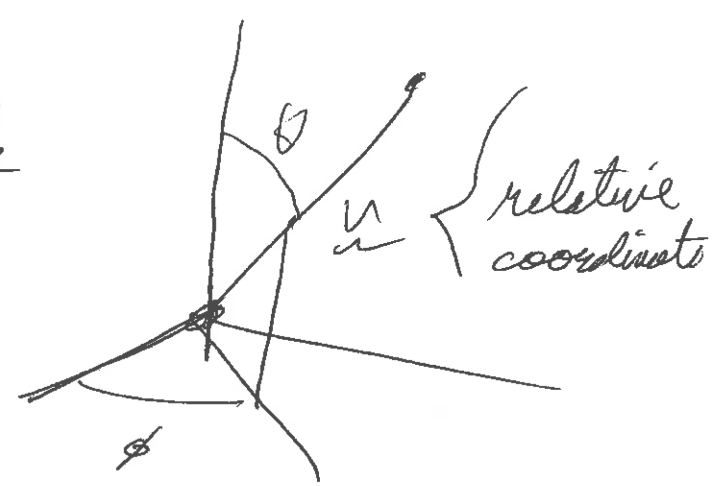
neglecting any spin components which should be symmetric too.

In fact the Hydrogenic stationary states are symmetric.

$$\Psi_{total}(\underset{\sim}{r}_1, \underset{\sim}{r}_2) = \Psi(\underset{\sim}{r}_2 - \underset{\sim}{r}_1)\, \Psi_{CM}\left(\frac{m_1 \underset{\sim}{r}_1 + m_2 \underset{\sim}{r}_2}{M}\right)$$

(see p. 5-95)

$$\Psi = \underbrace{R(|\underset{\sim}{r}_2 - \underset{\sim}{r}_1|)}_{\text{radial part just depends on magnitude}}\, Y_{lm}(\theta, \phi)$$



If $\underset{\sim}{r}_1$ & $\underset{\sim}{r}_2$ interchange,

$$\underset{\sim}{r} \longrightarrow -\underset{\sim}{r}$$

and $\theta_{new} = \pi - \theta$, $\phi_{new} = \phi + \cancel{\pi}$

5-182

$$Y_{lm}(180-\theta, \phi + \pi)$$

$$= (-1)^l \, Y_{lm}(\theta, \phi)$$

so even l symmetric
odd l antisymmetric

So eigenstates are symmetrized
but since that have both
kinds of symmetrization,
mixed states are
NOT necessarily symmetrized.

~~Now that I think~~

I know of ~~no the restrictions~~

~~on A~~ no other symmetrization restrictions on the states of positronium, but I don't know much about it actually

Come to think of it

$$[H, P_{ji}] = 0$$

implies a common orthonormal basis can be found (CT-1~~4~~0).

But if the particles aren't identical both ~~symmetrized & an~~ symmetric & antisymmetric eigenstates allowed.

5-184a

Symmetrization and linearity

The Sch. eqn is a linear DE

$$H\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

∴ any linear combination of solutions is a solution of Sch eqn

The symmetrization principle rules out some solutions as being non-physical for identical particles

So in a sense QM is a non-linear theory.

— Is this a profound sense?

I don't know —

Composite Particles

The symmetrization principle ~~applies only to to~~ as a law of nature applies to fundamental particles — those that are not made of other particles

These are identical in all respects except position and spin orientation (I think).

But what of composite particles?

Well it's pretty easy

Composite particles are made of fundamental ones.

So a wave function for a composite particle can be written in terms of fundamental particles:

Say $\Psi(\ldots \{\underline{r}\ m\}_i \ldots\ldots \{\underline{r}\ m\}_j \ldots\ldots)$

is wave function for a set of identical composite particle — identical in properties — the state of the composite could be anything.

set of coordinates for composite particle i/j's componenent fundamental particles

5-184c

If you exchange sets i and j, Symmetrization requires that the wave function change value by only

$$(-1)^n$$

where n is the number of fermions in the composite particle.

So composite particles are bosons if n even and fermions if n odd.

What of composites of composites? Well in the above argument nothing changes if one says the components are NOT fundamental but are merely known bosons and fermions.

So the symmetrization principle applies to all identical particles.

5-18d

The symmetrization nature of composite particles can have very striking effects.

99.99986% of all He on Earth

He-4 is a boson 2e, 2p, 2n

He-3 is a fermion 2e, 2p, 1n

stable but low abundance on Earth

He-4 exhibits superfluidity which is a complex manifestation of Bose-Einstein condensation (only shown by bosons where all ~~the bosons~~ or a large fraction of bosons get in the same single-particle state)

He-3 doesn't show superfluidity since a Fermion and obey the Pauli exclusion principle which ~~forbids~~ "allows only on fermion in a single particle state"

5-184d

— or more exactly
allows only one fermion
single particle state
in a product state
from which symmetrized
states are constructed

or one can "measure" only
one fermion in a
single-particle state
on wave function
collapse.

We get to the Pauli exclusion principle
result.

Another example of a composite particle
with great properties are Cooper Pairs.
two electrons weakly bound in
side ~~of~~ some solids at low T
— they are bosons and superconductivity
(at least of low T superconductors) rely
on their bosonic nature.

This suggests some
unique (samique
specification of
c_i and c_j that
kills the exchange degeneracy

Of course $c_i = c_j = \frac{1}{\sqrt{2}}$
for Bosons
~~and c_i c_j~~
and $c_i = \frac{1}{\sqrt{2}}$, $c_j = -\frac{1}{\sqrt{2}}$
for fermions

In both cases multiplication by a common phase factor $e^{i\phi}$ is allowed and doesn't change the physical state.

Maybe some way to make
the exchange degeneracy
argument more
cogent but I don't know it
(CT—1375-1377
kind of flub I
think)

5-18~~6~~ 8

Pauli Exclusion Principle

which is just for Fermions

— Historically introduced by Pauli in ~1925 or so just to explain atoms

— the modern version is more general and also is not ~~really~~ an independent law — It's a consequence of the symmetrization principle

"No two fermions can occupy the same single particle eigen state"

~~General~~ ~~[illegible]~~

Perhaps a more correct, but less memorable way to say it is

~~No two~~

"A single particle eigenstate cannot be included twice in the ~~[illegible]~~ product state of a symmetrized fermion state!"

We can prove ~~the~~ Pauli exclusion principle from the ~~sy~~ ~~x~~ ~~mmetrization~~ ~~principle~~ ~~for~~ fermions

Say $|1u_1, 2u_2, \ldots k u_k \ldots l u_l \ldots\rangle$ is a symmetrized multi-~~ple~~-particle ~~fermion~~ state

u_i are set of eigenvalue ~~that~~ fully character the state.

$$P_{lk} | \ldots k u_k \ldots l u_l \ldots \rangle$$

$$= | \ldots k u_l \ldots l u_k \ldots \rangle$$

$$= - | \ldots k u_k \ldots l u_l \rangle$$

Necessary symmetrization on ~~inter~~ particle exchange.

But if $u_l = u_k$, then ~~the state can only~~

the equality only holds if the state is the null state $|0\rangle$

— which can't be normalized and can't be a physical state

<u>What</u> if the overall state <u>cannot</u> be expressed ~~as a tensor pro~~ in terms of single particle states?

Can't for Hydrogenic atom for example — see p. 5-95

No one addresses this question — it's what John Bell refers to as "<u>unspeakable</u> in QM".

<u>But</u> actually any state I think can be expressed in terms of single particle

eigenstates of space + spin

or wavenumber space & spin

$\underbrace{}$ AKA momentum & spin eigenstates

So

$$\Psi(\ldots \underset{\sim}{k}_i m_i \ldots\ldots \underset{\sim}{k}_j m_j \ldots)$$

$$= -\Psi(\ldots \underset{\sim}{k}_j m_j \ldots\ldots \underset{\sim}{k}_i m_i \ldots)$$

But if $\underset{\sim}{k}_i = \underset{\sim}{k}_j$ and $m_i = m_j$ } particular values

then $\Psi = 0$

an antibunching effect. — Not a real force since it arises from a restriction on wave-functions — at least in standard QM. But called exchange force anyway.

No two identical fermions can be found at the same point with same spin state

or the probability density for two identical fermions at one point with same spin is zero.

5-190

Remember spatial wave functions are smooth i.e., continuous.

~~So if $\Psi($~~

So if $\Psi = 0$

for $\underset{\sim}{r}_i = \underset{\sim}{r}_j$

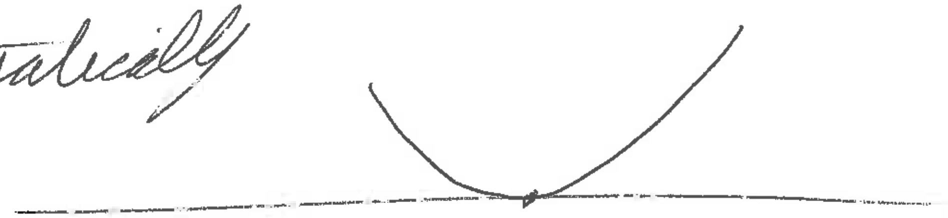
$m_i = m_j$

it must be nearly zero

for $\underset{\sim}{r}_i \triangleq \underset{\sim}{r}_j$

$m_i = m_j$

Schematically



$\underset{\sim}{r}_i = \underset{\sim}{r}_j$

So in the neighborhood of zero the wave function, the wave function and probability density will be small.

<u>But</u> what of other kinds of single-particle eigenstates.

— Well if they exist exactly then the <u>exclusion Principle</u> applies exactly.

— if they exist as an <u>approximation</u>, then the exclusion principle is right for them as far as they go.

— <u>What</u> if single-particle eigenstates are NOT even a good <u>approximation</u>?

Well maybe some other way to use the <u>exclusion principle</u> to characterize <u>the state</u> (but this is apparently unspeakable).

5-192

Examples of Spin 1/2 states

Example 1

T~~wo~~ spin 1/2 particles
(e.g., electrons)
~~and two states eigenstates.~~

and two single-particle stationary states

They are orthonormal

$$\psi_a(\mathbf{r})\chi_+ \quad \text{and} \quad \psi_b(\mathbf{r})\chi_+$$

both up states for simplicity.

Two <u>product states</u> can be constructed

$$\psi_a(\mathbf{r}_1)\chi_{+1}\,\psi_b(\mathbf{r}_2)\chi_{+2}$$

and $\psi_a(\mathbf{r}_2)\chi_{+2}\,\psi_b(\mathbf{r}_1)\chi_{+1}$

Question: Construct a symmetrized state.

Ans:
$$\Psi(1,2) = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) - \psi_a(x_2)\psi_b(x_1)\right]$$

$$* \; \chi_{+1} \chi_{+2}$$

$$\Psi(2,1) = -\Psi(1,2)$$

and so antisymmetric as ~~antisymmetrized~~ required.

Normalization? $|\Psi(1,2)|^2 = 1$?

— The cross terms vanish since $\int \psi_a^*(x)\psi_b(x)\,dx = 0$

etc.

$$|\Psi(1\,2)|^2 = \frac{1}{2}\left[1\cdot 1 + 1\cdot 1\right]\chi_{+1}^{\dagger}\chi_{+1} * \chi_{+2}^{\dagger}\chi_{+2}$$

$$= 1$$

5-194

~~Example 7.1~~

Note if $v = v_1 = v_2$

then $\Psi(12) \Rightarrow 0$.

The exchange force repulsion.

~~If~~ $a = b$, $\Psi(12) = 0$

everywhere for any v_1, v_2 value

We couldn't construct a 2-particle state in this case.

Both examples of the Pauli ~~exchange p~~ exclusion principle.