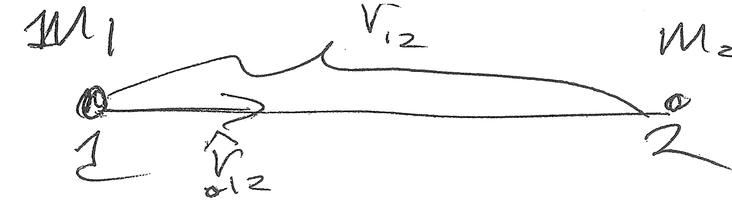
Ch. 13 Gravity

§13.1

Newton, of course, didn't discover gravity in the sense of apples falling from trees. — What he discovered was ~~that~~ that force (the apple-falling causing one) was the same as the force that was needed to make moons, planets, stars — and not known in his day — galaxies orbit.

He discovered universal gravitation and the law of universal gravitation

13-2



Force of 1 on 2

$$\mathbf{F}_{12} = -\frac{G\, m_1 m_2}{r_{12}^2}\,\hat{r}_{12}$$

depends linearly on masses

and $\underset{\sim}{F}_{21} = -F_{12}$ by 3rd law

depends inversely on the distance between the particles

points from 2 to 1

— the law of gravity is an inverse-square law force

The force is always attractive — there is no antigravity (in an ordinary sense — although something like that exists cosmologically, but people don't call it antigravity)

→ A very special kind of force with very interesting properties that we can't fully explicate here.

G is the gravitational constant. (modern value)

$$G = 6.67428(67) \times 10^{-11} \frac{N \cdot m^2}{kg^2}$$

(Wik 2008apr27)

units needed to make F_{12} a force.

known to only about 4 digits

~~One of~~ the poorest known universal constant.

It's hard to measure because gravity is so weak in a sense as we'll see.

but one hardly needs to remember if one always uses MKS in a calculation as one nearly always does

A pretty small number.

In magnitude

$$F_{12} = \frac{G m_1 m_2}{r^2}$$

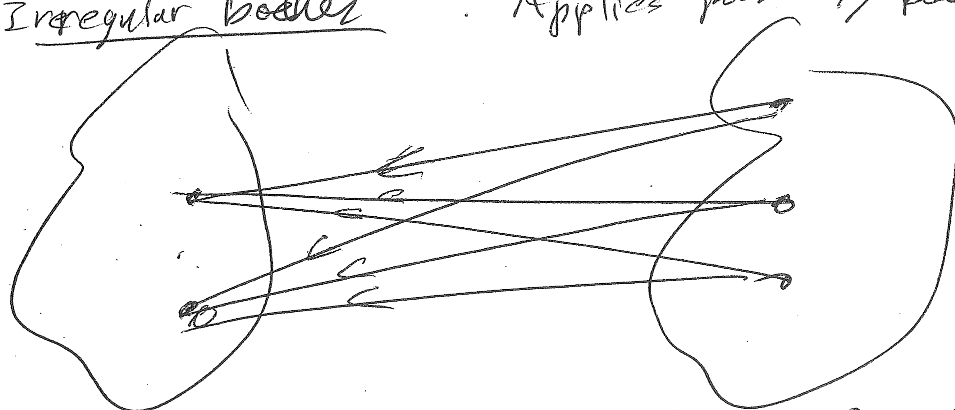
We often say force between $m_1 m_2$ as a shorthand

13-4 The law as written is for idealized points.

But we in fact don't have those. We have finite bodies.

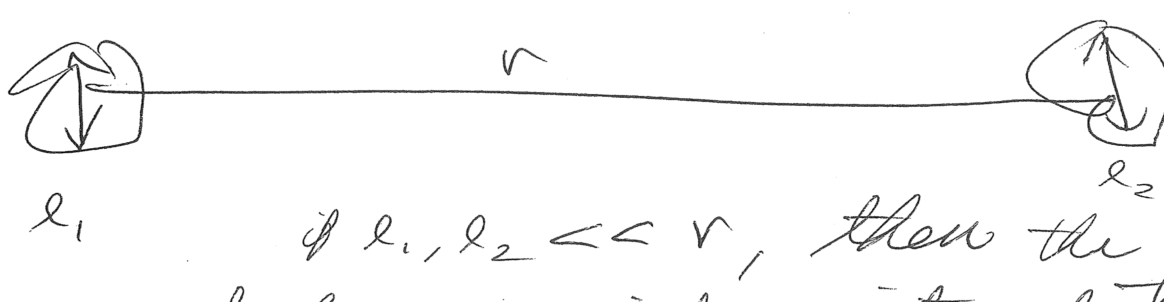
Cases

(1) Irregular bodies : Applies point by point



Do an integration of attraction ~~we~~ between all points using the universal law to get the net gravity between them — in general tough — and we won't do it.

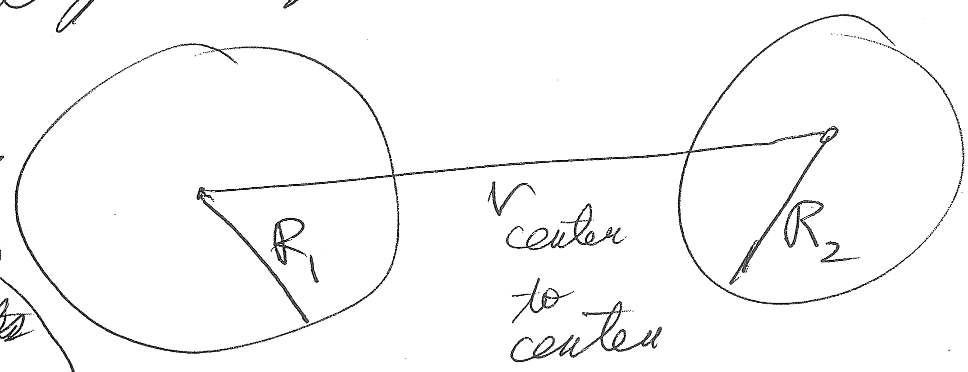
(2) Applies Far apart bodies



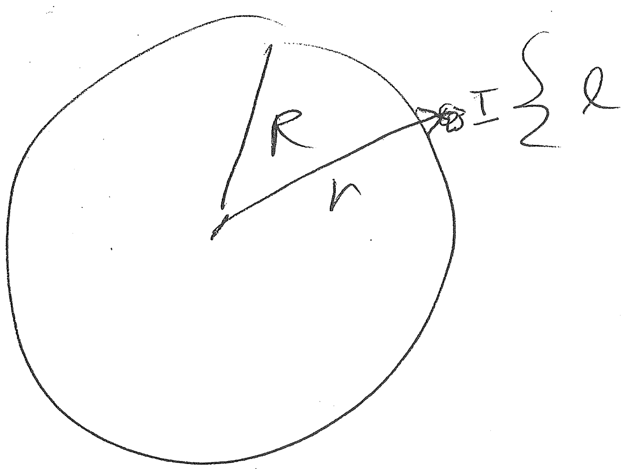
if $l_1, l_2 << r$, then the bodies approximates points and the point-law formula applies

(3) Applies
Spherically symmetric bodies
by integration one can show that the point law applies in this case as long as $r > R_1 + R_2$ (the bodies don't touch) In fact, Each one acts as a point mass (Follows from Gauss law).

A great discovery of Newton meant he could apply his law to Sun + planets with ease



(4) Applies
Spherically symmetric body & a minute body by comparison



$l << R$

— really a corollary of cases (2) & (3)

Gravity is weak in a sense

Ex $r_{12} = 1\,m$
$m_1 = m_2 = 1\,kg$
The fiducial case.

$$F_{12} = \frac{G m_1 m_2}{r_{12}} \doteq \frac{~~6.67~~\ 7\times10^{-11}\cdot 1 \cdot 1}{1^2}$$

$$\cong 7\times10^{-11}\,N.$$

We in ~~this~~ room ~~all~~ are all mutually attracted to each other
— but so weakly we never notice.
— good thing too or we'd all be one mass of arms, legs, and torso in the center of the room. — pretty gross really

But nevertheless it is gravity that dominates or codominates (along with pressure) large structures. → ~~cluster~~ ~~pla~~ planets, stars, ~~often~~ galaxies clusters of galaxies, universe as a whole it seems.

— Why?
— a) there's a lot of mass
b) gravity is always attractive.

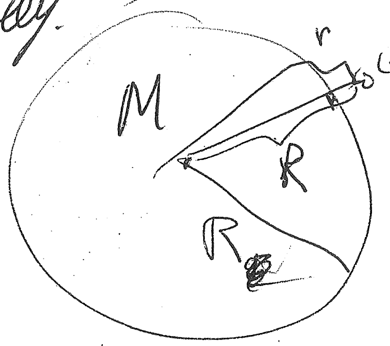
and has only one kind of "charge" is mass

— ~~there~~ is no cancellation as for example ~~t~~ with the electric force where negative + positive charges give rise to repulsion & attraction and in neutral matter largely cancel out at the macro-level.

§13.2 Gravity Near surface of the Earth

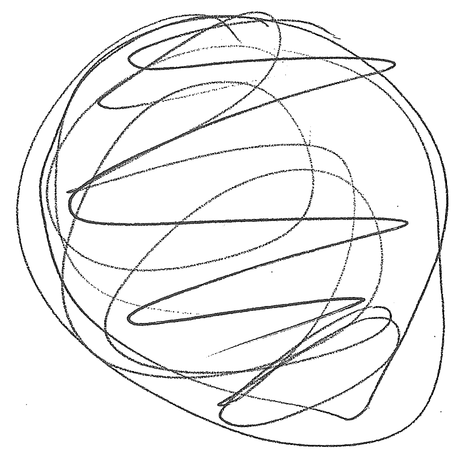
Earth or any large spherical body.

$$F = \frac{G M_{Earth}\, m}{r^2}$$



m small body.

always points to center



$$r = \Delta r + R_E$$

$$F = \frac{G M m}{(R_E + \Delta r)^2} = \frac{G M m}{R_E^2 \left(1 + \frac{\Delta r}{R_E}\right)^2}$$

Taylor series of $\frac{1}{\left(1 + \frac{\Delta r}{R_E}\right)^2} = 1 - 2\frac{\Delta r}{R_E} + \dots$

One can truncate series for $\Delta r / R \ll 1$

Grav Field (grav force per unit mass) $g = \frac{F}{m} \simeq \frac{G M_{Earth}}{R^2}\left(1 - 2\frac{\Delta r}{R}\right)$

13-8

$$\frac{G M_{\oplus}}{R^2} \doteq \frac{7 \times 10^{-11} \cdot 6 \times 10^{24}\,\text{kg}}{(6.37 \times 10^{6}\,\text{m})^2}$$

$$= \frac{10^{-11} \cdot 10^{24}}{10^{12}}$$

$$= 10 \ \frac{\text{N}}{\text{kg}}$$

A more precise calculation gives $g = 9.8$ N/kg our fiducial value of free-fall acceleration.

$$g(y = \Delta r) \doteq 10 \cdot \left(1 - 2\frac{y}{R}\right)$$

$$= 10\left(1 - \frac{2 \cdot 1000^{6} \cdot y\,\text{m}}{6 \times 10^{6}}\right)$$

megameter = 1000 km

$$= 10\left(1 - \tfrac{1}{3}\, y_{Mm}\right)$$

7 (7.33) for $y_{Mn} = 1$ OK

3 for $y_{Mn} = 2$ (5.68) } our Taylor expansion approximation has f...

§13.3 Kepler's Laws of Planetary Motion

Kepler discovered his 3 laws in the early 17th century.

— they are semi-empirical in that they were found by fitting observed planet data

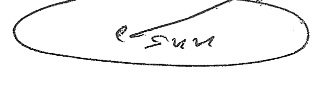
but not totally empirically. He had a strong guiding principle that the planet-Sun distance was the key physical parameter — as indeed it is and that had not been understood clearly before

They are actually approximation — but very good ones. — they'd be exact for two bodies orbiting alone in empty space. (aside from GR effects)

But now they can be derived from Newtonian physics which we'll do only in part.

1st The planets move in elliptical orbits about the Sun with the Sun at one focus

13-10a) of the ellipse.



— we won't go into ellipse details

— we are running low on time.

Suffice to say the planet orbits are all highly circular — only slightly to somewhat elliptical.

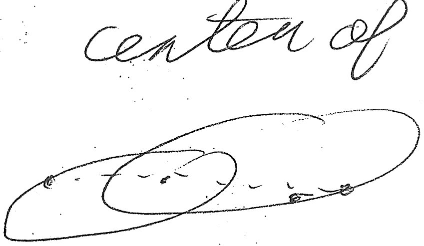
(Comet orbits are quite elliptical)

We'll concentrate on circular orbits mainly which are easy to understand.

— and won't try to prove this law.

Actually to be a bit more exact, both bodies ~~on~~ of a 2-body system orbit their joint center of Mass (CM) in ellipses

— ~~each at the focus of one ellipse~~.



It's at ~~the~~ a focus of both elliptical orbits.

But if one of the bodies ~~is~~ much more massive than the other, then the massive body approximately coincides with the CM ~~and the~~ and defines an inertial frame.

— the small body can ~~then~~ be said to orbit the large one on an ellipse.

This system is true for the Sun and each planet (Sun is much more massive) and the Earth and Moon with Earth as the more massive body.

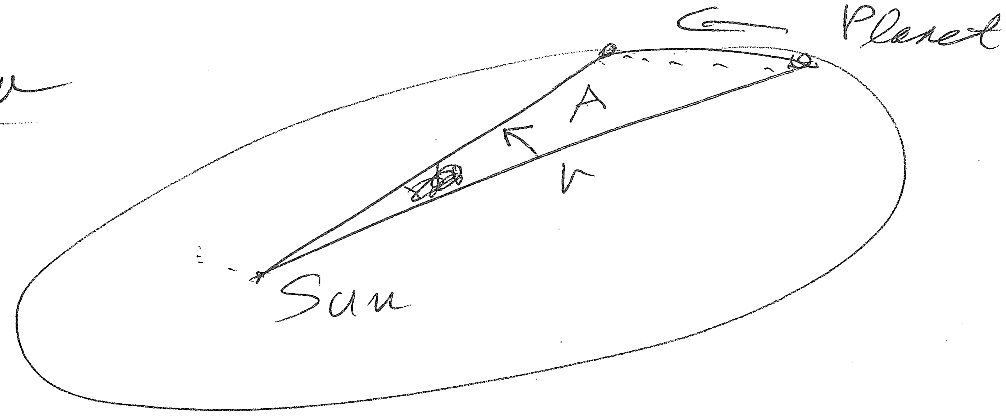
We can't go into the proof of the ellipse law — it does depend on the inverse square law nature of the grav. force.

3-10c

13-11

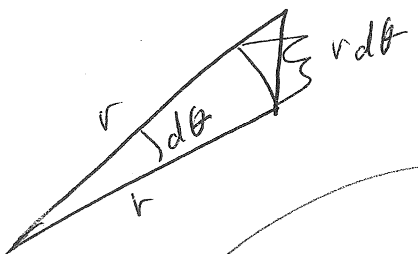
13-12

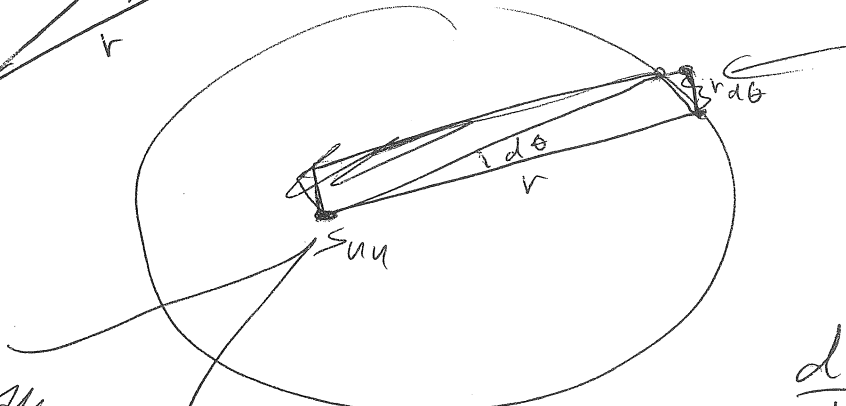
2nd law



Planet -Sun line sweeps out equal areas in equal times as it

Or what is called the areal velocity is constant.





$$dA = \frac{1}{2}\, r\, r\, d\theta$$

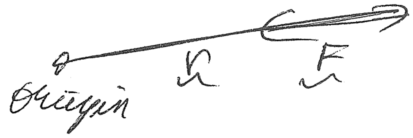
exact in a differential sense. Just triangle area rule $A = \frac{1}{2} b h$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \omega$$

areal velocity

We assume the is so massive that is essentially the CM and defines an inertial frame.

This we can prove easily.



$$\tau_{\text{about origin}} = \underset{\sim}{r} \times \underset{\sim}{F} = 0$$

since they are aligned.

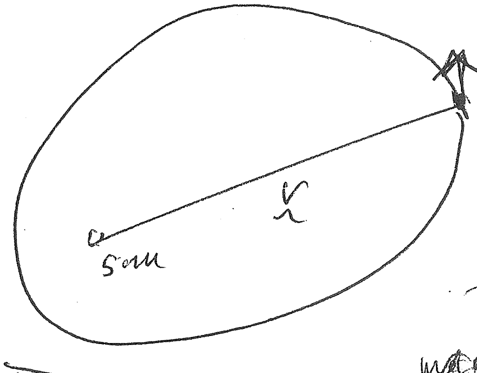
13-13

— gravity is called a central force in that ~~for point~~ ~~source at the origin,~~ ~~t points to origin at the various body where~~ ~~the grav. force always points toward~~ is the origin (or center)

for a point masses it always points along the line between the points. If one point mass is dominant it defines the origin or center

$\therefore \underset{\sim}{L} = \text{Constant}$

$L_z = \text{Constant} \Rightarrow$ our z-axis comes out of paper



$\underset{\sim}{r}$ } all in x-y plane

$L_z = | \underset{\sim}{r} \times \underset{\sim}{p} |$ assuming counterclock motion

magnitude of L_z

$= | \underset{\sim}{r} \times m (\underset{\sim}{v}_{tan} + \underset{\sim}{v}_{radial}) |$

$= | mrv_{tan} + 0 |$

since $\underset{\sim}{r} \times \underset{\sim}{v}_{radial} = 0$.

$= mrv_{tan}$

$= mr^2\omega$

Unlike Laws 1 & 3 which depend on gravity be an inverse square-law force, the 2nd law only depends on the force being central and so is true for all central forces.

$\therefore \frac{dA}{dt} = \frac{1}{2} r^2 \omega = \frac{1}{2} \frac{L_z}{m} = \text{Constant}$

QED.

13-14

③ Third law:

is ~~that~~ $\underbrace{T^2}_{\text{Period}} \propto \underbrace{a^3}_{\text{ellipse semi-major axis}}$

— but for a circle ~~this~~ is just the radius r

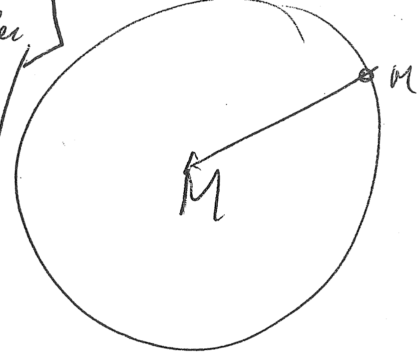
$\therefore T^2 \propto r^3$ for a circular orbit. (Go-101)

This we can prove in the approximation that the Sun is the CM ~~in its own rest frame~~ and defines an inertial frame.

By the way we've also proved that planets can orbit in circles confirming Kepler's 1st law in the circular case

$$F_c = \frac{mv^2}{r}$$

recall is a requirement



$$F = \frac{GMm}{r^2}$$

is the force that provides the centripetal force.

$$\therefore \frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\frac{GM}{r} = v^2 = \left(\frac{2\pi r}{T}\right)^2$$

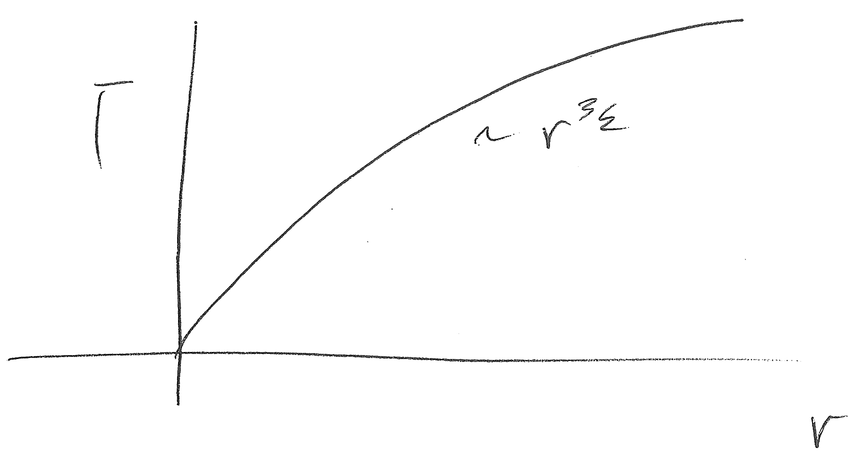
$$T^2 = \frac{(2\pi)^2 r^2}{GM/r}$$

QED. $$T^2 = \left(\frac{(2\pi)^2}{GM}\right) r^3$$

I actually prefer

$$T = \frac{2\pi}{\sqrt{GM}} r^{3/2}$$

which is a nice simple function



We can actually find a single comprehensible fiducial form.

The period of the Earth in years is 1 year.

13-16

The mean ~~Earth~~-Sun distance is defined to be 1 AU = 1 astronomical unit.

$$\therefore \; T_{yr} = r_{AU}^{3/2}$$ The constant must be 1 in this case

Omit

$$T_{yr}\left(\frac{T_s}{1\,yr}\right) = \frac{2\pi}{\sqrt{GM}}\, r_{AU}^{3/2}\left(\frac{d_m}{1\,AU}\right)^{3/2}$$

$$T_y = \underbrace{\frac{2\pi}{\sqrt{GM}}\,\frac{1\,(AU)^{3/2}}{1\,y}}_{=1}\; r_{AU}^{3/2}$$

Planet or dwarf Planet	(Orbital Radius Mean AU)	Period 3rd law	Period exact (yr)
Me	.387	.241	.241
Ve	.723	.615	.615
Ea	1	1	1.
Ma	1.52	1.87	1.881
Ju	5.20	11.86	11.87
Sa	9.58	29.65	29.45
Ur	19.23	84.33	84.07
Ne	30.10	165.14	164.9
PL	39.48	248.1	248.1

13-17

All of Kepler's laws are approximate because ~~of the~~ each planet and the Sun are not isolated 2-body systems, but part of the whole solar system in which all bodies gravitationally interact with all others.

But detailed analysis confirms the law of gravitation & Newtonian physics aside ~~aside~~ from corrections to to special & general relativity

Ex 13.5 Period & velocity for ~~Geosynchronous~~ Earth satellite orbits & Geosynchronous Orbits

13-18

The ~~Earth~~ can be said to define a inertial frame for orbiting satellites to high accuracy.

$$F_c = \frac{m v^2}{r} = \frac{GMm}{r^2}$$

$$\therefore \quad v = \sqrt{\frac{GM}{r}}$$ is orbital speed.

$$v = \underbrace{\sqrt{\frac{GM_E}{R_E}}}_{v_{fid} \text{ fiducial value.}} \sqrt{\frac{M}{M_e}} \sqrt{\frac{R_E}{R}}$$

$= 7.9107 \times 10^3$ m/s

$= 7.9107$ km/s

$\cong 8$ km/s

$M_e = 5.9736 \times 10^{24}$ kg

$R_{E\,mean} = 6371.0$ km

$= 6.3710 \times 10^6$ m

Kepler's 3rd law again

$$p = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

$$= 2\pi \frac{R_E^{3/2}}{\sqrt{GM_E}} \sqrt{\frac{M}{M_E}} \left(\frac{r}{R_E}\right)^{3/2}$$

13-19

$P_{sid} = 5060.2\,s$

$= 84$ min and $20.2\,s$

an orbit at ground level which is impossible — extra forces ~~are~~ present — air resistance

— ground friction

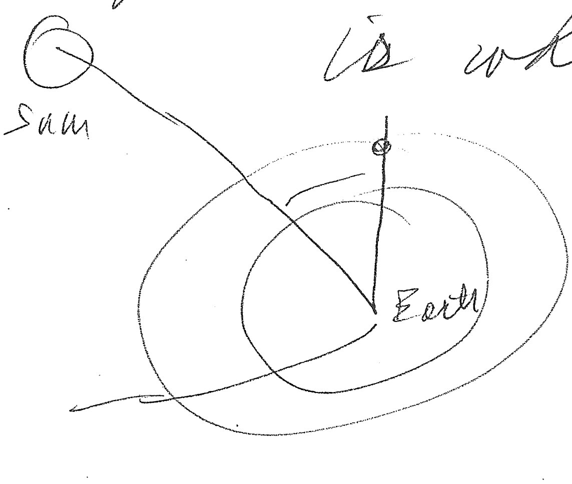
— the normal forces of mountains (a real killer).

but low Earth orbit $P \sim 90$ min.

— so the astronauts ~~roll~~ orbit around the ~ 90 min.

— it's day, then it's night, it's day, ...

<u>Geosynchronous orbit</u> for satellites is where $P = 1$ sidereal day



— time for Earth to rotate once ~~on~~ in frame of <u>fixed stars</u>

— which is a more inertial frame than that of Sun.

13-20

P = 1 sidereal day
$= 86164.1\ s < \underline{86400\ s}$

Reduced solar day.
— Actual mean solar day is a bit longer due to slowing of Earth's rotation.

$$r^{3/2} = \frac{P}{2\pi}\sqrt{GM}$$

$$r = \left[\frac{P}{2\pi}\sqrt{GM}\right]^{2/3}$$

$= 42.167 \times 10^6\ m$

$= 42167\ km$ or $\underline{35000\ km}$

$\cong 7\ R_{Earth}$ from Earth center

Altitude — the distance above the ground

Geostationary orbit

If the satellite orbit is in the plane in the equator, it will just ~~rotay~~ stay at one place in the sky relative to ground.

— ~~It will go west by it.~~

If you saw it over head at Zenith, it stay there.

— Such satellites are very useful in <u>telecommunications</u>

Geostationary orbits for communication were first popularized by Arthur C Clarke (1917dec16 – 2008mar19) in 1945.