

# Quantum Mechanics

**Exam 1**

**2011 March 7, Monday**

**NAME:**

**SIGNATURE:**

**Instructions:** There are 10 multiple-choice questions each worth 2 marks for a total of 20 marks altogether. Choose the **BEST** answer, completion, etc., and darken fully the appropriate circle on the table provided below. Read all responses carefully. **NOTE** long detailed preambles and responses won't depend on hidden keywords: keywords in such preambles and responses are bold-faced capitalized.

There are **THREE** full answer questions each worth 10 marks for a total of 30 marks altogether. Answer them all on the paper provided. It is important that you **SHOW (SHOW, SHOW, SHOW)** how you got the answer.

This is a **CLOSED-BOOK** exam. **NO** cheat sheets allowed. An equation sheet is provided. Calculators are permitted for calculations. Cell phones **MUST** be turned off. The test is out of 50 marks altogether.

This a 50-minute test. Remember your name (and write it down on the exam too).

## Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
1.	<input type="radio"/>	6.	<input type="radio"/>								
2.	<input type="radio"/>	7.	<input type="radio"/>								
3.	<input type="radio"/>	8.	<input type="radio"/>								
4.	<input type="radio"/>	9.	<input type="radio"/>								
5.	<input type="radio"/>	10.	<input type="radio"/>								

---

011 qmult 00100 1 4 3 easy deducto-memory: central force

1. A central force is one which always points radially inward or outward from a fixed point which is the center of the central force. The magnitude of central force depends only on:
  - a) the angle of the particle.
  - b) the vector  $\vec{r}$  from the center to the particle.
  - c) the radial distance  $r$  from the center to the particle.
  - d) the magnetic quantum number of the particle.
  - e) the uncertainty principle.

**SUGGESTED ANSWER:** (c)

Wikipedia confirms this definition of a central force. Mathematically, one can write the force

$$\vec{F}(\vec{r}) = F(r)\hat{r} .$$

But what would a force like

$$\vec{F}(\vec{r}) = F(\vec{r})\hat{r}$$

be called. It's not officially a central force since the magnitude depends on direction. But its torque about the center is also zero, and so it conserves angular momentum. Perhaps, such forces are rare, and therefore not much studied.

**Wrong Answers:**

- a) Nah.
- b) Exactly wrong.

**Redaction:** Jeffery, 2001jan01

---

011 qmult 00212 1 1 3 easy memory: separation of variables 3

2. Say you have a partial differential equation with independent variables  $x_i$  and you want to look for solutions that can be factorized thusly

$$f(\{x_i\}) = \prod f_i(x_i) .$$

Now you substitute the factored form into the differential equation and find that it is possible to reorder differential equation into the form

$$g = \sum_j g_j(x_j) = 0 ,$$

where  $g_j(x_j)$  is some formula depending for  $f_j(x_j)$  only out of the set of functions  $f_i(x_i)$ . If we differentiate  $g$  with respect to general  $x_i$ , we find

$$\frac{\partial g}{\partial x_i} = \sum_j \frac{\partial g_j(x_j)}{\partial x_i} = \frac{\partial g_i(x_i)}{\partial x_i} = 0 ,$$

where we have used the fact that the variables are independent. Since  $x_i$  was general, we conclude that  $g_i(x_i)$  is actually independent of  $x_i$  as well as all other independent

variables. So all the  $g_i(x_i)$  expressions in fact equal to constants. These constants are called constants of separation. Thus, we have the set of ordinary differential equations

$$g_i(x_i) = C_i ,$$

where  $C_i$  is the constant of separation for variable  $i$ . The solutions for all  $f_i(x_i)$  can now be looked for. The solutions  $f(\{x_i\})$  that cannot be factorized are not obtained, of course, by the described procedure. However, if one obtains complete sets of solutions for each of the ordinary differential equation, then any solution  $f(\{x_i\})$  can be constructed at least to within some approximation (Arf-443). Thus, the described procedure is very general and powerful. It is called:

- a) separation of the  $g$ 's.    b) partitioning.    c) separation of the variables.  
d) the solution factorization.    e) the King Lear method.

**SUGGESTED ANSWER:** (c)

In quantum mechanics, it is a postulate that a complete set of eigenstates exists for any observable and that any physical state defined for the same space as the observable can be expanded exactly in those eigenstates in principle. The whole paradigm of quantum mechanics relies on this postulate—and quantum mechanics has never failed. Thus, separation of the variables when it can be applied to quantum mechanics is a very important procedure.

**Wrong answers:**

- d) Seems reasonable.  
e) Metaphorical names due turn up in physics like the Monte Carlo method (named after a famous casino in Monaco) and the Urca process (named after a casino in Rio de Janeiro). One sometimes gets the feeling that theoretical physicists spend a lot of time in casinos. I used to wander through them all the time in my Vegas years.

**Redaction:** Jeffery, 2008jan01

011 qmult 00300 1 4 2 easy deducto-memory: relative/cm reduction

3. “Let’s play *Jeopardy!* For \$100, the answer is: By writing the two-body Schrödinger equation in relative/center-of-mass coordinates.”

How do you \_\_\_\_\_, Alex?

- a) reduce a **ONE-BODY** problem to a **TWO-BODY** problem  
b) reduce a **TWO-BODY** problem to a **ONE-BODY** problem  
c) solve a one-dimensional infinite square well problem  
d) solve for the simple harmonic oscillator eigenvalues  
e) reduce a **TWO-BODY** problem to a **TWO-BODY** problem

**SUGGESTED ANSWER:** (b)

**Wrong answers:**

- e) Seems a bit pointless.

**Redaction:** Jeffery, 2001jan01

011 qmult 00310 1 4 4 easy deducto-memory: reduced mass

4. The formula for the reduced mass  $m$  for two-body system (with bodies labeled 1 and 2) is:

$$\begin{array}{llll} \text{a) } m = m_1 m_2. & \text{b) } m = \frac{1}{m_1 m_2}. & \text{c) } m = \frac{m_1 + m_2}{m_1 m_2}. & \text{d) } m = \frac{m_1 m_2}{m_1 + m_2}. \\ \text{e) } m = \frac{1}{m_1}. & & & \end{array}$$

**SUGGESTED ANSWER:** (d)

**Wrong Answers:**

- a) Dimensionally wrong.
- b) Dimensionally wrong.
- c) Dimensionally wrong.
- e) Dimensionally wrong and it only refers to one mass.

**Redaction:** Jeffery, 2001jan01

011 qmult 00410 1 4 4 easy deducto-memory: spherical harmonics 2

**Extra keywords:** mathematical physics

5. “Let’s play *Jeopardy!* For \$100, the answer is: They form a basis or complete set for the 2-dimensional space of the surface of a sphere which is usually described by the angular coordinates of spherical polar coordinates.”

What are the \_\_\_\_\_, Alex?

- a) Hermite polynomials
- b) Laguerre polynomials
- c) associated Laguerre polynomials
- d) spherical harmonics
- e) Chebyshev polynomials

**SUGGESTED ANSWER:** (d)

**Wrong answers:**

- a) These turn up as factors in the solution of the 1-dimensional simple harmonic oscillator problem.
- c) These turn up as factors in the solution of the radial part of the Coulomb potential 2-body problem
- e) These have uses in numerical computation.

**Redaction:** Jeffery, 2008jan01

011 qmult 00420 1 4 3 easy deducto memory: spherical harmonic Y00

6. Just about the only spherical harmonic that people remember—and they really should remember it too—is  $Y_{00}$  =:

$$\text{a) } e^{im\phi}. \quad \text{b) } r^2. \quad \text{c) } \frac{1}{\sqrt{4\pi}}. \quad \text{d) } \theta^2. \quad \text{e) } 2a^{-3/2}e^{-r/a}.$$

**SUGGESTED ANSWER:** (c)

**Wrong Answers:**

- a) This is the general azimuthal component of the spherical harmonics:  
 $m = 0, \pm 1, \pm 2, \dots, \pm \ell$ .

- b) This is radial and it's not normalizable.
- d) Except for  $Y_{00}$  itself, the spherical harmonics are all combinations of sinusoidal functions of the  $\theta$  and  $\phi$ .
- e) This is the  $R_{10}$  hydrogenic radial wave function where  $a$  is the scale radius

$$a = a_0 \frac{m_e}{m} \frac{1}{Z},$$

where  $m_e$  is the electron mass,  $m$  is the reduced mass,  $Z$  is the number of unit charges of the central particle, and  $a_0$  is the Bohr radius (Gr2005-137). The Bohr radius in MKS units is given by

$$a_0 = \frac{\hbar^2}{m_e [e^2 / (4\pi\epsilon_0)]} = \frac{\lambda_C}{2\pi} \frac{1}{\alpha} = 0.52917720859(36) \text{ \AA},$$

where  $e$  is the elementary charge,  $\lambda_C = \hbar / (m_e c)$  is the Compton wavelength, and  $\alpha \approx 1/137$  is the fine structure constant.

**Redaction:** Jeffery, 2001jan01

011 qmult 00510 1 4 3 easy deducto-memory: s electrons

7. "Let's play *Jeopardy!* For \$100, the answer is: What the  $\ell = 0$  electrons (or zero orbital angular momentum electrons) are called in spectroscopic notation."

What are \_\_\_\_\_, Alex?

- a) the Hermitian conjugates
- b) Herman's Hermits
- c)  $s$  electrons
- d)  $p$  electrons
- e)  $h$  electrons

**SUGGESTED ANSWER:** (c)

**Wrong answers:**

- b) A British rock band of the 1960s: not the Beatles.
- d) These are the  $\ell = 1$  electrons.
- e) These are the  $\ell = 5$  electrons.

**Redaction:** Jeffery, 2001jan01

019 qmult 00110 1 1 3 easy memory: exchange degeneracy and symmetrization principle

8. As strange as the symmetrization principle seems at first, quantum mechanics would be inconsistent without it since then you could create infinitely many physically distinct states by superpositions of the same state. This inconsistency is called the:

- a) symmetrization paradox.
- b) symmetrization degeneracy.
- c) exchange degeneracy.
- d) baffling degeneracy.
- e) baffling paradox.

**SUGGESTED ANSWER:** (c)

**Wrong answers:**

- e) By Gad, Holmes, baffled again.

**Redaction:** Jeffery, 2008jan01

020 qmult 00100 1 1 1 easy memory: atom defined

9. An atom is a stable bound system of electrons and:

- a) a single nucleus.    b) two nuclei.    c) three nuclei.    d) a single quark.  
e) two quarks.

**SUGGESTED ANSWER:** (a)

**Wrong Answers:**

- b) This is a diatomic molecule.

**Redaction:** Jeffery, 2001jan01

---

020 qmult 01000 1 4 1 easy deducto-memory: central potential

10. "Let's play *Jeopardy!* For \$100, the answer is: A favored approximation in the simpler solutions for the electronic structure of atoms in quantum mechanics."

What is the \_\_\_\_\_, Alex?

- a) central potential approximation    b) non-central potential approximation  
c) grand central approximation    d) atom-approximated-as-molecule method  
e) electrons-as-bosons approximation

**SUGGESTED ANSWER:** (a)

**Wrong answers:**

- d) Doesn't seem to likely to work.  
e) Off hand I can't think of a poorer approximation.

**Redaction:** Jeffery, 2001jan01

019 qfull 02000 2 5 0 moderate thinking: symmetrization of 4 orthonormal single-particle states

11. Say  $|ai\rangle$  and  $|bi\rangle$  are **ORTHONORMAL** single-particle states, where  $i$  is a particle label. The label can be thought of as labeling the coordinates to be integrated or summed over in an inner product: see below. The symbolic combination of such states for two particles, one in  $a$  and one in  $b$  is

$$|12\rangle = |a1\rangle|b2\rangle ,$$

where 1 and 2 are particle labels. This combination is actually a tensor product, but let's not worry about that now. The inner product of such a combined state is written

$$\langle 12|12\rangle = \langle a1|a1\rangle\langle b2|b2\rangle .$$

If one expanded the inner product in the position and spinor representation assuming the wave function and spinor parts can be separated (which in general is not the case),

$$\begin{aligned} \langle 12|12\rangle = & \left[ \int \psi_a(x_1)^* \psi_a(x_1) dx_1 (c_{a+}^* \quad c_{a-}^*)_1 \begin{pmatrix} c_{a+} \\ c_{a-} \end{pmatrix}_1 \right] \\ & \times \left[ \int \psi_b(x_2)^* \psi_b(x_2) dx_2 (c_{b+}^* \quad c_{b-}^*)_2 \begin{pmatrix} c_{b+} \\ c_{b-} \end{pmatrix}_2 \right] . \end{aligned}$$

A lot of conventions go into the last expression: don't worry too much about them.

- Let particles 1 and 2 be distinct particles. What are the two simplest and most obvious normalized 2-particle states that can be constructed from states  $a$  and  $b$ ? What happens if  $a = b$  (i.e., the two single-particle states are only one state actually)?
- Say particles 1 and 2 are identical bosons or identical fermions. What is the only normalized physical 2-particle state that can be constructed in either case allowing for the possibility that  $a = b$  (i.e., the two single-particle states are only one state actually)? What happens if  $a = b$  for fermions and what does it mean?

**SUGGESTED ANSWER:**

- a) Behold:

$$|12\rangle = |a1\rangle|b2\rangle \quad \text{and} \quad |21\rangle = |a2\rangle|b1\rangle$$

which are just the allowed product states. More complicated states can be constructed if the particles are in mixtures of the two states just given. If  $a = b$ , then one can construct only one state

$$|12\rangle = |a1\rangle|a2\rangle .$$

- b) Behold:

$$|12\rangle = \frac{1}{\sqrt{2(1 + \delta_{ab})}} (|a1\rangle|b2\rangle \pm |a2\rangle|b1\rangle) ,$$

where the upper case is for bosons and the lower case is for fermions.

I don't think there are any other possible physical states that can be constructed. There are only two mathematical product states obtained by permuting the permutation of particle labels. But those states are physically identical since the particles are identical. Thus, there is only one unpermuted product state which is not an exact physical state since it is not symmetrized. Product states are useful approximations in some cases. From the one unpermuted product state, only one symmetrized state can be constructed.

The Kronecker delta allows for the case that  $a = b$  for bosons. Obviously, we never had to symmetrize the product state at all for bosons if  $a = b$ . The product state is already symmetrized. It is

$$|12\rangle = |a1\rangle|a2\rangle .$$

If  $a = b$  for fermions, the state is null, and thus no physical state can be constructed in this case. The nullness is a manifestation of the Pauli exclusion principle (a corollary of the symmetrization postulate): two fermions cannot be found in the same single-particle state (as specified by a C.S.C.O.: i.e., a complete set of commuting observables (CT-143)). "Cannot be found" has to be interpreted as the probability for finding two fermions in the same single-particle state is zero or, in other words, that the probability of collapsing the wave function to having two fermions in the same single-particle state is zero. So if  $a = b$  for fermions, then a physical symmetrized state cannot be created from product states.

**Redaction:** Jeffery, 2001jan01

020 qfull 00200 1 3 0 easy math: electronic configurations to Ca

12. Write down the ground state electronic configurations of the neutral atoms from hydrogen (element 1) to calcium (element 20).

**SUGGESTED ANSWER:**

Ground State Electronic Configurations of Atoms from H to Ca

Atom	Electronic Configuration
H <sup>1</sup>	(1s)
He <sup>2</sup>	(1s) <sup>2</sup>
Li <sup>3</sup>	(He)(2s)
Be <sup>4</sup>	(He)(2s) <sup>2</sup>
B <sup>5</sup>	(He)(2s) <sup>2</sup> (2p)
C <sup>6</sup>	(He)(2s) <sup>2</sup> (2p) <sup>2</sup>
N <sup>7</sup>	(He)(2s) <sup>2</sup> (2p) <sup>3</sup>
O <sup>8</sup>	(He)(2s) <sup>2</sup> (2p) <sup>4</sup>
F <sup>9</sup>	(He)(2s) <sup>2</sup> (2p) <sup>5</sup>
Ne <sup>10</sup>	(He)(2s) <sup>2</sup> (2p) <sup>6</sup>
Na <sup>11</sup>	(Ne)(3s)
Mg <sup>12</sup>	(Ne)(3s) <sup>2</sup>
Al <sup>13</sup>	(Ne)(3s) <sup>2</sup> (3p)

Si <sup>14</sup>	(Ne)(3s) <sup>2</sup> (3p) <sup>2</sup>
P <sup>15</sup>	(Ne)(3s) <sup>2</sup> (3p) <sup>3</sup>
S <sup>16</sup>	(Ne)(3s) <sup>2</sup> (3p) <sup>4</sup>
Cl <sup>17</sup>	(Ne)(3s) <sup>2</sup> (3p) <sup>5</sup>
Ar <sup>18</sup>	(Ne)(3s) <sup>2</sup> (3p) <sup>6</sup>
K <sup>19</sup>	(Ar)(4s)
Ca <sup>20</sup>	(Ar)(4s) <sup>2</sup>

**Redaction:** Jeffery, 2008jan01

019 qfull 02500 2 3 0 mod math: coupled simple harmonic oscillator, coupled SHOs

**Extra keywords:** On tests going to part f might be sufficient

13. There are two particles subject to separate simple harmonic oscillator (SHO) potentials. Initially, we assume that they are distinct particles. They are also coupled by a mutual SHO potential. The full Hamiltonian is:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}m_1\omega^2x_1^2 + \frac{1}{2}m_2\omega^2x_2^2 + \frac{1}{2}k(x_1 - x_2)^2 ,$$

where  $k > 0$  which in this context means the interaction is attractive. The problem is 1-dimensional: it is in the  $x$  dimension only.

- Write down the formulae for the center-of-mass (CM) and relative (REL) coordinate and their inverses (i.e.,  $x_1$  and  $x_2$  expressed in terms of the CM coordinate  $X$  and the relative  $x$ ).
- Transform the Hamiltonian to the center-of-mass-relative (CM-REL) coordinates (showing all the steps).
- Now show that the time-independent Schrödinger equation for the Hamiltonian separates into CM and REL time-independent Schrödinger equations. Define

$$\tilde{\omega} = \sqrt{\omega^2 + \frac{k}{\mu}} = \omega \sqrt{1 + \frac{k}{\mu\omega^2}}$$

(where  $\mu$  is the reduced mass) in order to simplify the REL equation. Does the overall time-independent Schrödinger equation have an exact solution?

- Write down the general expression for the eigen-energies of the total stationary states in terms of the SHO quantum numbers  $n_{\text{CM}}$  and  $n_{\text{REL}}$  for the respective CM and REL parts.
- Next write the expression for the eigen-energies in the case that  $k = 0$ . Define a new quantum number  $n$  that alone gives the eigen-energy and the degeneracy of the eigen-energy. What is the degeneracy of an eigen-energy of quantum number  $n$ ?

- f) Now assume that  $k > 0$ , but that  $k/(\mu\omega^2) \ll 1$ . Write down a 1st order correct expression for the energy in terms of  $n$  and  $n_{\text{REL}}$ . Give a schematic energy-level diagram.
- g) Now assume that  $k/(\mu\omega^2) \gg 1$ . Give a schematic energy-level diagram in this case.
- h) Now assume that the two particles are identical spin-0 bosons. Note that identical means they now have the same mass. Given the symmetry requirement for boson states, which solutions (specified by the  $n_{\text{CM}}$  and  $n_{\text{REL}}$  quantum numbers) are not physically allowed?
- i) Now assume that the two particles are identical spin-1/2 fermions. Note again that identical means they now have the same mass. But also note they aren't electrons. Their interactions are determined by the given Hamiltonian only. Because the particles are spin-1/2 fermions, the stationary state wave functions for system must be multiplied by appropriate eigen-spinors to specify the full stationary state. Given the antisymmetry requirement for fermion states, what restrictions are put on the wave function and spinor quantum numbers of an eigenstate?

**SUGGESTED ANSWER:**

- a) For the sake of generality, let's work in three dimensions for awhile even though the problem is 1-dimensional. We can specialize when we need to to the 1-dimensional case.

Define the relative radius

$$\vec{r} = \vec{r}_2 - \vec{r}_1 ,$$

the total mass

$$M = m_1 + m_2$$

and the CM radius

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} .$$

For the inverses, we substitute for  $\vec{r}_2$  in the formula for  $\vec{R}$  the expression  $\vec{r}_2 = \vec{r} + \vec{r}_1$  which gives

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} = \frac{m_1\vec{r}_1 + m_2(\vec{r} + \vec{r}_1)}{M} = \vec{r}_1 + \frac{m_2}{M}\vec{r} = \vec{r}_1 + \frac{m_2}{M}\vec{r} .$$

Thus, we find

$$\vec{r}_1 = \vec{R} - \frac{m_2}{M}\vec{r}$$

and

$$\vec{r}_2 = \vec{r} + \vec{r}_1 = \vec{r} + \vec{R} - \frac{m_2}{M}\vec{r} = \vec{R} + \left(1 - \frac{m_2}{M}\right)\vec{r} = \vec{R} + \frac{m_1}{M}\vec{r} .$$

The specializations to the 1-dimensional case are obvious.

- b) For the sake of generality, let's work in three dimensions for awhile even though the problem is 1-dimensional. We can specialize when we need to to the 1-dimensional case.

First, we need to transform kinetic energy observables. We find

$$\frac{\partial}{\partial x_i} = \frac{\partial x}{\partial x_i} \frac{\partial}{\partial x} + \frac{\partial X}{\partial x_i} \frac{\partial}{\partial X} = \mp \frac{\partial}{\partial x} + \frac{m_i}{M} \frac{\partial}{\partial X} ,$$

where  $i = 1$  for the upper case and  $i = 2$  for the lower case. Next we find

$$\frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial x^2} + \left(\frac{m_i}{M}\right)^2 \frac{\partial^2}{\partial X^2} \mp 2 \frac{m_i}{M} \frac{\partial^2}{\partial x \partial X} .$$

Exactly analogous expressions hold for  $y$  and  $z$  coordinates. Thus, we find that

$$\frac{1}{m_1} \nabla_1^2 + \frac{1}{m_2} \nabla_2^2 = \frac{(m_1 + m_2)}{M^2} \nabla_{\text{CM}}^2 + \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \nabla_{\text{REL}}^2 = \frac{1}{M} \nabla_{\text{CM}}^2 + \frac{1}{\mu} \nabla_{\text{REL}}^2 ,$$

where for simplicity here and below we only distinguish the operators with a subscript CM and REL when needed and where we define the reduced mass by

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad \text{or} \quad \mu = \frac{m_1 m_2}{M} .$$

Second, we need to find the conversion relations needed for the potential terms. We find

$$\begin{aligned} \frac{1}{2} m_1 \omega^2 r_1^2 + \frac{1}{2} m_2 \omega^2 r_2^2 &= \frac{1}{2} \omega \left[ m_1 \left( \vec{R} - \frac{m_2}{M} \vec{r} \right) \cdot \left( \vec{R} - \frac{m_2}{M} \vec{r} \right) + m_2 \left( \vec{R} + \frac{m_1}{M} \vec{r} \right) \cdot \left( \vec{R} + \frac{m_1}{M} \vec{r} \right) \right] \\ &= \frac{1}{2} \omega \left[ (m_1 + m_2) R^2 + \frac{(m_1 m_2^2 + m_2 m_1^2)}{M^2} r^2 - 2 \frac{m_1 m_2}{M} \vec{R} \cdot \vec{r} + 2 \frac{m_1 m_2}{M} \vec{R} \cdot \vec{r} \right] \\ &= \frac{1}{2} M \omega R^2 + \frac{1}{2} \mu \omega r^2 . \end{aligned}$$

The corresponding relations for any component are analogous. Note that the nice separation into a center-of-mass SHO potential and relative SHO potential would not have been possible if the angular frequencies had been different.

In the present case, the Hamiltonian is confined to the  $x$  dimension only. We see this Hamiltonian in CM-REL coordinates must be

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + \frac{1}{2} M \omega^2 X^2 + \frac{1}{2} \mu \omega^2 x^2 + \frac{1}{2} k x^2 .$$

- c) With the transformed Hamiltonian, the time-independent Schrödinger equation becomes

$$H\psi(X, x) = H_{\text{CM}}\psi(X, x) + H_{\text{REL}}\psi(X, x) = E\psi(X, x) ,$$

where we have specialized our transformations to the 1-dimensional case.

If we separate the wave function

$$\psi(X, x) = \psi(X)\psi(x)$$

and divide the Schrödinger equation by the wave function, we obtain

$$\frac{H_{\text{CM}}\psi(X)}{\psi(X)} + \frac{H_{\text{REL}}\psi(x)}{\psi(x)} = E .$$

Now  $X$  and  $x$  are independent, and so we can vary one holding the other constant. If we do that, we see that both terms on the right-hand side must equal constants which we identify as  $E_{\text{CM}}$  and  $E_{\text{REL}}$ . The total energy is

$$E = E_{\text{CM}} + E_{\text{REL}} .$$

We can now write down the separate Schrödinger equations

$$H_{\text{CM}}\psi(X) = \left( \frac{P^2}{2M} + \frac{1}{2}M\omega^2 X^2 \right) \psi(X) = E_{\text{CM}}\psi(X) ,$$

$$H_{\text{REL}}\psi(x) = \left( \frac{p^2}{2\mu} + \frac{1}{2}\mu\tilde{\omega}^2 x^2 \right) \psi(x) = E_{\text{REL}}\psi(x) ,$$

where, as suggested, we have defined

$$\tilde{\omega} = \sqrt{\omega^2 + \frac{k}{\mu}} = \omega \sqrt{1 + \frac{k}{\mu\omega^2}} .$$

Actually, to the experienced, the separate Schrödinger equations are obvious since the potential is separable for the two variables. But the question asks for a demonstration.

Both the CM and REL parts are SHO equations, and thus have exact solutions. Ergo the total Schrödinger equation also has an exact solution.

- d) The eigen-energies of the total solutions (i.e., the total stationary states) are given by

$$E = E_{\text{CM}} + E_{\text{REL}} = \left( n_{\text{CM}} + \frac{1}{2} \right) \hbar\omega + \left( n_{\text{REL}} + \frac{1}{2} \right) \hbar\tilde{\omega} ,$$

where both the  $n_{\text{CM}}$  and  $n_{\text{REL}}$  quantum numbers run  $0, 1, 2, 3, \dots$

- e) If  $k = 0$ , then

$$E = (n_{\text{CM}} + n_{\text{REL}} + 1) \hbar\omega .$$

We can define a new quantum number

$$n = n_{\text{CM}} + n_{\text{REL}}$$

which labels the distinct energy levels and runs  $0, 1, 2, 3, \dots$ . Thus,

$$E = (n + 1)\hbar\omega .$$

The degenerate states can be labeled by

$$n_{\text{REL}} = n - n_{\text{CM}}$$

which for fixed  $n$  runs over  $0, 1, 2, 3, \dots, n$  as  $n_{\text{CM}}$  runs over  $n, n - 1, n - 2, n - 3, \dots, 0$ . The degeneracy is  $n + 1$ . Only the ground state is not degenerate.

If one made a square array states with  $n_{\text{CM}}$  and  $n_{\text{REL}}$  labeling, respectively, infinite rows and columns, then  $n$  runs along the vertical axis labeling diagonals made of degenerate states and  $n_{\text{REL}}$  runs along the diagonal labeling the degenerate states for each  $n$ .

f) If  $k/(\mu\omega^2) \ll 1$ , then

$$\tilde{\omega} \approx \omega \left( 1 + \frac{k}{2\mu\omega^2} \right)$$

to 1st order. We now see that

$$\begin{aligned} E &= E_{\text{CM}} + E_{\text{REL}} = \left( n_{\text{CM}} + \frac{1}{2} \right) \hbar\omega + \left( n_{\text{REL}} + \frac{1}{2} \right) \hbar\tilde{\omega} \\ &\approx \left( n_{\text{CM}} + \frac{1}{2} \right) \hbar\omega + \left( n_{\text{REL}} + \frac{1}{2} \right) \hbar\omega + \left( n_{\text{REL}} + \frac{1}{2} \right) \hbar\omega \left( \frac{k}{2\mu\omega^2} \right) \\ &= (n + 1)\hbar\omega + \left( n_{\text{REL}} + \frac{1}{2} \right) \hbar\omega \left( \frac{k}{2\mu\omega^2} \right) \end{aligned}$$

to 1st order. The  $k$  term is now a perturbation which splits the degeneracy of the part (e) case. The  $n$  quantum number now labels a group of closely spaced states that run from low energy when  $n_{\text{REL}} = 0$  to high when  $n_{\text{REL}} = n$ . The number of states in a group is  $n + 1$ .

The energy-level diagram should look like a potential well with walls rising to infinity. For  $n = 0$ , one has just one low-lying level. For  $n = 1$ , there are two higher closely spaced levels. For  $n = 2$ , there are three still higher closely spaced levels. And so on. I need to leave the diagram to your imagination.

g) If  $k/(\mu\omega^2) \gg 1$ , then the  $n$  quantum number is not useful. In this case one can only use the original quantum numbers  $n_{\text{CM}}$  and  $n_{\text{REL}}$  and let them run over  $0, 1, 2, 3, \dots$ . However, since  $n_{\text{REL}}$  accounts for large energy steps it makes sense to define orders of energy levels by their  $n_{\text{REL}}$  quantum number. Thus given  $n_{\text{REL}}$ , the zeroth level of the  $n_{\text{REL}}$  order has  $n_{\text{CM}} = 0$ , the 1st level has  $n_{\text{CM}} = 1$ , the 2nd  $n_{\text{CM}} = 2$ , and so on. Each order  $n_{\text{REL}}$  will be overlapped by the high levels of all lower orders. In general the levels of different

orders will not be coincident however: they will be scattered about higgedly-piggedly.

If we do, however, equate to energy levels from different orders

$$\left(n_{\text{CM}} + \frac{1}{2}\right) \hbar\omega + \left(n_{\text{REL}} + \frac{1}{2}\right) \hbar\tilde{\omega} = \left(n'_{\text{CM}} + \frac{1}{2}\right) \hbar\omega + \left(n'_{\text{REL}} + \frac{1}{2}\right) \hbar\tilde{\omega}$$

and rearrange to get

$$n'_{\text{CM}} - n_{\text{CM}} = (n_{\text{REL}} - n'_{\text{REL}}) \frac{\tilde{\omega}}{\omega},$$

then we only get an acceptable integral solution for  $n'_{\text{CM}} - n_{\text{CM}}$  for any  $n_{\text{REL}} - n'_{\text{REL}}$  if  $\tilde{\omega}/\omega$  is an integer. For definiteness, let's say that the primes indicate the lower order. Thus  $n_{\text{REL}} - n'_{\text{REL}} \geq 0$  and  $n'_{\text{CM}} - n_{\text{CM}} \geq 0$ . This means that any solution of order  $n_{\text{REL}}$  is degenerate with an energy solution of each of lower orders  $n'_{\text{REL}}$ . Thus the degeneracy of any energy level is  $n_{\text{REL}} + 1$  counting orders 0 through  $n_{\text{REL}}$  itself.

- h) The symmetry requirement for spin-0 boson states is that on the interchange of any pair of particle coordinates, the state function stay the same. In this case the state function in terms of the individual coordinates is

$$\psi_{n_{\text{CM}}} \left( \frac{m_1 x_1 + m_2 x_2}{M} \right) \psi_{n_{\text{REL}}} (x_2 - x_1),$$

where each component state function is a one-dimensional SHO wave function. The CM component is, in fact, already symmetric on pair interchange, and so imposes no new restrictions. Since SHO wave functions are even for even quantum number and odd for odd, the REL component is symmetric for  $n_{\text{REL}}$  even and antisymmetric for  $n_{\text{REL}}$  odd. No combination of antisymmetric functions can ever be symmetric, so we in must exclude all odd  $n_{\text{REL}}$  solutions: they arn't physically realizable for spin-0 bosons. Note the remaining solutions still constitute a complete set, but only for spin-0 bosons.

- i) The standard eigen spinors for a 2 spin-1/2 fermion system are

$$\begin{aligned} \chi_{1,1} &= \alpha(1)\alpha(2) \\ \chi_{1,0} &= \frac{1}{\sqrt{2}}[\alpha(1)\beta(2) + \beta(1)\alpha(2)] \\ \chi_{1,-1} &= \beta(1)\beta(2) \end{aligned}$$

and

$$\chi_{0,0} = \frac{1}{\sqrt{2}}[\alpha(1)\beta(2) - \beta(1)\alpha(2)],$$

where the first three spinors constitute the symmetric triplet state and the last one the antisymmetric singlet state and where

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(e.g., Morrison et al. 1991, p. 188). Since the overall eigenstate must be antisymmetric, the singlet state requires  $n_{\text{REL}}$  even and the triplet state requires  $n_{\text{REL}}$  odd. This is the only restriction.

**Redaction:** Jeffery, 2001jan01

## Appendix 2 Quantum Mechanics Equation Sheet

**Note:** This equation sheet is intended for students writing tests or reviewing material. Therefore it neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things.

---

### 1 Constants not to High Accuracy

Constant Name	Symbol	Derived from CODATA 1998
Bohr radius	$a_{\text{Bohr}} = \frac{\lambda_{\text{Compton}}}{2\pi\alpha}$	$= 0.529 \text{ \AA}$
Boltzmann's constant	$k$	$= 0.8617 \times 10^{-6} \text{ eV K}^{-1}$ $= 1.381 \times 10^{-16} \text{ erg K}^{-1}$
Compton wavelength	$\lambda_{\text{Compton}} = \frac{h}{m_e c}$	$= 0.0246 \text{ \AA}$
Electron rest energy	$m_e c^2$	$= 5.11 \times 10^5 \text{ eV}$
Elementary charge squared	$e^2$	$= 14.40 \text{ eV \AA}$
Fine Structure constant	$\alpha = \frac{e^2}{\hbar c}$	$= 1/137.036$
Kinetic energy coefficient	$\frac{\hbar^2}{2m_e}$	$= 3.81 \text{ eV \AA}^2$
	$\frac{\hbar^2}{m_e}$	$= 7.62 \text{ eV \AA}^2$
Planck's constant	$h$	$= 4.15 \times 10^{-15} \text{ eV}$
Planck's h-bar	$\hbar$	$= 6.58 \times 10^{-16} \text{ eV}$
Rydberg Energy	$hc$	$= 12398.42 \text{ eV \AA}$
	$\hbar c$	$= 1973.27 \text{ eV \AA}$
	$E_{\text{Ryd}} = \frac{1}{2} m_e c^2 \alpha^2$	$= 13.606 \text{ eV}$

---

### 2 Some Useful Formulae

Leibniz's formula 
$$\frac{d^n(fg)}{dx^n} = \sum_{k=0}^n \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}$$

Normalized Gaussian 
$$P = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - \langle x \rangle)^2}{2\sigma^2}\right]$$

---

### 3 Schrödinger's Equation

$$H\Psi(x, t) = \left[ \frac{p^2}{2m} + V(x) \right] \Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}$$

$$H\psi(x) = \left[ \frac{p^2}{2m} + V(x) \right] \psi(x) = E\psi(x)$$

$$H\Psi(\vec{r}, t) = \left[ \frac{p^2}{2m} + V(\vec{r}) \right] \Psi(\vec{r}, t) = i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} \quad H|\Psi\rangle = i\hbar \frac{\partial}{\partial t} |\Psi\rangle$$

$$H\psi(\vec{r}) = \left[ \frac{p^2}{2m} + V(\vec{r}) \right] \psi(\vec{r}) = E\psi(\vec{r}) \quad H|\psi\rangle = E|\psi\rangle$$

#### 4 Some Operators

$$p = \frac{\hbar}{i} \frac{\partial}{\partial x} \quad p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

$$H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$$

$$p = \frac{\hbar}{i} \nabla \quad p^2 = -\hbar^2 \nabla^2$$

$$H = \frac{p^2}{2m} + V(\vec{r}) = -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r})$$

$$\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

#### 5 Kronecker Delta and Levi-Civita Symbol

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & \text{otherwise} \end{cases} \quad \varepsilon_{ijk} = \begin{cases} 1, & ijk \text{ cyclic}; \\ -1, & ijk \text{ anticyclic}; \\ 0, & \text{if two indices the same.} \end{cases}$$

$$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \quad (\text{Einstein summation on } i)$$

---

## 6 Time Evolution Formulae

$$\text{General} \quad \frac{d\langle A \rangle}{dt} = \left\langle \frac{\partial A}{\partial t} \right\rangle + \frac{1}{\hbar} \langle i[H(t), A] \rangle$$

$$\text{Ehrenfest's Theorem} \quad \frac{d\langle \vec{r} \rangle}{dt} = \frac{1}{m} \langle \vec{p} \rangle \quad \text{and} \quad \frac{d\langle \vec{p} \rangle}{dt} = -\langle \nabla V(\vec{r}) \rangle$$

$$|\Psi(t)\rangle = \sum_j c_j(0) e^{-iE_j t/\hbar} |\phi_j\rangle$$

---

## 7 Simple Harmonic Oscillator (SHO) Formulae

$$V(x) = \frac{1}{2} m \omega^2 x^2 \quad \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \right) \psi = E \psi$$

$$\beta = \sqrt{\frac{m\omega}{\hbar}} \quad \psi_n(x) = \frac{\beta^{1/2}}{\pi^{1/4} \sqrt{2^n n!}} H_n(\beta x) e^{-\beta^2 x^2/2} \quad E_n = \left( n + \frac{1}{2} \right) \hbar \omega$$

$$H_0(\beta x) = H_0(\xi) = 1 \quad H_1(\beta x) = H_1(\xi) = 2\xi$$

$$H_2(\beta x) = H_2(\xi) = 4\xi^2 - 2 \quad H_3(\beta x) = H_3(\xi) = 8\xi^3 - 12\xi$$

---

## 8 Position, Momentum, and Wavenumber Representations

$$p = \hbar k \quad E_{\text{kinetic}} = E_T = \frac{\hbar^2 k^2}{2m}$$

$$|\Psi(p, t)|^2 dp = |\Psi(k, t)|^2 dk \quad \Psi(p, t) = \frac{\Psi(k, t)}{\sqrt{\hbar}}$$

$$x_{\text{op}} = x \quad p_{\text{op}} = \frac{\hbar}{i} \frac{\partial}{\partial x} \quad Q \left( x, \frac{\hbar}{i} \frac{\partial}{\partial x}, t \right) \quad \text{position representation}$$

$$x_{\text{op}} = -\frac{\hbar}{i} \frac{\partial}{\partial p} \quad p_{\text{op}} = p \quad Q \left( -\frac{\hbar}{i} \frac{\partial}{\partial p}, p, t \right) \quad \text{momentum representation}$$

$$\delta(x) = \int_{-\infty}^{\infty} \frac{e^{ipx/\hbar}}{2\pi\hbar} dp \quad \delta(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{2\pi} dk$$

$$\Psi(x, t) = \int_{-\infty}^{\infty} \Psi(p, t) \frac{e^{ipx/\hbar}}{(2\pi\hbar)^{1/2}} dp \quad \Psi(x, t) = \int_{-\infty}^{\infty} \Psi(k, t) \frac{e^{ikx}}{(2\pi)^{1/2}} dk$$

$$\Psi(p, t) = \int_{-\infty}^{\infty} \Psi(x, t) \frac{e^{-ipx/\hbar}}{(2\pi\hbar)^{1/2}} dx \quad \Psi(k, t) = \int_{-\infty}^{\infty} \Psi(x, t) \frac{e^{-ikx}}{(2\pi)^{1/2}} dx$$

$$\Psi(\vec{r}, t) = \int_{\text{all space}} \Psi(\vec{p}, t) \frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}} d^3p \quad \Psi(\vec{r}, t) = \int_{\text{all space}} \Psi(\vec{k}, t) \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} d^3k$$

$$\Psi(\vec{p}, t) = \int_{\text{all space}} \Psi(\vec{r}, t) \frac{e^{-i\vec{p}\cdot\vec{r}/\hbar}}{(2\pi\hbar)^{3/2}} d^3r \quad \Psi(\vec{k}, t) = \int_{\text{all space}} \Psi(\vec{r}, t) \frac{e^{-i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} d^3r$$

## 9 Commutator Formulae

$$[A, BC] = [A, B]C + B[A, C] \quad \left[ \sum_i a_i A_i, \sum_j b_j B_j \right] = \sum_{i,j} a_i b_j [A_i, B_j]$$

$$\text{if } [B, [A, B]] = 0 \quad \text{then } [A, F(B)] = [A, B]F'(B)$$

$$[x, p] = i\hbar \quad [x, f(p)] = i\hbar f'(p) \quad [p, g(x)] = -i\hbar g'(x)$$

$$[a, a^\dagger] = 1 \quad [N, a] = -a \quad [N, a^\dagger] = a^\dagger$$

## 10 Uncertainty Relations and Inequalities

$$\sigma_x \sigma_p = \Delta x \Delta p \geq \frac{\hbar}{2} \quad \sigma_Q \sigma_R = \Delta Q \Delta R \geq \frac{1}{2} |\langle i[Q, R] \rangle|$$

$$\sigma_H \Delta t_{\text{scale time}} = \Delta E \Delta t_{\text{scale time}} \geq \frac{\hbar}{2}$$

## 11 Probability Amplitudes and Probabilities

$$\Psi(x, t) = \langle x | \Psi(t) \rangle \quad P(dx) = |\Psi(x, t)|^2 dx \quad c_i(t) = \langle \phi_i | \Psi(t) \rangle \quad P(i) = |c_i(t)|^2$$

## 12 Spherical Harmonics

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}} \quad Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos(\theta) \quad Y_{1,\pm 1} = \mp \left(\frac{3}{8\pi}\right)^{1/2} \sin(\theta) e^{\pm i\phi}$$

$$L^2 Y_{\ell m} = \ell(\ell+1) \hbar^2 Y_{\ell m} \quad L_z Y_{\ell m} = m \hbar Y_{\ell m} \quad |m| \leq \ell \quad m = -\ell, -\ell+1, \dots, \ell-1, \ell$$

0	1	2	3	4	5	6	...
<i>s</i>	<i>p</i>	<i>d</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>	...

## 13 Hydrogenic Atom

$$\psi_{n\ell m} = R_{n\ell}(r) Y_{\ell m}(\theta, \phi) \quad \ell \leq n-1 \quad \ell = 0, 1, 2, \dots, n-1$$

$$a_z = \frac{a}{Z} \left( \frac{m_e}{m_{\text{reduced}}} \right) \quad a_0 = \frac{\hbar}{m_e c \alpha} = \frac{\lambda_C}{2\pi\alpha} \quad \alpha = \frac{e^2}{\hbar c}$$

$$R_{10} = 2a_Z^{-3/2} e^{-r/a_Z} \quad R_{20} = \frac{1}{\sqrt{2}} a_Z^{-3/2} \left( 1 - \frac{1}{2} \frac{r}{a_Z} \right) e^{-r/(2a_Z)}$$

$$R_{21} = \frac{1}{\sqrt{24}} a_Z^{-3/2} \frac{r}{a_Z} e^{-r/(2a_Z)}$$

$$R_{n\ell} = - \left\{ \left( \frac{2}{na_Z} \right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3} \right\}^{1/2} e^{-\rho/2} \rho^\ell L_{n+\ell}^{2\ell+1}(\rho) \quad \rho = \frac{2r}{nr_Z}$$

$$L_q(x) = e^x \left( \frac{d}{dx} \right)^q (e^{-x} x^q) \quad \text{Rodrigues's formula for the Laguerre polynomials}$$

$$L_q^j(x) = \left( \frac{d}{dx} \right)^j L_q(x) \quad \text{Associated Laguerre polynomials}$$

$$\langle r \rangle_{n\ell m} = \frac{a_Z}{2} [3n^2 - \ell(\ell+1)]$$

$$\text{Nodes} = (n-1) - \ell \quad \text{not counting zero or infinity}$$

$$E_n = -\frac{1}{2} m_e c^2 \alpha^2 \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} = -E_{\text{Ryd}} \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} = -13.606 \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} \text{ eV}$$

## 14 General Angular Momentum Formulae

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k \quad (\text{Einstein summation on } k) \quad [J^2, \vec{J}] = 0$$

$$J^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle \quad J_z |jm\rangle = m\hbar |jm\rangle$$

$$J_\pm = J_x \pm iJ_y \quad J_\pm |jm\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} |jm \pm 1\rangle$$

$$J_{\{x\}} = \left\{ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2i} \end{array} \right\} (J_+ \pm J_-) \quad J_\pm^\dagger J_\pm = J_\mp J_\pm = J^2 - J_z(J_z \pm \hbar)$$

$$[J_{fi}, J_{gj}] = \delta_{fg} i\hbar \varepsilon_{ijk} J_k \quad \vec{J} = \vec{J}_1 + \vec{J}_2 \quad J^2 = J_1^2 + J_2^2 + J_{1+}J_{2-} + J_{1-}J_{2+} + 2J_{1z}J_{2z}$$

$$J_{\pm} = J_{1\pm} + J_{2\pm} \quad |j_1 j_2 j m\rangle = \sum_{m_1 m_2, m=m_1+m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle$$

$$|j_1 - j_2| \leq j \leq j_1 + j_2 \quad \sum_{|j_1 - j_2|}^{j_1 + j_2} (2j + 1) = (2j_1 + 1)(2j_2 + 1)$$

### 15 Spin 1/2 Formulae

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\pm\rangle_x = \frac{1}{\sqrt{2}} (|+\rangle \pm |-\rangle) \quad |\pm\rangle_y = \frac{1}{\sqrt{2}} (|+\rangle \pm i|-\rangle) \quad |\pm\rangle_z = |\pm\rangle$$

$$|++\rangle = |1, +\rangle |2, +\rangle \quad |+-\rangle = \frac{1}{\sqrt{2}} (|1, +\rangle |2, -\rangle \pm |1, -\rangle |2, +\rangle) \quad |--\rangle = |1, -\rangle |2, -\rangle$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} + i \varepsilon_{ijk} \sigma_k \quad [\sigma_i, \sigma_j] = 2i \varepsilon_{ijk} \sigma_k \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$$(\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) = \vec{A} \cdot \vec{B} + i(\vec{A} \times \vec{B}) \cdot \vec{\sigma}$$

$$\frac{d(\vec{S} \cdot \hat{n})}{d\alpha} = -\frac{i}{\hbar} [\vec{S} \cdot \hat{\alpha}, \vec{S} \cdot \hat{n}] \quad \vec{S} \cdot \hat{n} = e^{-i\vec{S} \cdot \hat{\alpha}} \vec{S} \cdot \hat{n}_0 e^{i\vec{S} \cdot \hat{\alpha}} \quad |\hat{n}_{\pm}\rangle = e^{-i\vec{S} \cdot \hat{\alpha}} |\hat{z}_{\pm}\rangle$$

$$e^{ixA} = \mathbf{1} \cos(x) + iA \sin(x) \quad \text{if } A^2 = \mathbf{1} \quad e^{-i\vec{\sigma} \cdot \hat{\alpha}/2} = \mathbf{1} \cos(x) - i\vec{\sigma} \cdot \hat{\alpha} \sin(x)$$

$$\sigma_i f(\sigma_j) = f(\sigma_j) \sigma_i \delta_{ij} + f(-\sigma_j) \sigma_i (1 - \delta_{ij})$$

$$\mu_{\text{Bohr}} = \frac{e\hbar}{2m} = 0.927400915(23) \times 10^{-24} \text{ J/T} = 5.7883817555(79) \times 10^{-5} \text{ eV/T}$$

$$g = 2 \left( 1 + \frac{\alpha}{2\pi} + \dots \right) = 2.0023193043622(15)$$

$$\vec{\mu}_{\text{orbital}} = -\mu_{\text{Bohr}} \frac{\vec{L}}{\hbar} \quad \vec{\mu}_{\text{spin}} = -g\mu_{\text{Bohr}} \frac{\vec{S}}{\hbar} \quad \vec{\mu}_{\text{total}} = \vec{\mu}_{\text{orbital}} + \vec{\mu}_{\text{spin}} = -\mu_{\text{Bohr}} \frac{(\vec{L} + g\vec{S})}{\hbar}$$

$$H_{\mu} = -\vec{\mu} \cdot \vec{B} \quad H_{\mu} = \mu_{\text{Bohr}} B_z \frac{(L_z + gS_z)}{\hbar}$$

## 16 Time-Independent Approximation Methods

$$H = H^{(0)} + \lambda H^{(1)} \quad |\psi\rangle = N(\lambda) \sum_{k=0}^{\infty} \lambda^k |\psi_n^{(k)}\rangle$$

$$H^{(1)} |\psi_n^{(m-1)}\rangle (1 - \delta_{m,0}) + H^{(0)} |\psi_n^{(m)}\rangle = \sum_{\ell=0}^m E^{(m-\ell)} |\psi_n^{(\ell)}\rangle \quad |\psi_n^{(\ell>0)}\rangle = \sum_{m=0, m \neq n}^{\infty} a_{nm} |\psi_n^{(0)}\rangle$$

$$|\psi_n^{1\text{st}}\rangle = |\psi_n^{(0)}\rangle + \lambda \sum_{\text{all } k, k \neq n} \frac{\langle \psi_k^{(0)} | H^{(1)} | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_k^{(0)}} |\psi_k^{(0)}\rangle$$

$$E_n^{1\text{st}} = E_n^{(0)} + \lambda \langle \psi_n^{(0)} | H^{(1)} | \psi_n^{(0)} \rangle$$

$$E_n^{2\text{nd}} = E_n^{(0)} + \lambda \langle \psi_n^{(0)} | H^{(1)} | \psi_n^{(0)} \rangle + \lambda^2 \sum_{\text{all } k, k \neq n} \frac{|\langle \psi_k^{(0)} | H^{(1)} | \psi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$E(\phi) = \frac{\langle \phi | H | \phi \rangle}{\langle \phi | \phi \rangle} \quad \delta E(\phi) = 0$$

$$H_{kj} = \langle \phi_k | H | \phi_j \rangle \quad H\vec{c} = E\vec{c}$$

## 17 Time-Dependent Perturbation Theory

$$\pi = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx$$

$$\Gamma_{0 \rightarrow n} = \frac{2\pi}{\hbar} |\langle n | H_{\text{perturbation}} | 0 \rangle|^2 \delta(E_n - E_0)$$

## 18 Interaction of Radiation and Matter

$$\vec{E}_{\text{op}} = -\frac{1}{c} \frac{\partial \vec{A}_{\text{op}}}{\partial t} \quad \vec{B}_{\text{op}} = \nabla \times \vec{A}_{\text{op}}$$

## 19 Box Quantization

$$kL = 2\pi n, \quad n = 0, \pm 1, \pm 2, \dots \quad k = \frac{2\pi n}{L} \quad \Delta k_{\text{cell}} = \frac{2\pi}{L} \quad \Delta k_{\text{cell}}^3 = \frac{(2\pi)^3}{V}$$

$$dN_{\text{states}} = g \frac{k^2 dk d\Omega}{(2\pi)^3/V}$$

## 20 Identical Particles

$$|a, b\rangle = \frac{1}{\sqrt{2}} (|1, a; 2, b\rangle \pm |1, b; 2, a\rangle)$$

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}} (\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\psi_a(\vec{r}_2))$$

## 21 Second Quantization

$$[a_i, a_j^\dagger] = \delta_{ij} \quad [a_i, a_j] = 0 \quad [a_i^\dagger, a_j^\dagger] = 0 \quad |N_1, \dots, N_n\rangle = \frac{(a_n^\dagger)^{N_n}}{\sqrt{N_n!}} \dots \frac{(a_1^\dagger)^{N_1}}{\sqrt{N_1!}} |0\rangle$$

$$\{a_i, a_j^\dagger\} = \delta_{ij} \quad \{a_i, a_j\} = 0 \quad \{a_i^\dagger, a_j^\dagger\} = 0 \quad |N_1, \dots, N_n\rangle = (a_n^\dagger)^{N_n} \dots (a_1^\dagger)^{N_1} |0\rangle$$

$$\Psi_s(\vec{r})^\dagger = \sum_{\vec{p}} \frac{e^{-i\vec{p}\cdot\vec{r}}}{\sqrt{V}} a_{\vec{p}s}^\dagger \quad \Psi_s(\vec{r}) = \sum_{\vec{p}} \frac{e^{i\vec{p}\cdot\vec{r}}}{\sqrt{V}} a_{\vec{p}s}$$

$$[\Psi_s(\vec{r}), \Psi_{s'}(\vec{r}')]_{\mp} = 0 \quad [\Psi_s(\vec{r})^\dagger, \Psi_{s'}(\vec{r}')^\dagger]_{\mp} = 0 \quad [\Psi_s(\vec{r}), \Psi_{s'}(\vec{r}')^\dagger]_{\mp} = \delta(\vec{r} - \vec{r}')\delta_{ss'}$$

$$|\vec{r}_1 s_1, \dots, \vec{r}_n s_n\rangle = \frac{1}{\sqrt{n!}} \Psi_{s_n}(\vec{r}_n)^\dagger \dots \Psi_{s_1}(\vec{r}_1)^\dagger |0\rangle$$

$$\Psi_s(\vec{r})^\dagger |\vec{r}_1 s_1, \dots, \vec{r}_n s_n\rangle \sqrt{n+1} |\vec{r}_1 s_1, \dots, \vec{r}_n s_n, \vec{r} s\rangle$$

$$|\Phi\rangle = \int d\vec{r}_1 \dots d\vec{r}_n \Phi(\vec{r}_1, \dots, \vec{r}_n) |\vec{r}_1 s_1, \dots, \vec{r}_n s_n\rangle$$

$$1_n = \sum_{s_1 \dots s_n} \int d\vec{r}_1 \dots d\vec{r}_n |\vec{r}_1 s_1, \dots, \vec{r}_n s_n\rangle \langle \vec{r}_1 s_1, \dots, \vec{r}_n s_n| \quad 1 = |0\rangle \langle 0| + \sum_{n=1}^{\infty} 1_n$$

$$N = \sum_{\vec{p}s} a_{\vec{p}s}^\dagger a_{\vec{p}s} \quad T = \sum_{\vec{p}s} \frac{p^2}{2m} a_{\vec{p}s}^\dagger a_{\vec{p}s}$$

$$\rho_s(\vec{r}) = \Psi_s(\vec{r})^\dagger \Psi_s(\vec{r}) \quad N = \sum_s \int d\vec{r} \rho_s(\vec{r}) \quad T = \frac{1}{2m} \sum_s \int d\vec{r} \nabla \Psi_s(\vec{r})^\dagger \cdot \nabla \Psi_s(\vec{r})$$

$$\vec{j}_s(\vec{r}) = \frac{1}{2im} [\Psi_s(\vec{r})^\dagger \nabla \Psi_s(\vec{r}) - \Psi_s(\vec{r}) \nabla \Psi_s(\vec{r})^\dagger]$$

$$G_s(\vec{r} - \vec{r}') = \frac{3n \sin(x) - x \cos(x)}{2x^3} \quad g_{ss'}(\vec{r} - \vec{r}') = 1 - \delta_{ss'} \frac{G_s(\vec{r} - \vec{r}')^2}{(n/2)^2}$$

$$v_{2\text{nd}} = \frac{1}{2} \sum_{ss'} \int d\vec{r} d\vec{r}' v(\vec{r} - \vec{r}') \Psi_s(\vec{r})^\dagger \Psi_{s'}(\vec{r}')^\dagger \Psi_{s'}(\vec{r}') \Psi_s(\vec{r})$$

$$v_{2\text{nd}} = \frac{1}{2V} \sum_{pp'qq'} \sum_{ss'} v_{\vec{p}-\vec{p}'} \delta_{\vec{p}+\vec{q}, \vec{p}'+\vec{q}'} a_{\vec{p}s}^\dagger a_{\vec{q}s'}^\dagger a_{\vec{q}'s'} a_{\vec{p}'s} \quad v_{\vec{p}-\vec{p}'} = \int d\vec{r} e^{-i(\vec{p}-\vec{p}') \cdot \vec{r}} v(\vec{r})$$


---

## 22 Klein-Gordon Equation

$$E = \sqrt{p^2 c^2 + m^2 c^4} \quad \frac{1}{c^2} \left( i\hbar \frac{\partial}{\partial t} \right)^2 \Psi(\vec{r}, t) = \left[ \left( \frac{\hbar}{i} \nabla \right)^2 + m^2 c^2 \right] \Psi(\vec{r}, t)$$

$$\left[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \left( \frac{mc}{\hbar} \right)^2 \right] \Psi(\vec{r}, t) = 0$$

$$\rho = \frac{i\hbar}{2mc^2} \left( \Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right) \quad \vec{j} = \frac{\hbar}{2im} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*)$$

$$\frac{1}{c^2} \left( i\hbar \frac{\partial}{\partial t} - e\Phi \right)^2 \Psi(\vec{r}, t) = \left[ \left( \frac{\hbar}{i} \nabla - \frac{e}{c} \vec{A} \right)^2 + m^2 c^2 \right] \Psi(\vec{r}, t)$$

$$\Psi_+(\vec{p}, E) = e^{i(\vec{p} \cdot \vec{r} - Et)/\hbar} \quad \Psi_-(\vec{p}, E) = e^{-i(\vec{p} \cdot \vec{r} - Et)/\hbar}$$