Quantum Mechanics

Practice Exam **2011 March 4, Friday**

NAME: **SIGNATURE:**

Instructions: There are 10 multiple-choice questions each worth 2 marks for a total of 20 marks altogether. Choose the **BEST** answer, completion, etc., and darken fully the appropriate circle on the table provided below. Read all responses carefully. **NOTE** long detailed preambles and responses won't depend on hidden keywords: keywords in such preambles and responses are bold-faced capitalized.

There are **THREE** full answer questions each worth 10 marks for a total of 30 marks altogether. Answer them all on the paper provided. It is important that you **SHOW** (**SHOW, SHOW, SHOW**) how you got the answer.

This is a **CLOSED-BOOK** exam. **NO** cheat sheets allowed. An equation sheet is provided. Calculators are permitted for calculations. Cell phones **MUST** be turned off. The test is out of 50 marks altogether.

This a 50-minute test. Remember your name (and write it down on the exam too).

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	6.	O	O	O	O	O
2.	O	O	O	O	O	7.	O	O	O	O	O
3.	O	O	O	O	O	8.	O	O	O	O	O
4.	O	O	O	O	O	9.	O	O	O	O	O
5.	O	O	O	O	O	10.	O	O	O	O	O

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011 qmult 00100 1 4 3 easy deducto-memory: central force

1. A central force is one which always points radially inward or outward from a fixed point which is the center of the central force. The magnitude of central force depends only on:

   a) the angle of the particle.
   b) the vector $\vec{r}$ from the center to the particle.
   c) the radial distance r from the center to the particle.
   d) the magnetic quantum number of the particle.
   e) the uncertainty principle.

   SUGGESTED ANSWER: (c)

   Wikipedia confirms this definition of a central force. Mathemetically, one can write the force

   $$\vec{F}(\vec{r}) = F(r)\hat{r} \; .$$

   But what would a force like

   $$\vec{F}(\vec{r}) = F(\vec{r})\hat{r}$$

   be called. It's not officially a central force since the magnitude depends on direction. But its torque about the center is also zero, and so it conserves angular momentum. Perhaps, such forces are rare, and therefore not much studied.

   Wrong Answers:

   a) Nah.
   b) Exactly wrong.

   Redaction: Jeffery, 2001jan01

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011 qmult 00210 1 1 3 easy memory: separation of variables 2

2. Say you have a differential equation of two independent variables x and y and you want to look for solutions that can be factorized thusly $f(x,y) = g(x)h(y)$. Say then it is possible to reorder equation into the form

   $$\text{LHS}(x) = \text{RHS}(y) \; ,$$

   where LHS stands for left-hand side and RHS for right-hand side. Well LHS is explicitly independent of y and implicitly independent of x:

   $$\frac{\partial \text{LHS}}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial \text{LHS}}{\partial x} = \frac{\partial \text{RHS}}{\partial x} = 0 \; .$$

   Thus, LHS is equal to a constant C and necessarily RHS is equal to the same constant C which is called the constant of separation (e.g., Arf-383). The solutions for $g(x)$ and $h(y)$ can be found separately and are related to each other through C. The solutions for $f(x,y)$ that cannot be factorized are not obtained, of course, by the described procedured. However, if one obtains complete sets of $g(x)$ and $h(y)$ solutions for the x-y region of interest, then any solution $f(x,y)$ can be constructed at least to within some approximation (Arf-443). Thus, the generalization of the described procedure is very general and powerful. It is called:

a) separation of the left- and right-hand sides. b) partitioning.
c) separation of the variables. d) solution factorization.
e) the King Lear method.

SUGGESTED ANSWER: (c)

In quantum mechanics, it is a postulate that a complete set of eigenstates exists for any observable and that any physical state defined for the same space as the observable can be expanded exactly in those eigenstates in principle. The whole paradigm of quantum mechanics relies on this postulate—and quantum mechanics has never failed.

Wrong answers:

d) Seems reasonable.
e) Metaphorical names due turn up in physics like the Monte Carlo method (named after a famous casino in Monaco) and the Urca process (named after a casino in Rio de Janeiro). One sometimes gets the feeling that theoretical physicists spend a lot of time in casinos. I used to wander through them all the time in my Vegas years.

Redaction: Jeffery, 2008jan01

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011 qmult 00300 1 4 2 easy deducto-memory: relative/cm reduction

3. "Let's play *Jeopardy*! For \$100, the answer is: By writing the two-body Schrödinger equation in relative/center-of-mass coordinates."

How do you ____________, Alex?

a) reduce a **ONE-BODY** problem to a **TWO-BODY** problem
b) reduce a **TWO-BODY** problem to a **ONE-BODY** problem
c) solve a one-dimensional infinite square well problem
d) solve for the simple harmonic oscillator eigenvalues
e) reduce a **TWO-BODY** problem to a **TWO-BODY** problem

SUGGESTED ANSWER: (b)

Wrong answers:

e) Seems a bit pointless.

Redaction: Jeffery, 2001jan01

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011 qmult 00310 1 4 4 easy deducto-memory: reduced mass

4. The formula for the reduced mass m for two-body system (with bodies labeled 1 and 2) is:

a) $m = m_1 m_2$. b) $m = \dfrac{1}{m_1 m_2}$. c) $m = \dfrac{m_1 + m_2}{m_1 m_2}$. d) $m = \dfrac{m_1 m_2}{m_1 + m_2}$.
e) $m = \dfrac{1}{m_1}$.

SUGGESTED ANSWER: (d)

Wrong Answers:

a) Dimensionally wrong.
b) Dimensionally wrong.
c) Dimensionally wrong.
e) Dimensionally wrong and it only refers to one mass.

Redaction: Jeffery, 2001jan01

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011 qmult 00400 1 4 2 easy deducto memory: spherical harmonics 1

5. The eigensolutions of the angular part of the Hamiltonian for the central force problem are the:

   a) linear harmonics. b) spherical harmonics. c) square harmonics.
   d) Pythagorean harmonics. e) Galilean harmonics.

   SUGGESTED ANSWER: (b)

   Wrong Answers:

   d) Legend has it that Pythagoras discovered the harmonic properties of strings.
   e) Vincenzo Galileo, father of the other Galileo, was a scientist too and studied music scientifically.

   Redaction: Jeffery, 2001jan01

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011 qmult 00420 1 4 3 easy deducto memory: spherical harmonic Y00

6. Just about the only spherical harmonic that people remember—and they really should remember it too—is $Y_{00} =$:

   a) $e^{im\phi}$. b) r^2. c) $\dfrac{1}{\sqrt{4\pi}}$. d) θ^2. e) $2a^{-3/2}e^{-r/a}$.

   SUGGESTED ANSWER: (c)

   Wrong Answers:

   a) This is the general azimuthal component of the spherical harmonics: $m = 0, \pm 1, \pm 2, \ldots$
   b) This is radial and it's not normalizable.
   d) Except for Y_{00} itself, the spherical harmonics are all combinations of sinusoidal functions of the θ and ϕ.
   e) This is the R_{10} hydrogenic radial wave function where a is the scale radius

   $$a = a_0 \frac{m_e}{m} \frac{1}{Z} ,$$

   where m_e is the electron mass, m is the reduced mass, Z is the number of unit charges of the central particle, and a_0 is the Bohr radius (Gr2005-137). The Bohr radius in MKS units is given by

   $$a_0 = \frac{\hbar^2}{m_e[e^2/(4\pi\varepsilon_0)]} = \frac{\lambda_{\rm C}}{2\pi} \frac{1}{\alpha} = 0.52917720859(36)\,\text{Å} ,$$

where e is the elementary charge, $\lambda_C = \hbar/(m_e c)$ is the Compton wavelength, and $\alpha \approx /137$ is the fine structure constant.

Redaction: Jeffery, 2001jan01

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011 qmult 00500 1 4 2 easy deducto-memory: spdf designations

7. Conventionally, the spherical harmonic eigenstates for angular momentum quantum numbers

$$\ell = 0, 1, 2, 3, 4, ...$$

are designated by:

a) a, b, c, d, e, etc.
b) s, p, d, f, and then alphabetically following f: i.e., g, h, etc.
c) x, y, z, xx, yy, zz, xxx, etc.
d) A, C, B, D, E, etc.
e) $@%&*!!

SUGGESTED ANSWER: (b)

Wrong Answers:

a) This is the way it should be, not the way it is.
e) Only in Tasmanian devilish.

Redaction: Jeffery, 2001jan01

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019 qmult 00110 1 1 3 easy memory: exchange degeneracy and symmetrization principle

8. As strange as the symmetrization principle seems at first, quantum mechanics would be inconsistent without it since then you could create infinitely many physically distinct states by superpositions of the same state. This inconsistency is called the:

a) symmetrization paradox. b) symmetrization degeneracy.
c) exhcange degeneracy. d) baffling degeneracy. e) baffling paradox.

SUGGESTED ANSWER: (c)

Wrong answers:

e) By Gad, Holmes, baffled again.

Redaction: Jeffery, 2008jan01

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020 qmult 00100 1 1 1 easy memory: atom defined

9. An atom is a stable bound system of electrons and:

a) a single nucleus. b) two nuclei. c) three nuclei. d) a single quark.
e) two quarks.

SUGGESTED ANSWER: (a)

Wrong Answers:

b) This is a diatomic molecule.

Redaction: Jeffery, 2001jan01

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020 qmult 01000 1 4 1 easy deducto-memory: central potential

10. "Let's play *Jeopardy*! For $100, the answer is: A favored approximation in the simpler solutions for the electronic structure of atoms in quantum mechanics."

    What is the ____________, Alex?

    a) central potential approximation b) non-central potential approximation
    c) grand central approximation d) atom-approximated-as-molecule method
    e) electrons-as-bosons approximation

    SUGGESTED ANSWER: (a)

    Wrong answers:

    d) Doesn't seem to likely to work.

    e) Off hand I can't think of a poorer approximation.

    Redaction: Jeffery, 2001jan01

011 qfull 00100 2 5 0 moderate thinking: 2-body reduced to 1-body problem

Extra keywords: (Gr-178:5.1)

11. The 2-body time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V\psi = E_{\rm total}\psi \ .$$

If the V depends only on $\vec{r} = \vec{r}_2 - r_1$ (the relative vector), then the problem can be separated into two problems: a relative problem 1-body equivalent problem and a center-of-mass 1-body equivalent problem. The center of mass vector is

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} \ ,$$

where $M = m_1 + m_2$.

a) Determine the expressions for $\vec{r}_1$ and $\vec{r}_2$ in terms of $\vec{R}$ and $\vec{r}$.

b) Determine the expressions for ∇_1^2 and ∇_2^2 in terms of $\nabla_{\rm cm}^2$ (the center-of-mass Laplacian operator) and ∇^2 (the relative Laplacian operator). Then re-express the kinetic operator

$$-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2$$

in terms of $\nabla_{\rm cm}^2$ and ∇^2. **HINTS:** The x, y, and z direction components of vectors can all be treated separately and identically since x components of $\vec{R}$ and $\vec{r}$) (i.e., X and x) depend only on x_1 and x_2, etc. You can introduce a reduced mass to make the transformed kinetic energy operator simpler.

c) Now separate the 2-body Schrödinger equation assuming $V = V(\vec{r}\,) + V_{\rm cm}(\vec{R}\,)$. What are the solutions of the center-of-mass problem if $V_{\rm cm}(\vec{R}) = 0$? How would you interpret the solutions of the relative problem? **HINT:** I'm only looking for a short answer to the interpretation question.

SUGGESTED ANSWER:

a) Well substituting for $\vec{r}_2$ using the relative expression gives

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} = \vec{r}_1 + \frac{m_2}{M}\vec{r} \ ,$$

and so

$$\vec{r}_1 = \vec{R} - \frac{m_2}{M}\vec{r}$$

and

$$\vec{r}_2 = \vec{R} + \frac{m_1}{M}\vec{r} \ .$$

b) Well

$$\frac{\partial}{\partial x_{\binom{1}{2}}} = \frac{\partial X}{\partial x_{\binom{1}{2}}}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_{\binom{1}{2}}}\frac{\partial}{\partial x} = \frac{m_{\binom{1}{2}}}{M}\frac{\partial}{\partial X} \mp \frac{\partial}{\partial x} \ .$$

Thus

$$\frac{\partial^2}{\partial x^2_{\binom{1}{2}}} = \left[\frac{m_{\binom{1}{2}}}{M}\right]^2 \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial x^2} \mp 2\frac{m_{\binom{1}{2}}}{M}\frac{\partial}{\partial X}\frac{\partial}{\partial x} \ .$$

The other coordinate directions are treated identically *mutatis mutandis*. We then find that

$$-\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 = -\frac{\hbar^2}{M}\nabla^2_{\rm cm} - \frac{\hbar^2}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\nabla^2 = -\frac{\hbar^2}{2M}\nabla^2_{\rm cm} - \frac{\hbar^2}{2m}\nabla^2 \ ,$$

where define the reduced mass by

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2} \qquad \text{or} \qquad m = \frac{m_1 m_2}{m_1 + m_2} \ .$$

The symbol μ is often used for reduced mass, but I think that is unnecessarily obscure myself. Note

$$\frac{1}{m} \geq \frac{1}{m_i} \ ,$$

where i stands for 1 or 2 and equality only holds if the dropped mass is infinite. Thus

$$m \leq m_i \qquad \text{or} \qquad m \leq \min(m_1, m_2) \ .$$

If $m_1 = m_2$, then

$$m = \frac{m_1}{2} \ .$$

If $m_1/m_2 < 1$, then one can expand the reduced mass expression in the power series (e.g., Ar-238)

$$m = \frac{m_1}{1 + m_1/m_2} = m_1 \sum_k (-1)^k \left(\frac{m_1}{m_2}\right)^k \approx m_1\left(1 - \frac{m_1}{m_2}\right) \ ,$$

where the last expression holds for $m_1/m_2 << 1$.

c) We make the anzatz that we can set

$$\psi_{\rm total}(\vec{r}_1, \vec{r}_2) = \psi_{\rm cm}(\vec{R})\psi(\vec{r}) \ .$$

The Schrödinger equation can then be written at once as

$$-\frac{\hbar^2}{2M}\frac{\nabla^2_{\rm cm}\psi_{\rm cm}(\vec{R})}{\psi(\vec{R})} - \frac{\hbar^2}{2m}\frac{\nabla^2\psi(\vec{r})}{\psi(\vec{r})} + V(\vec{r}) + V_{\rm cm}(\vec{R}) = E_{\rm total} \ .$$

For the differential equation to hold for all $\vec{R}$ and $\vec{r}$, we must have

$$-\frac{\hbar^2}{2M}\frac{\nabla^2_{\rm cm}\psi_{\rm cm}(\vec{R})}{\psi(\vec{R})} + V_{\rm cm}(\vec{R}) = E_{\rm cm} \qquad \text{and} \qquad -\frac{\hbar^2}{2m}\frac{\nabla^2\psi(\vec{r})}{\psi(\vec{r})} + V(\vec{r}) = E \ ,$$

where $E_{\rm cm}$ and E are constants of separation that sum to $E_{\rm total}$. We then have two 1-body Schrödinger equation problems:

$$\frac{\hbar^2}{2M}\nabla^2_{\rm cm}\psi_{\rm cm}(\vec{R}\,) + V_{\rm cm}(\vec{R}\,)\psi(\vec{R}\,) = E_{\rm cm}\psi(\vec{R}\,)$$

and

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}\,) + V(\vec{r}\,)\psi(\vec{r}\,) = E\psi(\vec{r}\,) \ .$$

The center-of-mass problem is just the free particle Schrödinger equation if $V_{\rm cm}(\vec{R}\,) = 0$.

The relative problem is just the central force Schrödinger equation. The wave functions that solve the relative problem give the position of particle 2 relative to particle 1. Of course, one can get the reverse by a change of sign of the relative vector. The relative problem is not in an inertial frame, but it can be treated as if it were because it is a lawful equation derived from the Schrödinger equation. I always think that the reduced mass must account for the non-inertiality, but no textbook I know of spits out that notion.

The classical 2-body problem with only a central force separates in analogous way to the quantum 2-body problem. For example, the identical formula of the reduced mass appears.

Redaction: Jeffery, 2001jan01

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019 qfull 02000 2 5 0 moderate thinking: symmetrization of 4 orthonormal single-particle states

12. Say $|ai\rangle$ and $|bi\rangle$ are **ORTHONORMAL** single-particle states, where i is a particle label. The label can be thought of as labeling the coordinates to be integrated or summed over in an inner product: see below. The symbolic combination of such states for two particles, one in a and one in b is

$$|12\rangle = |a1\rangle|b2\rangle \ ,$$

where 1 and 2 are particle labels. This combination is actually a tensor product, but let's not worry about that now. The inner product of such a combined state is written

$$\langle 12|12\rangle = \langle a1|a1\rangle\langle b2|b2\rangle \ .$$

If one expanded the inner product in the position and spinor representation assuming the wave function and spinor parts can be separated (which in general is not the case),

$$\langle 12|12\rangle = \left[\int \psi_a(x_1)^*\psi_a(x_1)\,dx_1\,(\,c^*_{a+}\quad c^*_{a-}\,)_1 \begin{pmatrix} c_{a+} \\ c_{a-} \end{pmatrix}_1\right] \times \left[\int \psi_b(x_2)^*\psi_b(x_2)\,dx_2\,(\,c^*_{b+}\quad c^*_{b-}\,)_2 \begin{pmatrix} c_{b+} \\ c_{b-} \end{pmatrix}_2\right] \ .$$

A lot of conventions go into the last expression: don't worry too much about them.

a) Let particles 1 and 2 be **NON-**identical particles. What are the two simplest and most obvious normalized 2-particle states that can be constructed from states a and b? What happens if $a = b$ (i.e., the two single-particle states are only one state actually)?

b) Say particles 1 and 2 are identical bosons or identical fermions. What is the only normalized physical 2-particle state that can be constructed in either case allowing for the possibility that $a = b$ (i.e., the two single-particle states are only one state actually)? What happens if $a = b$ for fermions?

SUGGESTED ANSWER:

a) Behold:

$$|12\rangle = |a1\rangle|b2\rangle \qquad \text{and} \qquad |21\rangle = |a2\rangle|b1\rangle$$

which are just the allowed product states. More complicated states can be constructed if the particles are in mixtures of the two states just given. If $a = b$, then one can construct only one state

$$|12\rangle = |a1\rangle|a2\rangle \ .$$

b) Behold:

$$|12\rangle = \frac{1}{\sqrt{2(1+\delta_{ab})}}\left(|a1\rangle|b2\rangle \pm |a2\rangle|b1\rangle\right) \ ,$$

where the upper case is for bosons and the lower case is for fermions.

I don't think there are any other possible physical states that can be constructed. There are only two possible product states. And only a symmetrized mixed state is allowed.

The Kronecker delta allows for the case that $a = b$ for bosons. Obviously, we never had to symmetrize at all for bosons if $a = b$. If $a = b$ for fermions, the state is null and thus no physical state can be constructed in this case. The nullness is a manifestation of the Pauli exclusion principle (a corollary of the symmetrization postulate): two fermions cannot be in found in single-particle state (as specified by a C.S.C.O.: i.e., a complete set of commuting observables (CT-143)). "Cannot be found" has to be interpreted as the probability for two fermions in the same single-particle state is zero or that the probability of collapsing the wave function to having two fermions in the same single-particle stat is zero. So if $a = b$ for fermions, then physical symmetrized state cannot be created from product states.

Redaction: Jeffery, 2001jan01

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019 qfull 02300 3 5 0 tough thinking: exchange force

Extra keywords: (Gr-182)

13. Say we have orthonormal single-particle states $|a\rangle$ and $|b\rangle$. If we have distinct particles 1 and 2 in, respectively, $|a\rangle$ and $|b\rangle$, the net state is

$$|a1, b2\rangle = |a1\rangle|b2\rangle \ .$$

Of course, each of particles 1 and 2 could be in linear combinations of the two states if the states physically allowed the distinct particles to be in either one. In that case the linear combined state would be a four term state. But we have no interest in pursuing that digression at the moment.

Now two identical particles in states $|a\rangle$ and $|b\rangle$ have no choice, but to be in a symmetrized state by the symmetry postulate:

$$|1,2\rangle = \frac{1}{\sqrt{2(1+\delta_{ab})}}\left(|a1,b2\rangle \pm |a2,b1\rangle\right) ,$$

where the upper case is for identical bosons and the lower case for identical fermions. If the two states are actually the same state $|a\rangle$, then the state for bosons reduces to

$$|1,2\rangle = |a1,a2\rangle$$

and for fermions the state reduces to the null state $|0\rangle$ which is not a physical state, and thus the Pauli exclusion principle is incorporated in the state expression.

Note products of kets are actually tensor products (CT-154). In taking scalar products, the bras with index i (e.g., 1 or 2 above) act on the kets of index i. For example, for the state $|1a,2b\rangle = |a1\rangle|a2\rangle$ the norm squared is

$$\langle a1,b2|a1,b2\rangle = \langle a1|a1\rangle\langle a2|a2\rangle .$$

The fact that identical particles must be in symmetrized states means that their wave functions will be more or less clumped depending on whether they are bosons or fermions than if they could be fitted into simple product states like distinct particles. We are not bothering with the complication of spin for this problem. We will assume that all the particles are in the same spin state: e.g., they are all in the spin up state.

The clumping/declumping effect is called the **EXCHANGE FORCE**. Obviously, it is not really a force, but rather a result of the symmetrization principle requirements on physical states for identical particles. Still for some practical purposes one can certainly consider it as force. In this problem, we investigate the effect of the **EXCHANGE FORCE**.

a) Expand $\langle\Delta x^2\rangle = \langle(x_1 - x_2)^2\rangle$ into three terms that can be evaluated individually.

b) For the given two-particle state for **DISTINCT PARTICLES** $|a1,b2\rangle = |a1\rangle|b2\rangle$, formally show that

$$\langle\Delta x^2\rangle = \langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b ,$$

where the particle labels can be dropped from the single-particle state expectation values, but these values must be identified by the single-particle state that they are for: i.e., for states $|a\rangle$ and $|b\rangle$. What happens in the case that $|a\rangle = |b\rangle$? **HINT:** Remember that variance is defined by

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 .$$

c) There is an identity that is needed for part (d) and is useful in many other contexts. Say $|\alpha\rangle$ and $|\beta\rangle$ are general states (e.g., they could be one-particle or two-particle states). Say that

$$|\Psi\rangle = c_\alpha|\alpha\rangle + c_\beta|\beta\rangle$$

and we have general observable Q. We have the identity

$$\langle\Psi|\Psi\rangle = |c_\alpha|^2\langle\alpha|\alpha\rangle + |c_\beta|^2\langle\beta|\beta\rangle + 2\text{Re}(c_\alpha^* c_\beta\langle\alpha|Q|\beta\rangle) \ .$$

Prove the identity.

d) For the given two-particle state for **IDENTICAL PARTICLES**

$$|1,2\rangle = \frac{1}{\sqrt{2(1+\delta_{ab})}}\left(|a1,b2\rangle \pm |a2,b1\rangle\right) \ ,$$

determine $\langle\Delta x^2\rangle$ for identical bosons and fermions. What happens in the case that $|a\rangle = |b\rangle$? **HINT:** Recall that

$$\langle a|b\rangle = \delta_{ab} \ ,$$

since the states are orthonormal.

SUGGESTED ANSWER:

a) Behold:

$$\langle\Delta x^2\rangle = \langle(x_1 - x_2)^2\rangle = \langle x_1^2\rangle + \langle x_2^2\rangle - 2\langle x_1 x_2\rangle \ .$$

b) For distinct particles,

$$\begin{aligned}
\langle x_1^2\rangle &= \langle a1,b2|x_1^2|a1,b2\rangle = \langle a1|x_1^2|a1\rangle\langle b2|b2\rangle = \langle x^2\rangle_a \ , \\
\langle x_2^2\rangle &= \langle a1,b2|x_2^2|a1,b2\rangle = \langle a1|a1\rangle\langle b2|x_2^2|b2\rangle = \langle x^2\rangle_b \ , \\
\langle x_1 x_2\rangle &= \langle a1,b2|x_1 x_2|a1,b2\rangle = \langle a1|x_1|a1\rangle\langle b2|x_2|b2\rangle = \langle x\rangle_a\langle x\rangle_b \ ,
\end{aligned}$$

where the particle labels 1 and 2 are irrelevant to the single-particle state expectation values, and so we have dropped them. Thus,

$$\langle\Delta x^2\rangle = \langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b \ ,$$

In the case that $|a\rangle = |b\rangle$,

$$\langle\Delta x^2\rangle = 2\langle x^2\rangle_a - 2\langle x\rangle_a^2 = 2\sigma_a^2 \ .$$

So the relative variance is just twice the absolute variance of a single particle in this case.

c) Begorra:

$$\begin{aligned}
\langle\Psi|Q|\Psi\rangle &= |c_\alpha|^2\langle\alpha|Q|\alpha\rangle + |c_\beta|^2\langle\beta|Q|\beta\rangle + c_\alpha^* c_\beta\langle\alpha|Q|\beta\rangle + c_\alpha c_\beta^*\langle\beta|Q|\alpha\rangle \\
&= |c_\alpha|^2\langle\alpha|\alpha\rangle + |c_\beta|^2\langle\beta|\beta\rangle + 2\text{Re}(c_\alpha^* c_\beta\langle\alpha|Q|\beta\rangle) \ ,
\end{aligned}$$

where we have used the definition of the Hermitian conjugate and the Hermiticity of Q: i.e., we have used

$$\langle\beta|Q|\alpha\rangle = \langle\alpha|Q^\dagger|\beta\rangle^* = \langle\alpha|Q|\beta\rangle^* \ .$$

Thus, we have proven the identity

$$\langle\Psi|\Psi\rangle = |c_\alpha|^2\langle\alpha|\alpha\rangle + |c_\beta|^2\langle\beta|\beta\rangle + 2\mathrm{Re}(c_\alpha^* c_\beta\langle\alpha|Q|\beta\rangle) \ .$$

d) For identical particles and making use of the identity, we find

$$\begin{aligned}
\langle x_1^2\rangle &= \frac{1}{2(1+\delta_{ab})}[\langle a1|x_1^2|a1\rangle\langle b2|b2\rangle + \langle a2|a2\rangle\langle b1|x_1^2|b1\rangle \pm 2\mathrm{Re}(\langle a1|x_1^2|b1\rangle\langle b2|a2\rangle)] \\
&= \frac{1}{2(1+\delta_{ab})}(\langle x^2\rangle_a + \langle x^2\rangle_b \pm 2\langle x^2\rangle_a\delta_{ab}) \\
\langle x_2^2\rangle &= \frac{1}{2(1+\delta_{ab})}[\langle a1|a1\rangle\langle b2|x_2^2|b2\rangle + \langle a2|x_2^2|a2\rangle\langle b1|b1\rangle \pm 2\mathrm{Re}(\langle a1|b1\rangle\langle b2|x_2^2|a2\rangle)] \\
&= \frac{1}{2(1+\delta_{ab})}(\langle x^2\rangle_a + \langle x^2\rangle_b \pm 2\langle x^2\rangle_a\delta_{ab}) \\
\langle x_1 x_2\rangle &= \frac{1}{2(1+\delta_{ab})}[\langle a1|x_1|a1\rangle\langle b2|x_2|b2\rangle + \langle a2|x_2|a2\rangle\langle b1|x_1|b1\rangle \pm 2\mathrm{Re}(\langle a1|x_1|b1\rangle\langle b2|x_2|a2\rangle)] \\
&= \frac{1}{2(1+\delta_{ab})}(2\langle x\rangle_a\langle x\rangle_b \pm 2|\langle x\rangle_{ab}|^2) \\
&= \frac{1}{(1+\delta_{ab})}(\langle x\rangle_a\langle x\rangle_b \pm |\langle x\rangle_{ab}|^2) \ .
\end{aligned}$$

Thus,

$$\langle\Delta x^2\rangle = \frac{1}{(1+\delta_{ab})}(\langle x^2\rangle_a + \langle x^2\rangle_b \pm 2\langle x^2\rangle_a\delta_{ab} - 2\langle x\rangle_a\langle x\rangle_b \mp 2|\langle x\rangle_{ab}|^2) \ .$$

In the case of $|a\rangle \neq |b\rangle$, we get

$$\langle\Delta x^2\rangle = \langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b \mp 2|\langle x\rangle_{ab}|^2 \ .$$

Note for bosons the relative variance is smaller than for distinct particles by the term $-2|\langle x\rangle_{ab}|^2$ and for fermions it is larger by the term $2|\langle x\rangle_{ab}|^2$. Thus symmetrization clumps identical bosons more and identical fermions less that for the counterpart distinct particle case. This amounts to the exchange force.

If the $|a\rangle$ and $|b\rangle$ states don't spatially overlap, then $\langle b|x|a\rangle = 0$ and the identical-particle result is the same as distinct particle result.

In the case of $|a\rangle = |b\rangle$, we get

$$\langle\Delta x^2\rangle = \langle x^2\rangle_a \pm \langle x^2\rangle_a - \langle x\rangle_a^2 \mp \langle x\rangle_a^2 \ .$$

For bosons, we have

$$\langle\Delta x^2\rangle = 2\langle x^2\rangle_a - 2\langle x\rangle_a^2 = 2\sigma_a^2$$

which is the same result as for distinct particles obtained in the part (b) answer. For fermions, we have

$$\langle\Delta x^2\rangle = 0$$

which is what you would expect for a null state.

Redaction: Jeffery, 2001jan01

Appendix 2 Quantum Mechanics Equation Sheet

Note: This equation sheet is intended for students writing tests or reviewing material. Therefore it neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things.

1 Constants not to High Accuracy

Constant Name	Symbol	Derived from CODATA 1998
Bohr radius	$a_{\text{Bohr}} = \dfrac{\lambda_{\text{Compton}}}{2\pi\alpha}$	$= 0.529\,\text{Å}$
Boltzmann's constant	k	$= 0.8617 \times 10^{-6}\,\text{eV}\,\text{K}^{-1}$ $= 1.381 \times 10^{-16}\,\text{erg}\,\text{K}^{-1}$
Compton wavelength	$\lambda_{\text{Compton}} = \dfrac{h}{m_e c}$	$= 0.0246\,\text{Å}$
Electron rest energy	$m_e c^2$	$= 5.11 \times 10^5\,\text{eV}$
Elementary charge squared	e^2	$= 14.40\,\text{eV}\,\text{Å}$
Fine Structure constant	$\alpha = \dfrac{e^2}{\hbar c}$	$= 1/137.036$
Kinetic energy coefficient	$\dfrac{\hbar^2}{2m_e}$	$= 3.81\,\text{eV}\,\text{Å}^2$
	$\dfrac{\hbar^2}{m_e}$	$= 7.62\,\text{eV}\,\text{Å}^2$
Planck's constant	h	$= 4.15 \times 10^{-15}\,\text{eV}$
Planck's h-bar	$\hbar$	$= 6.58 \times 10^{-16}\,\text{eV}$
	hc	$= 12398.42\,\text{eV}\,\text{Å}$
	$\hbar c$	$= 1973.27\,\text{eV}\,\text{Å}$
Rydberg Energy	$E_{\text{Ryd}} = \dfrac{1}{2} m_e c^2 \alpha^2$	$= 13.606\,\text{eV}$

2 Some Useful Formulae

Leibniz's formula
$$\frac{d^n(fg)}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k f}{dx^k} \frac{d^{n-k} g}{dx^{n-k}}$$

Normalized Gaussian
$$P = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - \langle x \rangle)^2}{2\sigma^2}\right]$$

3 Schrödinger's Equation

$$H\Psi(x,t) = \left[\frac{p^2}{2m} + V(x)\right]\Psi(x,t) = i\hbar \frac{\partial \Psi(x,t)}{\partial t}$$

$$H\psi(x) = \left[\frac{p^2}{2m} + V(x)\right]\psi(x) = E\psi(x)$$

$$H\Psi(\vec{r},t) = \left[\frac{p^2}{2m} + V(\vec{r})\right]\Psi(\vec{r},t) = i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t} \qquad H|\Psi\rangle = i\hbar\frac{\partial}{\partial t}|\Psi\rangle$$

$$H\psi(\vec{r}) = \left[\frac{p^2}{2m} + V(\vec{r})\right]\psi(\vec{r}) = E\psi(\vec{r}) \qquad H|\psi\rangle = E|\psi\rangle$$

---

4 **Some Operators**

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad p^2 = -\hbar^2\frac{\partial^2}{\partial x^2}$$

$$H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$p = \frac{\hbar}{i}\nabla \qquad p^2 = -\hbar^2\nabla^2$$

$$H = \frac{p^2}{2m} + V(\vec{r}) = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})$$

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\theta}\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

---

5 **Kronecker Delta and Levi-Civita Symbol**

$$\delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & \text{otherwise} \end{cases} \qquad \varepsilon_{ijk} = \begin{cases} 1, & ijk \text{ cyclic;} \\ -1, & ijk \text{ anticyclic;} \\ 0, & \text{if two indices the same.} \end{cases}$$

$$\varepsilon_{ijk}\varepsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell} \qquad \text{(Einstein summation on } i\text{)}$$

---

6 Time Evolution Formulae

General $$\frac{d\langle A\rangle}{dt} = \left\langle \frac{\partial A}{\partial t} \right\rangle + \frac{1}{\hbar}\langle i[H(t), A]\rangle$$

Ehrenfest's Theorem $$\frac{d\langle \vec{r}\rangle}{dt} = \frac{1}{m}\langle \vec{p}\rangle \qquad \text{and} \qquad \frac{d\langle \vec{p}\rangle}{dt} = -\langle \nabla V(\vec{r})\rangle$$

$$|\Psi(t)\rangle = \sum_j c_j(0) e^{-iE_j t/\hbar} |\phi_j\rangle$$

---

7 Simple Harmonic Oscillator (SHO) Formulae

$$V(x) = \frac{1}{2}m\omega^2 x^2 \qquad \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right)\psi = E\psi$$

$$\beta = \sqrt{\frac{m\omega}{\hbar}} \qquad \psi_n(x) = \frac{\beta^{1/2}}{\pi^{1/4}}\frac{1}{\sqrt{2^n n!}} H_n(\beta x) e^{-\beta^2 x^2/2} \qquad E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

$$H_0(\beta x) = H_0(\xi) = 1 \qquad H_1(\beta x) = H_1(\xi) = 2\xi$$
$$H_2(\beta x) = H_2(\xi) = 4\xi^2 - 2 \qquad H_3(\beta x) = H_3(\xi) = 8\xi^3 - 12\xi$$

---

8 Position, Momentum, and Wavenumber Representations

$$p = \hbar k \qquad E_{\text{kinetic}} = E_T = \frac{\hbar^2 k^2}{2m}$$

$$|\Psi(p,t)|^2\, dp = |\Psi(k,t)|^2\, dk \qquad \Psi(p,t) = \frac{\Psi(k,t)}{\sqrt{\hbar}}$$

$$x_{\text{op}} = x \qquad p_{\text{op}} = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad Q\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}, t\right)$$ position representation

$$x_{\text{op}} = -\frac{\hbar}{i}\frac{\partial}{\partial p} \qquad p_{\text{op}} = p \qquad Q\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}, p, t\right)$$ momentum representation

$$\delta(x)=\int_{-\infty}^{\infty}\frac{e^{ipx/\hbar}}{2\pi\hbar}\,dp \qquad \delta(x)=\int_{-\infty}^{\infty}\frac{e^{ikx}}{2\pi}\,dk$$

$$\Psi(x,t)=\int_{-\infty}^{\infty}\Psi(p,t)\frac{e^{ipx/\hbar}}{(2\pi\hbar)^{1/2}}\,dp \qquad \Psi(x,t)=\int_{-\infty}^{\infty}\Psi(k,t)\frac{e^{ikx}}{(2\pi)^{1/2}}\,dk$$

$$\Psi(p,t)=\int_{-\infty}^{\infty}\Psi(x,t)\frac{e^{-ipx/\hbar}}{(2\pi\hbar)^{1/2}}\,dx \qquad \Psi(k,t)=\int_{-\infty}^{\infty}\Psi(x,t)\frac{e^{-ikx}}{(2\pi)^{1/2}}\,dx$$

$$\Psi(\vec r,t)=\int_{\text{all space}}\Psi(\vec p,t)\frac{e^{i\vec p\cdot\vec r/\hbar}}{(2\pi\hbar)^{3/2}}\,d^3p \qquad \Psi(\vec r,t)=\int_{\text{all space}}\Psi(\vec k,t)\frac{e^{i\vec k\cdot\vec r}}{(2\pi)^{3/2}}\,d^3k$$

$$\Psi(\vec p,t)=\int_{\text{all space}}\Psi(\vec r,t)\frac{e^{-i\vec p\cdot\vec r/\hbar}}{(2\pi\hbar)^{3/2}}\,d^3r \qquad \Psi(\vec k,t)=\int_{\text{all space}}\Psi(\vec r,t)\frac{e^{-i\vec k\cdot\vec r}}{(2\pi)^{3/2}}\,d^3r$$

9 Commutator Formulae

$$[A,BC]=[A,B]C+B[A,C] \qquad \left[\sum_i a_iA_i,\sum_j b_jB_j\right]=\sum_{i,j}a_ib_j[A_i,b_j]$$

$$\text{if} \qquad [B,[A,B]]=0 \qquad \text{then} \qquad [A,F(B)]=[A,B]F'(B)$$

$$[x,p]=i\hbar \qquad [x,f(p)]=i\hbar f'(p) \qquad [p,g(x)]=-i\hbar g'(x)$$

$$[a,a^\dagger]=1 \qquad [N,a]=-a \qquad [N,a^\dagger]=a^\dagger$$

10 Uncertainty Relations and Inequalities

$$\sigma_x\sigma_p=\Delta x\Delta p\ge\frac{\hbar}{2} \qquad \sigma_Q\sigma_Q=\Delta Q\Delta R\ge\frac{1}{2}\,|\langle i[Q,R]\rangle|$$

$$\sigma_H\Delta t_{\text{scale time}}=\Delta E\Delta t_{\text{scale time}}\ge\frac{\hbar}{2}$$

11 Probability Amplitudes and Probabilities

$$\Psi(x,t) = \langle x|\Psi(t)\rangle \qquad P(dx) = |\Psi(x,t)|^2\,dx \qquad c_i(t) = \langle \phi_i|\Psi(t)\rangle \qquad P(i) = |c_i(t)|^2$$

12 Spherical Harmonics

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}} \qquad Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2}\cos(\theta) \qquad Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin(\theta)e^{\pm i\phi}$$

$$L^2 Y_{\ell m} = \ell(\ell+1)\hbar^2 Y_{\ell m} \qquad L_z Y_{\ell m} = m\hbar Y_{\ell m} \qquad |m| \le \ell \qquad m = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

0	1	2	3	4	5	6	...
s	p	d	f	g	h	i	...

13 Hydrogenic Atom

$$\psi_{n\ell m} = R_{n\ell}(r)Y_{\ell m}(\theta,\phi) \qquad \ell \le n-1 \qquad \ell = 0,1,2,\ldots,n-1$$

$$a_z = \frac{a}{Z}\left(\frac{m_e}{m_{\text{reduced}}}\right) \qquad a_0 = \frac{\hbar}{m_e c\alpha} = \frac{\lambda_{\text{C}}}{2\pi\alpha} \qquad \alpha = \frac{e^2}{\hbar c}$$

$$R_{10} = 2a_Z^{-3/2}e^{-r/a_Z} \qquad R_{20} = \frac{1}{\sqrt{2}}a_Z^{-3/2}\left(1-\frac{1}{2}\frac{r}{a_Z}\right)e^{-r/(2a_Z)}$$

$$R_{21} = \frac{1}{\sqrt{24}}a_Z^{-3/2}\frac{r}{a_Z}e^{-r/(2a_Z)}$$

$$R_{n\ell} = -\left\{\left(\frac{2}{na_Z}\right)^3\frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right\}^{1/2} e^{-\rho/2}\rho^\ell L_{n+\ell}^{2\ell+1}(\rho) \qquad \rho = \frac{2r}{nr_Z}$$

$$L_q(x) = e^x\left(\frac{d}{dx}\right)^q\left(e^{-x}x^q\right)$$ Rodrigues's formula for the Laguerre polynomials

$$L_q^j(x) = \left(\frac{d}{dx}\right)^j L_q(x)$$ Associated Laguerre polynomials

$$\langle r\rangle_{n\ell m} = \frac{a_Z}{2}\left[3n^2 - \ell(\ell+1)\right]$$

$$\text{Nodes} = (n-1) - \ell \qquad \text{not counting zero or infinity}$$

$$E_n = -\frac{1}{2}m_e c^2\alpha^2\frac{Z^2}{n^2}\frac{m_{\text{reduced}}}{m_e} = -E_{\text{Ryd}}\frac{Z^2}{n^2}\frac{m_{\text{reduced}}}{m_e} = -13.606\frac{Z^2}{n^2}\frac{m_{\text{reduced}}}{m_e}\ \text{eV}$$

14 General Angular Momentum Formulae

$$[J_i, J_j] = i\hbar\varepsilon_{ijk}J_k \qquad \text{(Einstein summation on } k\text{)} \qquad [J^2, \vec{J}\,] = 0$$

$$J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle \qquad J_z|jm\rangle = m\hbar|jm\rangle$$

$$J_\pm = J_x \pm iJ_y \qquad J_\pm|jm\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}|jm\pm1\rangle$$

$$J_{\left\{\begin{smallmatrix}x\\y\end{smallmatrix}\right\}} = \left\{\begin{matrix}\frac{1}{2}\\ \frac{1}{2i}\end{matrix}\right\}(J_+ \pm J_-) \qquad J_\pm^\dagger J_\pm = J_\mp J_\pm = J^2 - J_z(J_z \pm \hbar)$$

$$[J_{fi}, J_{gj}] = \delta_{fg}i\hbar\varepsilon_{ijk}J_k \qquad \vec{J} = \vec{J}_1 + \vec{J}_2 \qquad J^2 = J_1^2 + J_2^2 + J_{1+}J_{2-} + J_{1-}J_{2+} + 2J_{1z}J_{2z}$$

$$J_\pm = J_{1\pm} + J_{2\pm} \qquad |j_1j_2jm\rangle = \sum_{m_1m_2,\, m=m_1+m_2}|j_1j_2m_1m_2\rangle\langle j_1j_2m_1m_2|j_1j_2jm\rangle j_1j_2jm\rangle$$

$$|j_1 - j_2| \le j \le j_1 + j_2 \qquad \sum_{|j_1-j_2|}^{j_1+j_2}(2j+1) = (2j_1+1)(2j_2+1)$$

15 Spin 1/2 Formulae

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$|\pm\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right) \qquad |\pm\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle \pm i|-\rangle\right) \qquad |\pm\rangle_z = |\pm\rangle$$

$$|++\rangle = |1,+\rangle|2,+\rangle \qquad |+-\rangle = \frac{1}{\sqrt{2}}\left(|1,+\rangle|2,-\rangle \pm |1,-\rangle|2,+\rangle\right) \qquad |--\rangle = |1,-\rangle|2,-\rangle$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k \qquad [\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

$$(\vec{A}\cdot\vec{\sigma})(\vec{B}\cdot\vec{\sigma}) = \vec{A}\cdot\vec{B} + i(\vec{A}\times\vec{B})\cdot\vec{\sigma}$$

$$\frac{d(\vec{S}\cdot\hat{n})}{d\alpha} = -\frac{i}{\hbar}[\vec{S}\cdot\hat{\alpha}, \vec{S}\cdot\hat{n}] \qquad \vec{S}\cdot\hat{n} = e^{-i\vec{S}\cdot\vec{\alpha}}\vec{S}\cdot\hat{n}_0 e^{i\vec{S}\cdot\vec{\alpha}} \qquad |\hat{n}_\pm\rangle = e^{-i\vec{S}\cdot\vec{\alpha}}|\hat{z}_\pm\rangle$$

$$e^{ixA} = \mathbf{1}\cos(x) + iA\sin(x) \quad \text{if } A^2 = \mathbf{1} \qquad\qquad e^{-i\vec{\sigma}\cdot\vec{\alpha}/2} = \mathbf{1}\cos(x) - i\vec{\sigma}\cdot\hat{\alpha}\sin(x)$$

$$\sigma_i f(\sigma_j) = f(\sigma_j)\sigma_i\delta_{ij} + f(-\sigma_j)\sigma_i(1-\delta_{ij})$$

$$\mu_{\text{Bohr}} = \frac{e\hbar}{2m} = 0.927400915(23)\times 10^{-24}\,\text{J/T} = 5.7883817555(79)\times 10^{-5}\,\text{eV/T}$$

$$g = 2\left(1 + \frac{\alpha}{2\pi} + \ldots\right) = 2.0023193043622(15)$$

$$\vec{\mu}_{\text{orbital}} = -\mu_{\text{Bohr}}\frac{\vec{L}}{\hbar} \qquad \vec{\mu}_{\text{spin}} = -g\mu_{\text{Bohr}}\frac{\vec{S}}{\hbar} \qquad \vec{\mu}_{\text{total}} = \vec{\mu}_{\text{orbital}} + \vec{\mu}_{\text{spin}} = -\mu_{\text{Bohr}}\frac{(\vec{L}+g\vec{S})}{\hbar}$$

$$H_\mu = -\vec{\mu}\cdot\vec{B} \qquad H_\mu = \mu_{\text{Bohr}}B_z\frac{(L_z + gS_z)}{\hbar}$$

16 Time-Independent Approximation Methods

$$H = H^{(0)} + \lambda H^{(1)} \qquad |\psi\rangle = N(\lambda) \sum_{k=0}^{\infty} \lambda^k |\psi_n^{(k)}\rangle$$

$$H^{(1)}|\psi_n^{(m-1)}\rangle(1-\delta_{m,0}) + H^{(0)}|\psi_n^{(m)}\rangle = \sum_{\ell=0}^{m} E^{(m-\ell)}|\psi_n^{(\ell)}\rangle \qquad |\psi_n^{(\ell>0)}\rangle = \sum_{m=0,\ m\neq n}^{\infty} a_{nm}|\psi_n^{(0)}\rangle$$

$$|\psi_n^{1\text{st}}\rangle = |\psi_n^{(0)}\rangle + \lambda \sum_{\text{all } k,\ k\neq n} \frac{\left\langle \psi_k^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle}{E_n^{(0)} - E_k^{(0)}}|\psi_k^{(0)}\rangle$$

$$E_n^{1\text{st}} = E_n^{(0)} + \lambda \left\langle \psi_n^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle$$

$$E_n^{2\text{nd}} = E_n^{(0)} + \lambda \left\langle \psi_n^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle + \lambda^2 \sum_{\text{all } k,\ k\neq n} \frac{\left|\left\langle \psi_k^{(0)}|H^{(1)}|\psi_n^{(0)}\right\rangle\right|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$E(\phi) = \frac{\langle\phi|H|\phi\rangle}{\langle\phi|\phi\rangle} \qquad \delta E(\phi) = 0$$

$$H_{kj} = \langle\phi_k|H|\phi_j\rangle \qquad H\vec{c} = E\vec{c}$$

17 Time-Dependent Perturbation Theory

$$\pi = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\, dx$$

$$\Gamma_{0\to n} = \frac{2\pi}{\hbar}|\langle n|H_{\text{perturbation}}|0\rangle|^2\delta(E_n - E_0)$$

18 Interaction of Radiation and Matter

$$\vec{E}_{\text{op}} = -\frac{1}{c}\frac{\partial\vec{A}_{\text{op}}}{\partial t} \qquad \vec{B}_{\text{op}} = \nabla\times\vec{A}_{\text{op}}$$

19 Box Quantization

$$kL = 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \qquad k = \frac{2\pi n}{L} \qquad \Delta k_{\text{cell}} = \frac{2\pi}{L} \qquad \Delta k^3_{\text{cell}} = \frac{(2\pi)^3}{V}$$

$$dN_{\text{states}} = g\frac{k^2\, dk\, d\Omega}{(2\pi)^3/V}$$

---

20 Identical Particles

$$|a, b\rangle = \frac{1}{\sqrt{2}} \left(|1, a; 2, b\rangle \pm |1, b; 2, a\rangle\right)$$

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}} \left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\right)$$

---

21 Second Quantization

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i, a_j] = 0 \qquad [a_i^\dagger, a_j^\dagger] = 0 \qquad |N_1, \ldots, N_n\rangle = \frac{(a_n^\dagger)^{N_n}}{\sqrt{N_n!}} \ldots \frac{(a_1^\dagger)^{N_1}}{\sqrt{N_1!}}|0\rangle$$

$$\{a_i, a_j^\dagger\} = \delta_{ij} \qquad \{a_i, a_j\} = 0 \qquad \{a_i^\dagger, a_j^\dagger\} = 0 \qquad |N_1, \ldots, N_n\rangle = (a_n^\dagger)^{N_n} \ldots (a_1^\dagger)^{N_1}|0\rangle$$

$$\Psi_s(\vec{r})^\dagger = \sum_{\vec{p}} \frac{e^{-i\vec{p}\cdot\vec{r}}}{\sqrt{V}} a_{\vec{p}s}^\dagger \qquad \Psi_s(\vec{r}) = \sum_{\vec{p}} \frac{e^{i\vec{p}\cdot\vec{r}}}{\sqrt{V}} a_{\vec{p}s}$$

$$[\Psi_s(\vec{r}), \Psi_{s'}(\vec{r}\,')]_\mp = 0 \qquad [\Psi_s(\vec{r})^\dagger, \Psi_{s'}(\vec{r}\,')^\dagger]_\mp = 0 \qquad [\Psi_s(\vec{r}), \Psi_{s'}(\vec{r}\,')^\dagger]_\mp = \delta(\vec{r} - \vec{r}\,')\delta_{ss'}$$

$$|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle = \frac{1}{\sqrt{n!}} \Psi_{s_n}(\vec{r}_n)^\dagger \ldots \Psi_{s_n}(\vec{r}_n)^\dagger|0\rangle$$

$$\Psi_s(\vec{r})^\dagger|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle \sqrt{n+1}|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n, \vec{r}s\rangle$$

$$|\Phi\rangle = \int d\vec{r}_1 \ldots d\vec{r}_n\, \Phi(\vec{r}_1, \ldots, \vec{r}_n)|\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle$$

$$1_n = \sum_{s_1 \ldots s_n} \int d\vec{r}_1 \ldots d\vec{r}_n \, |\vec{r}_1 s_1, \ldots, \vec{r}_n s_n\rangle\langle\vec{r}_1 s_1, \ldots, \vec{r}_n s_n| \qquad 1 = |0\rangle\langle 0| + \sum_{n=1}^{\infty} 1_n$$

$$N = \sum_{\vec{p}s} a^\dagger_{\vec{p}s} a_{\vec{p}s} \qquad T = \sum_{\vec{p}s} \frac{p^2}{2m} a^\dagger_{\vec{p}s} a_{\vec{p}s}$$

$$\rho_s(\vec{r}) = \Psi_s(\vec{r})^\dagger \Psi_s(\vec{r}) \qquad N = \sum_s \int d\vec{r}\, \rho_s(\vec{r}) \qquad T = \frac{1}{2m} \sum_s \int d\vec{r}\, \nabla\Psi_s(\vec{r})^\dagger \cdot \nabla\Psi_s(\vec{r})$$

$$\vec{j}_s(\vec{r}) = \frac{1}{2im} \left[\Psi_s(\vec{r})^\dagger \nabla\Psi_s(\vec{r}) - \Psi_s(\vec{r}) \nabla\Psi_s(\vec{r})^\dagger\right]$$

$$G_s(\vec{r} - \vec{r}\,') = \frac{3n}{2} \frac{\sin(x) - x\cos(x)}{x^3} \qquad g_{ss'}(\vec{r} - \vec{r}\,') = 1 - \delta_{ss'} \frac{G_s(\vec{r} - \vec{r}\,')^2}{(n/2)^2}$$

$$v_{\text{2nd}} = \frac{1}{2} \sum_{ss'} \int d\vec{r} d\vec{r}\,' \, v(\vec{r} - \vec{r}\,') \Psi_s(\vec{r})^\dagger \Psi_{s'}(\vec{r}\,')^\dagger \Psi_{s'}(\vec{r}\,') \Psi_s(\vec{r})$$

$$v_{\text{2nd}} = \frac{1}{2V} \sum_{pp'qq'} \sum_{ss'} v_{\vec{p}-\vec{p}\,'} \delta_{\vec{p}+\vec{q},\vec{p}'+\vec{q}'} a^\dagger_{\vec{p}s} a^\dagger_{\vec{q}s'} a_{\vec{q}\,'s'} a_{\vec{p}\,'s} \qquad v_{\vec{p}-\vec{p}\,'} = \int d\vec{r}\, e^{-i(\vec{p}-\vec{p}\,')\cdot\vec{r}} v(\vec{r})$$

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22 **Klein-Gordon Equation**

$$E = \sqrt{p^2c^2 + m^2c^4} \qquad \frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2 \Psi(\vec{r}, t) = \left[\left(\frac{\hbar}{i}\nabla\right)^2 + m^2c^2\right] \Psi(\vec{r}, t)$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right] \Psi(\vec{r}, t) = 0$$

$$\rho = \frac{i\hbar}{2mc^2}\left(\Psi^*\frac{\partial\Psi}{\partial t} - \Psi\frac{\partial\Psi^*}{\partial t}\right) \qquad \vec{j} = \frac{\hbar}{2im}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)$$

$$\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t} - e\Phi\right)^2 \Psi(\vec{r}, t) = \left[\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\vec{A}\right)^2 + m^2c^2\right] \Psi(\vec{r}, t)$$

$$\Psi_+(\vec{p}, E) = e^{i(\vec{p}\cdot\vec{r} - Et)/\hbar} \qquad \Psi_-(\vec{p}, E) = e^{-i(\vec{p}\cdot\vec{r} - Et)/\hbar}$$