Ch. 15 Oscillatory Motion

~~motion~~ — back & forth motion.

More ~~generally, or~~, any motion that repeats

— it can repeat exactly

on ideal at the macro level.
— more nearly so at the micro level in QM.

or approximately

if the motion repeats in equal or nearly equal time periods (which usual for oscillation though not always true)

then it's periodic or approximately periodic.

(15-2)

§15.~~2~~ ~~Moti~~ $F = ma$

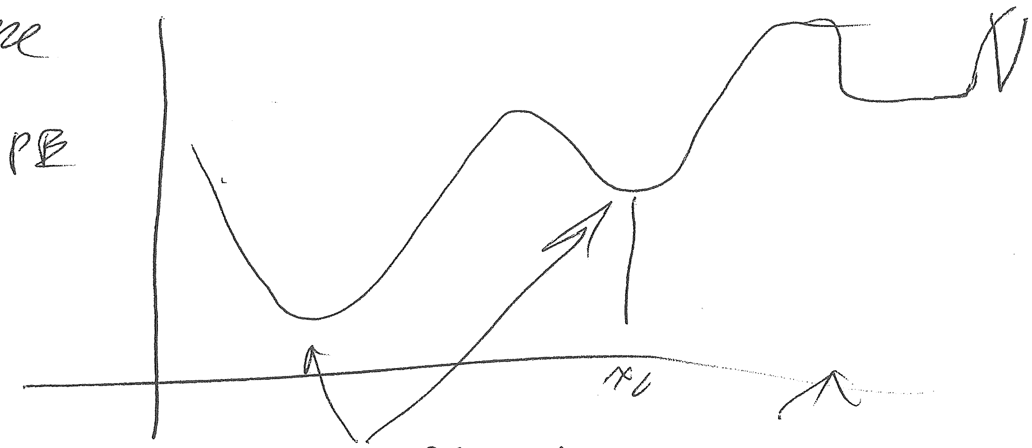
~~& the linear force~~
or linear restoring force
or spring force
or Hooke's law force
or simple-harmonic oscillator force.

1-d only in this chapter

$F = -kx$ it is a conservative force

$PE = \frac{1}{2}kx^2$ is the PE

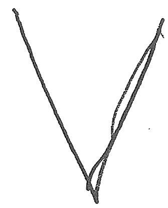
PE landscape



Many PE minima which are stable equilibria are approximately ~~$PE = \frac{1}{2}kx^2$~~

$$PE = \frac{1}{2}k(x-x_0)^2 + PE_0$$

In fact nearly all, except those that are flat-bottomed or kinked.



a Taylor's series expansion of PE around a minimum has 1st order term zero because the point is a minimum

$$PE = PE_0 + 0 + \frac{1}{2}(x-x_0)^2\frac{d^2P}{dx^2} + \ldots$$

— So these have

$$F = -\frac{dPE}{dx} = -kx$$

a linear restoring force.

this is true in analogous multi-d cases

So the linear force for small perturbations from stable equilibrium is extremely common and what makes the linear force so important and not just for springs.

There are usually frictional forces too that dissipate energy to waste heat.

Say $F_{net} = F = -kx$ } The only force is the linear force.

then from $F = ma$

$$ma = -kx$$

but $a = \frac{d^2x}{dt^2}$

$$m\frac{d^2x}{dt^2} = -kx$$

Called the simple harmonic oscillator ~~differential~~ equation since the resulting motion is SHM.

— How many have heard of DEs?

— So for some this is the first one — at least the first one you know is one.

— A DE is an equation involving a function ~~dx/dt~~ and its derivatives whose solution is ~~the~~ function which is the unknown.

This is a 2nd order linear DE ~~for~~ $x(t)$

2^{nd} order because
- ~~the~~ highest derivative is the 2^{nd} derivative

- linear because neither $x(t)$ nor it's derivatives appear as power explicitly or implicitly

~~In this cla~~

§15.2 Solution of the SHM DE

$$m \frac{d^2x}{dt^2} = -kx$$

we define $\omega = \sqrt{\frac{k}{m}}$

→ Greek small omega.
- we call it the angular frequency (not angular velocity) in this context.
- angular velocity & angular frequency are related sometimes

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

is our simplified DE

We solve it by knowing the solution in this case. $\underbrace{}$ this in fact is simple harmonic motion.

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\frac{dx}{dt} = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$$

$$\frac{d^2x}{dt^2} = -\omega^2\big(A\cos(\omega t) + B\sin(\omega t)\big)$$

$$= -\omega^2 x$$

and so the DE is satisfied.

A & B are arbitrary constants as far as the DE alone is concerned.

Initial conditions of the physical problem set them as we'll see in a minute.

Math tells us that $\cos(\omega t)$ and $\sin(\omega)$ are the 2 independent solution every 2nd order linear DE must have.

— independent means you can't make one out of the other.

But you can combine them as the text does into

$x = A\cos(\omega t + \phi)$ which gives a sinusoidal oscillation.

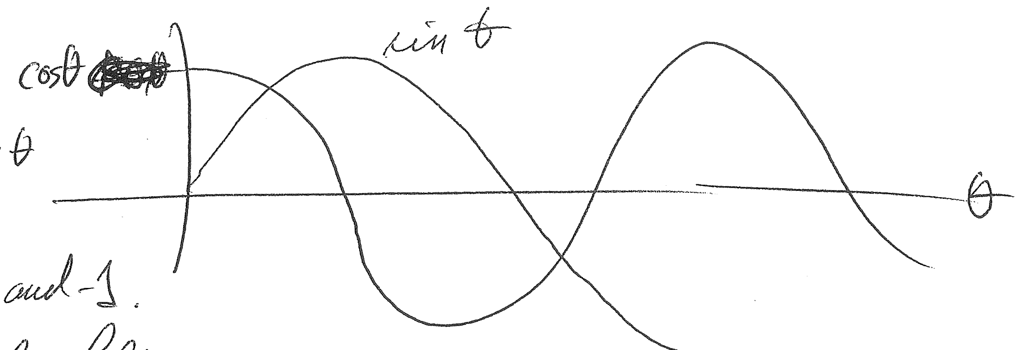
$= A[\cos\omega t\cos\phi - \sin\omega t\sin\phi)$

$= \underbrace{(A\cos\phi)}_{A}\cos\omega t + \underbrace{(\cancel{B}\ A\sin\phi)}_{B}\sin\omega t$

Let's analyze.

a) A is <u>amplitude</u> of the oscillation

~~cos~~
cos θ & sin θ oscillate between 1 and -1. sinusoidally



So $x(t)$ oscillates
oscillates sinusoidally
between A & $-A$.

i) How ω? the angular frequency.

$\cos\theta$ & $\sin\theta$ repeat
every 2π increase θ

$$\cos(\theta + 2\pi) = \cos(\theta)$$
$$\sin(\theta + 2\pi) = \sin(\theta)$$

So does $\cos(\cancel{\omega t + \phi + 2\pi})$ $\omega(t+T) + \phi$

$$= \cos(\omega t + \phi)$$

T is <u>independent of amplitude</u>. — A special feature of S.H.M.

if $\omega T = 2\pi$

$T = \dfrac{2\pi}{\omega}$ is the period

every T, the motion repeats.

The frequency is oscillations per unit time $f = \dfrac{1 \text{ oscillation}}{T} = \dfrac{1}{T}$

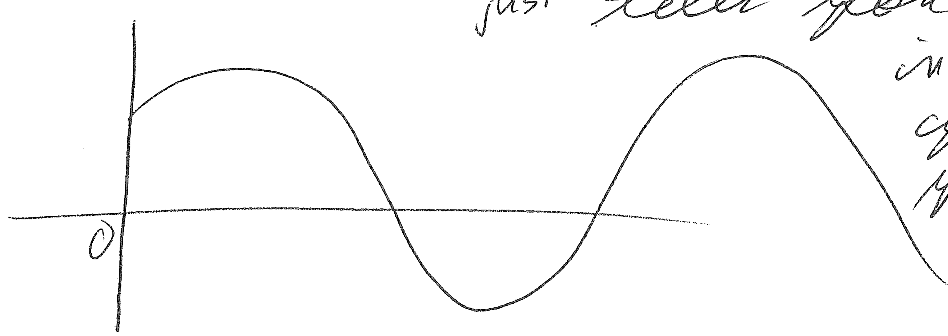
$\therefore\ f = \frac{1}{2\pi/\omega} = \frac{\omega}{2\pi}$ $\left\{ \text{f and } \omega \text{ are amplitude independent.} \right.$

or $\omega = 2\pi f$

So angular frequency is just frequency times 2π

– its the number of radians per unit time that the <u>argument</u> of cos (argument) goes thru. { Nothing necessarily is going around in an angular sense, but the ~~the~~ repeated motion is abstractly cyclic.

c) <u>ϕ is the phase</u>. It really just tells you where in the cycle you are at time zero.



<u>Examples</u>

1) $\phi = 0$

$x(t) = A\cos(\omega t)$

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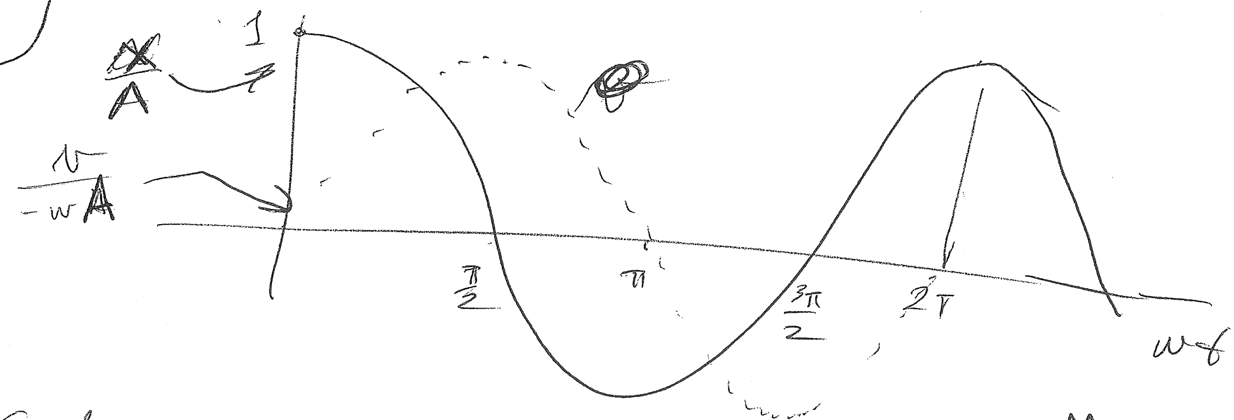


At $\omega t = 0$, $x(t) = A$

at $\omega t = \frac{\pi}{2}$, and so on.

What of velocity?

$$v = \frac{dx}{dt} = -\omega A \sin \omega t$$

It also oscillates with amplitude $v_0 = |-\omega A| = \omega A$ and the same angular frequency.

but the velocity is out of phase with x by $\frac{\pi}{2}$ or $90°$

i.e., $v = \omega A \cos(\omega t + \frac{\pi}{2})$

$= \omega A \left[\cos\omega t \cos\frac{\pi}{2} - \sin(\omega t) \sin\frac{\pi}{2}\right]$

$= -\omega A \sin \omega t$

The main qualitative result to keep in mind is

that when $|x| = A$

~~particle~~ body at rest at turning points of motion

$$v = 0$$

when $v_0 = |v| = \omega A$

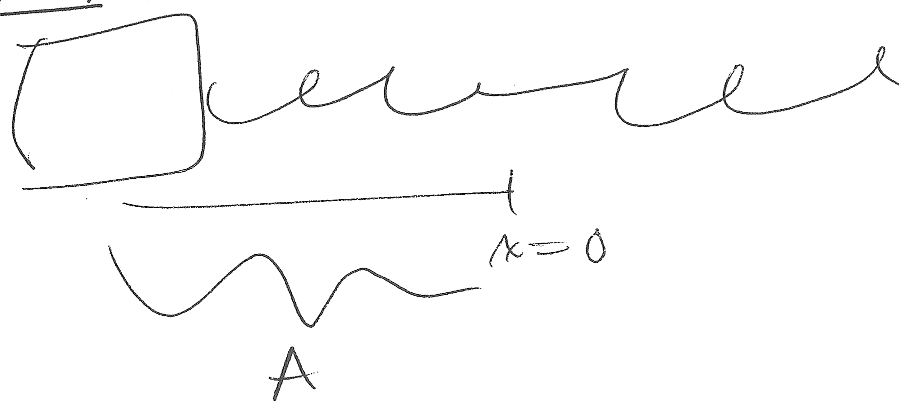
$$x = 0$$

body moving fastest at equilibrium point.

the initial amplitude sets the velocity amplitude.

Say you start a spring oscillator at $t = 0$

by releasing it from rest $x = A$ from the $x = 0$ equilibrium position.



then you've imposed the initial condition: A is amplitude and $\phi = 0$.

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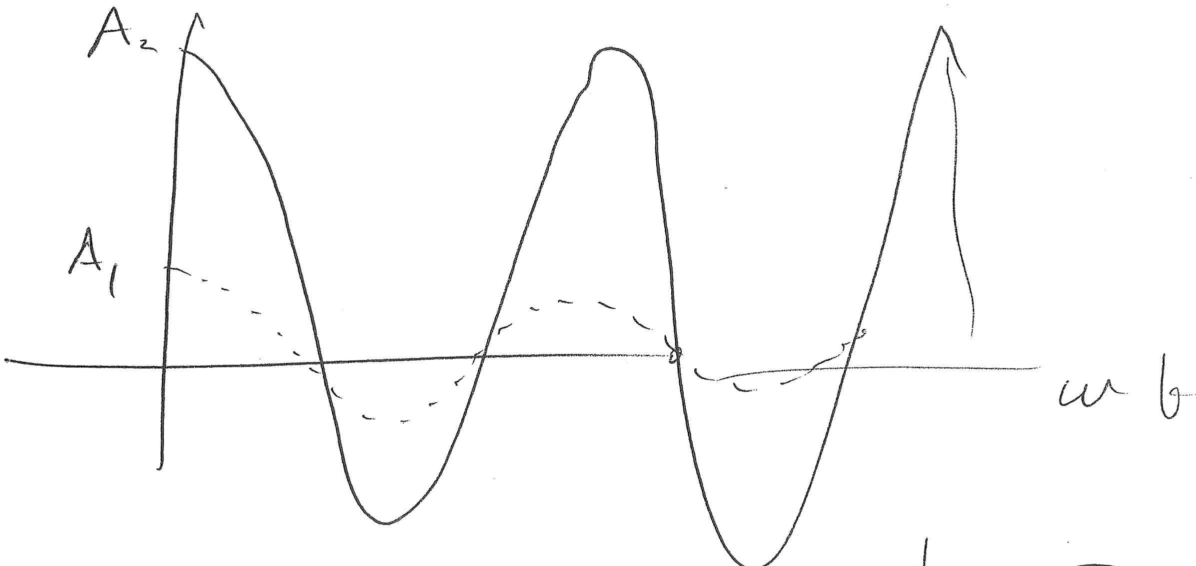
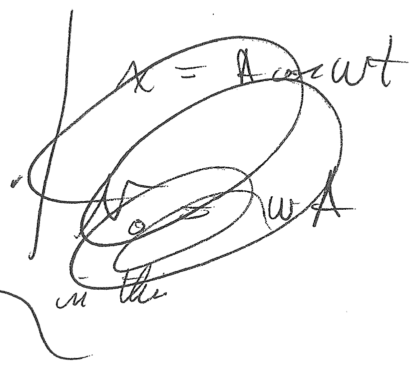
Again it's interesting to note ~~that~~ period T and ω are amplitude independent



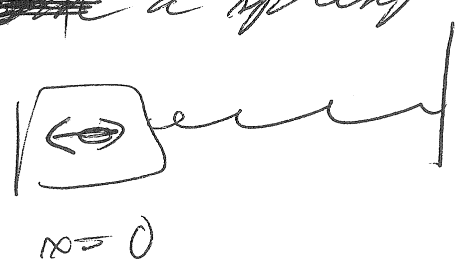
— a remarkable feature of SHM.



Ex 2 Say at $t = 0$

$x = 0$

and you give a block on a spring an initial velocity v_0



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$\cancel{x} = A \cancel{\sin}(\omega t)$

is the solution.

$v = \underbrace{\cancel{\omega} A}_{\substack{\omega \\ v_0}} \cancel{\cos}(\omega t)$

and so $A = \frac{v_0}{\omega}$ { the initial v_0 sets the amplitude in this case.

~~E~~x 3

$\left.\begin{array}{l} x = A \\ v = 0 \end{array}\right\}$ at $t = 0$

$A = .05\ m$

$m = .200\ kg$

$k = 5.00\ N/m$

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5}{.2}} = \sqrt{25}$

$= 5\ rad/s$

$T = \frac{1}{f} = \frac{1}{\omega/2\pi} = \frac{2\pi}{\omega}$

$\cong \frac{6}{5} = 1.2\ s$

Ans 1.26 s.

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velocity amplitud $\Big\}$ $v_{max} = \omega A$

$= 5 \cdot .05$

$= .25$ m/s

a_0 the accelleration amplitude.

$$a = \frac{d^2 x}{dt^2} = -\omega^2 A \cos \omega t$$

$$a_0 = \omega^2 A$$

$= 25 \cdot .05$

$= 1.25$ m/s^2

§15.3 Energy of SHM

- really simple

Since $F = -kx$ is a conservative force.

$E_{mech} = KE + PE$ should be constant.

15-15

– we don't need to do a calculation to prove this — it was proven in general long ago.

But to confirm and evaluate the E_{mech} let's solve

$$x(t) = A\cos(\omega t + \phi)$$

$$v = -\omega A \sin(\omega t + \phi)$$

$$E_{mech} = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

$$= \frac{1}{2} m\omega^2 A^2 \sin^2(\ldots) + \frac{1}{2} k A^2 \cos^2(\ldots)$$

$$m\omega^2 = m\left(\sqrt{\frac{k}{m}}\right)^2 = k^2$$

~~energy~~

$$= \frac{1}{2} \cancel{m\omega^2}\, k A^2 \sin^2(\ldots) + \frac{1}{2} k A^2 \cos^2(\ldots)$$

$$= \frac{1}{2} k A^2 \quad \text{since } \sin^2\theta + \cos^2\theta = 1$$

by trig identity

since $v_0 = \omega A$

$$E = \frac{1}{2} k \frac{v_0^2}{\ldots} = \frac{1}{2} m v_0^2$$

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What is v at $t=0$ — the instantaneous speed?

~~2-B~~
~~A-1~~

1~~B~~-41

Do all examples from hmk — and you all can think of any question on all material — as we are about to leave it.

$$v = \frac{dx}{dt} = 2 \cdot 2t' - 4$$

$$= 4t - 4$$

This is general. — true for all time

$$v(t=0) = -4$$

$$v(t = 2.5s) = 4\left(\frac{5}{2}\right) - 4$$

$$= 10 - 4$$

$$= 6 \, m/s \text{ as in text.}$$

Taylor's series ~~approx~~x.

OMIT?

Taylor's series Problem — Extraneous to Ch. 15 mostly.

At this point since we are on the subject of differentiation I'll do one of the hmk problems as an example.

— It's no problem for those who've already completed a calculus course,

~~2~~ 16 ~~#5-4-2~~ but for those just starting it's too much, too soon.

Taylor's series problem.

Say you have a general function $f(x)$ that is sufficiently differentiable (smooth)

— you can $\underbrace{\text{expand}}_{\text{The jargon word}}$ this function about a general point x_0

means lth order derivative

$$f(x) = \sum_{l=0}^{\infty} \frac{(x - x_0)^l}{l!} f^{(l)}(x_0)$$

$l! = l \cdot (l-1) \cdot (l-2) \ldots 1$

the factorial function

— no proof $\Longrightarrow$ Just accept it.

in general the equality holds only for $(x - x_0)$ sufficiently

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small.

How small?

Depends on the function
— anywhere from almost nothing to all x-axis.

Usually the series can be truncated to approximate $f(x)$ in a region around x_0

$f(x) \simeq \begin{cases} f(x_0) & 0^{th} \text{ approximation as a constant} \\ f(x_0) + (x-x_0)f'(x_0) & 1^{st} \text{ order — approximate as a line} \\ f(x_0) + (x-x_0)f'(x_0) + \frac{1}{2}(x-x_0)^2 f''(x_0) & 2^{nd} \text{ order or approximate as a quadratic} \end{cases}$

a) Schematic diagram of how these ~~approximation~~ approximation approximate a function

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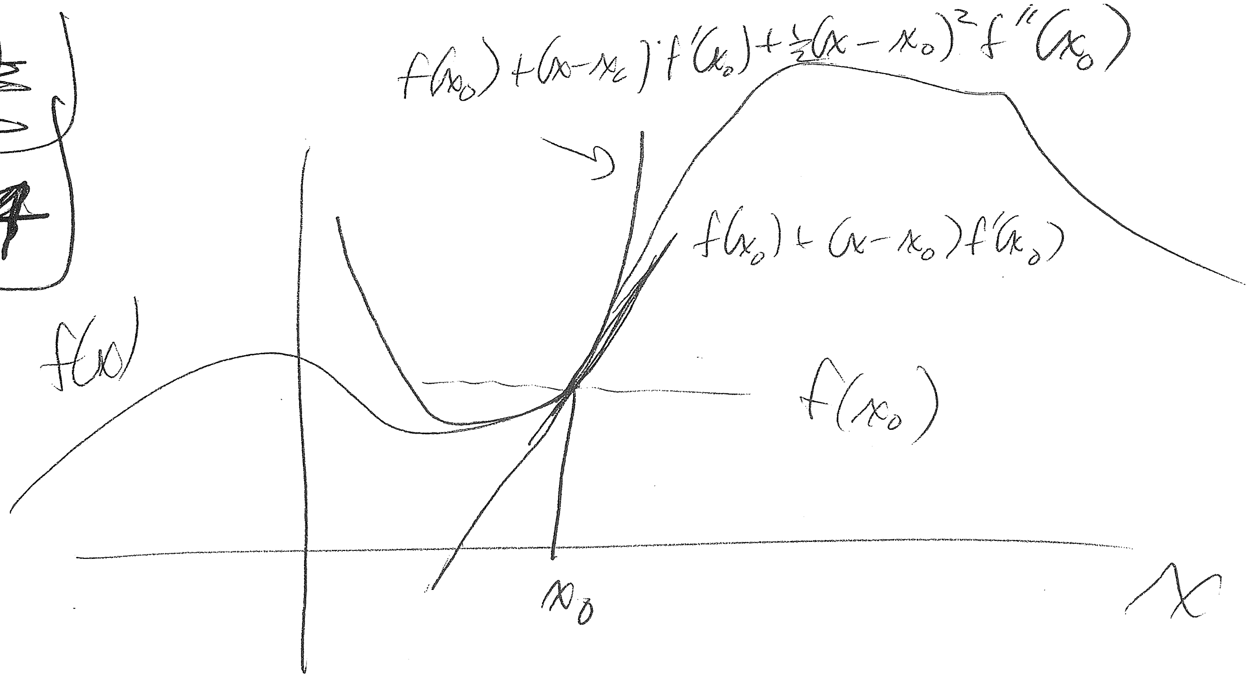

~~to each~~

- as you go up in order you get a better approximation in vicinity of x_0
- but far away from x_0 none of the approximation may be any good.

b) Taylor expand

$$f(x) = \frac{1}{1+x} \quad \text{to } 2^{nd} \text{ order about } x_0 = 0$$

$$\cancel{f = 1 + x +}$$

$$f = 1 + x\frac{1}{1!}(-1) + x^2\frac{1}{2!}(-1)(-2) + \ldots$$

$$= 1 - x + x^2 \ldots$$