Chapter 6

Circular Motion & Other Applications of Newton's Laws

— Just more examples with some interesting special cases

- — uniform circular motion
- — not so uniform circular motion.

But now with forces — the dynamics.

- — non-inertial frames & inertial forces. (we already anticipated ~~with~~ this with the elevator problem.
- — Resistive media
- — a differential equation and solution.

§ 6.1 Recall our general expression for acceleration in polar coordinates

$\boxed{6\text{-}2}$

$$\underset{\sim}{a} = \left(\frac{d^2 r}{dt} - r\omega^2\right)\hat{r} + \left(2\frac{dr}{dt}\,\omega + r\frac{d\omega}{dt}\right)\hat{\theta}$$

$$v_{tan} = r\omega \qquad \text{where} \quad \omega = \frac{d\theta}{dt}$$

$$= r\frac{d\theta}{dt}$$

$$= \frac{d s_{tan}}{dt} \qquad (\text{French-557})$$

If r is fixed for circular motion

$$\frac{dr}{dt} = 0, \quad \frac{d^2 r}{dt^2} = 0$$

$$\underset{\sim}{a} = \underbrace{(-r\omega^2)\hat{r}}_{a_{centripetal}} + \underbrace{\left(r\frac{d\omega}{dt}\right)\hat{\theta}}_{a_{tan}}$$

$$\underset{\sim}{a_c} = -r\omega^2\hat{r}$$

$$= -\frac{v^2}{r}\hat{r}$$

$$a_c = \frac{v^2}{r} \quad \text{for just magnitude}$$

Well $F = ma$ tells us if there is an non-zero acceleration, there must be a non-zero net force.

So For circular motion there must be a radial force component of magnitude

$$\underbrace{F_{\text{centripetal}}}_{\text{centripetal force}} = m\frac{v^2}{r}$$

→ this is the centripetal force

<u>Note</u> the centripetal force is <u>NOT</u> a magic force that turns on when there is circular motion → it is a <u>force requirement</u> needed to cause circular motion.

— A real actual force must supply this force.

A homely analogy may help. Say there is a truck that

6-4) delivers groceries.
— ~~the~~ truck's function is
to deliver groceries
So it is a delivery truck
by function.
But by nature it could
be a GM, Toyota, or even
a Chrysler (though
this seems unlikely)
So ~~the~~ centripetal ~~is~~ "centripetal
force" describes what
the force does, not what
it is.

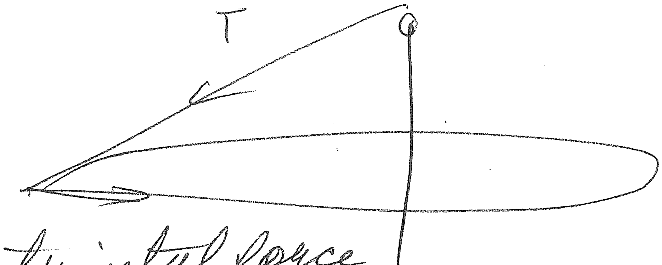
Or another perspective.

$$F_c = m\frac{v^2}{r}$$

is really $m\,a$
specialized for the radial
component of circular motion

— Conical pendulum

or sling.



— The string supplies the centripetal force

— it is a tension force

— DEMO. (students in front are permitted to cover)

→ ~~I~~ can

3rd law story
- Rope pulls on bob
- bob pulls on rope
- rope pulls on me
- I pull on rope

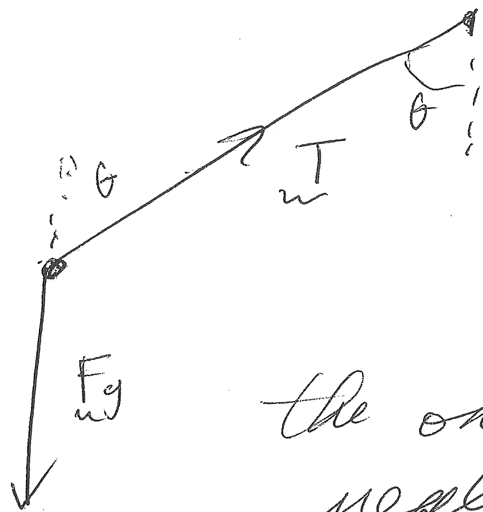
If there were no force to accelerate bob into a circular motion, it'd would fly off in a straight line

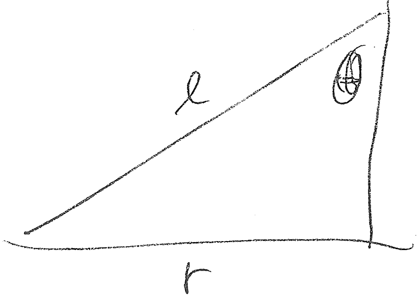
→ which is the whole point of the sling as a weapon actually.

Let's analyze the forces on the bob.

6-6

Bob as point mass in FBD





the only two forces neglecting air resistance.

Class mantra

F=ma is always true and it's always true component by component.

In r-direction $\sum F_{r} = -T \sin\theta = -m\frac{v^2}{r}$

$$T \sin\theta = m\frac{v^2}{r}$$

One could solve for T given $m, v, r, \sin\theta$

— recall we don't have an intrinsic force formula.

In y-direction $\sum F_y = T\cos\theta - mg = 0$

no acceleration

$$\therefore \quad T\cos\theta = mg$$

One can solve for angle by division

$$\tan\theta = \frac{v^2}{rg}$$

$$= \frac{v^2}{l \sin\theta \, g}$$

Note m cancelled out which often happens in simple gravity problems

$\tan\theta \sin\theta = \frac{v^2}{rg}$

$\theta^2 \simeq \frac{v^2}{\ldots}$ small angle approx because gravity is ~~a mas~~ linear in mass in and so is $F = ma$.

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

This formula recurs in several different contexts with somewhat different meanings

or $\theta^2 = \frac{v^2}{lg}$ for small angle approximation

Say $\cancel{r} = \cancel{\ \ }\ .5m$, $g = 9.8\ m/s^2 \cong 10$

$$v = \frac{2\pi r}{\Delta t}$$

10 swings in $\cancel{X}$ s means $\cancel{\Delta}t = \_$

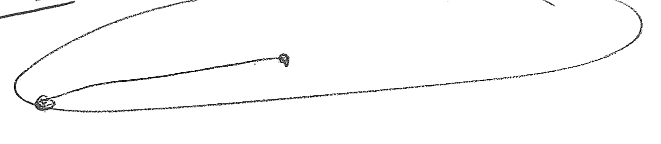
$$\cong \frac{6}{\Delta t}$$

$$\therefore \cancel{\theta = \tan^{-1}\left(\frac{v^2}{rg}\right) \cong \frac{v^2}{rg}}$$

$$\theta = \frac{v}{\sqrt{lg}}$$

→ using small angle approximation.

Ex 6.2 Horizontal circular motion



$m = .5\ kg$

$r = 1.5 m$

~~cord~~ breaks if $T > T_{max} = 50N$

~~How fast~~ can the mass be swung before breaking?

6-8)

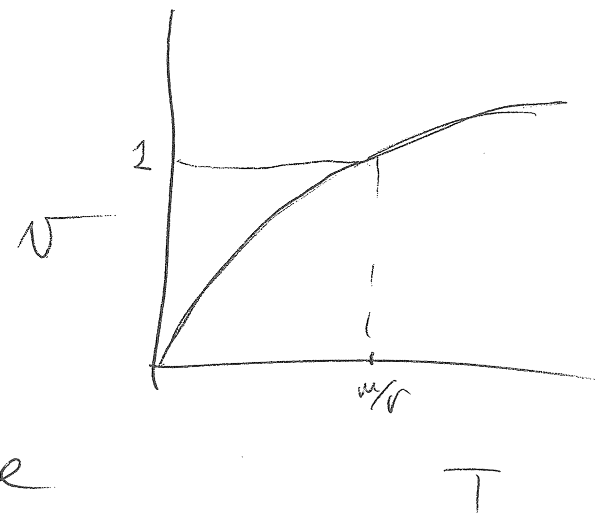
$$T \sin\theta = \frac{mv^2}{r}$$

But say we are on a frictionless horizontal plane

$$\theta = 90^\circ$$

$$T = \frac{mv^2}{r}$$

$$v = \sqrt{\frac{Tr}{m}}$$



v strictly rises with T.

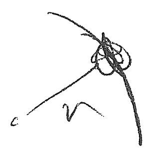
$\therefore$ at T_{max}, one has v_{max}

$$v_{max} = \sqrt{\frac{50\cdot 1.5}{.5}} = \sqrt{150}$$

$\cong 12.5$ m/s **Ans** 12.2 m/s

E 6.3 Car on a Horizontal Bend

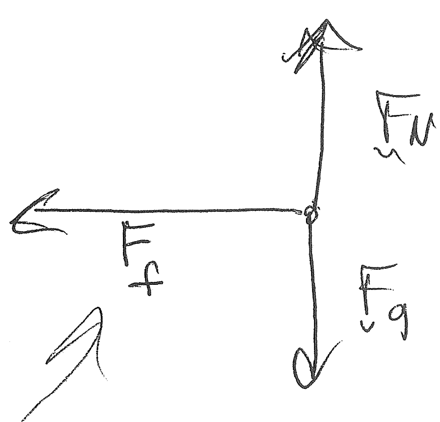
Such bends are rare — because good designers almost always bank roads for very good reasons as we'll soon see.



— Of course every ordinary intersection is a small horizontal bend that you take slowly for good reason also

Car moving into the page

center of ~~bend~~ circular bend



The only horizontal force on the car is static friction

Static because the wheels are not sliding on the roads

$$\sum_i F_{y,i} = F_N - mg = 0$$

$$F_N = mg$$

Magnitude $F_f = \text{Min}(F_{app}, \mu_s F_N) = \text{Min}(F_{app}, \mu_s mg)$

6-10

F_{app} is the direction force the car [illegible] applying to the ground in trying to turn

6-11

To be moving in a circle there must be a centripetal force

$$F_c = m\frac{v^2}{r}$$

Only <u>friction</u> can supply it.

But as $v \uparrow$ or $r \downarrow$ the force requirement gets bigger and the limit ~~[illegible]~~

$$\mu_s m g = m \frac{v_{max}^2}{r}$$ for a maximum bend ~~[illegible] turn~~ speed.

mass cancels

and one solves for the limiting speed.

$$v_{max} = \sqrt{\mu_s r g}$$

μ_s = .523, r = 35 m, g = 9.8 m/s²

$$\cong \sqrt{.5 \cdot 35 \cdot 10}$$

$$= \sqrt{175} \cong 13 \text{ m/s}$$

$$= 13 \text{ m/s} \; \frac{1 \text{ mi}}{1609 \text{ m}} \; \frac{3600 \text{ s}}{1 \text{ h}}$$

$$\cong 30 \text{ mi/h}$$

Text here but elsewhere rubber on concrete is general 1 — but this is rubber on asphalt.

This isn't very good actually

6-12

You can only ~~go~~ ~ 30 mi/h around a bend and even this is pushing it to the safety limit. → Actually the car becomes a bit hard to control if there

is a strong ~~[illegible]~~ sideways friction force on turned wheels ?

— you'd have to go much slower to be safe especially if friction were reduced by water or ice.

...nel ...ou ...to ...o ...low ...round ...rdinary ...nbanked ...orners.

~~This [illegible]~~

~~R~~. This leads to <u>banking</u>

Ex 6.4

— You know road bends are usually banked and empirically you sort of know why in part

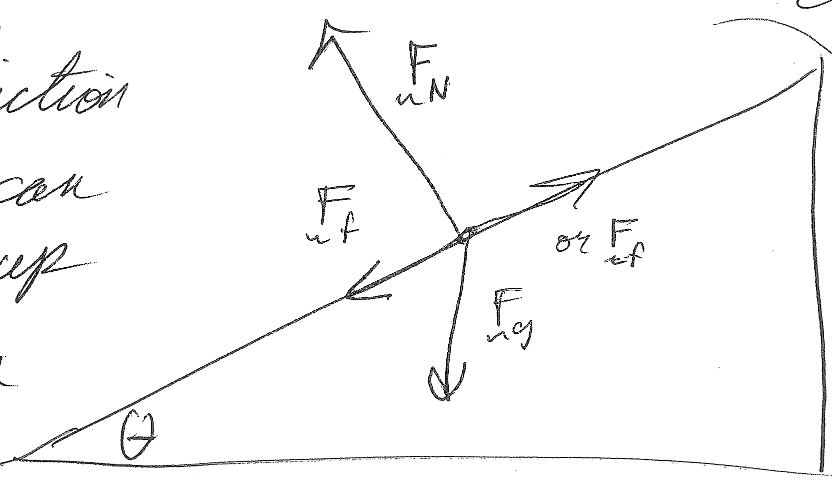
gravity + normal force ~~are~~ available to help supply the centripetal force.

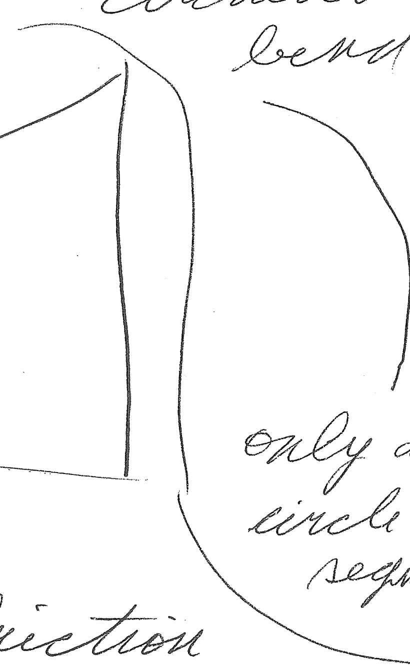
~~Force~~ But that's only ~~partially~~ the reason why The road normal force helps too.

Let's analyze the forces

Consider a circular bend

The friction force can point up or down the slope.





only a circle segment.

Why are there friction and normal forces?

~~The~~ Gravity (on the Earth's surface) is always turned on.

Static Friction is always helping to ~~accelerate and decelerate~~ change speed, but that along the direction of travel.

Rolling Friction also ~~slows~~ tries to slow car TM-130

6-14) The normal force helps against gravity — on level ground it ~~does~~ does all the support against gravity.

But if you *are* making a bend, the car is pushing outward on the road using the car's normal force and friction force *and* the road pushes back to accelerate the car around the bend.

Going fast with wheel — slowing into pavement

But friction is not too trustworthy.

It can fail

It varies because of road conditions — wet, icy, dry

It's actually a bit hard on car and road.

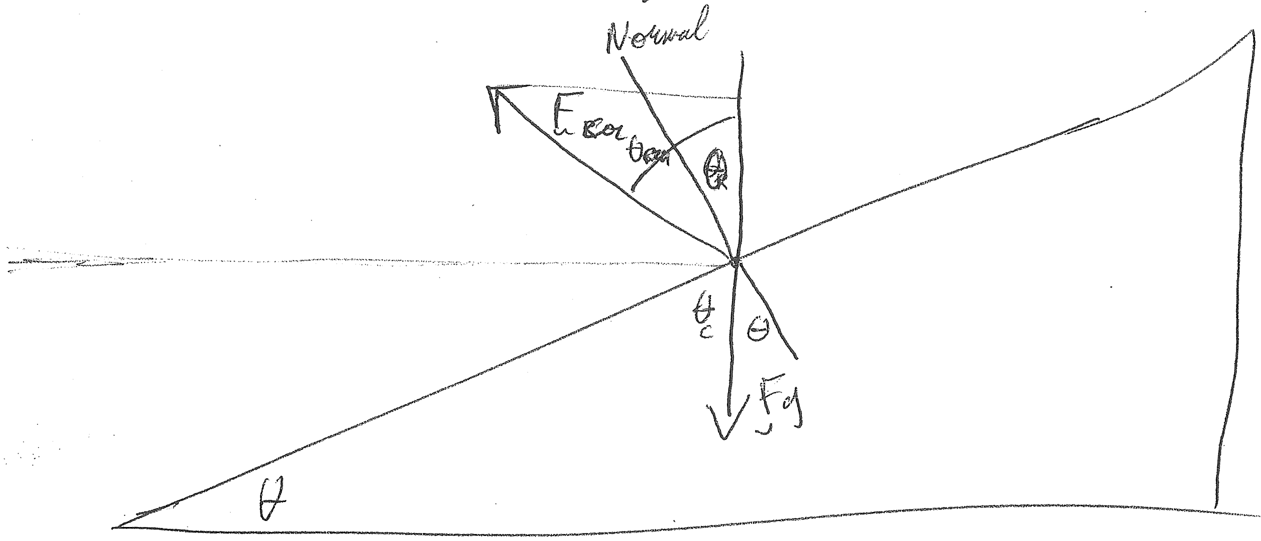
Car is a bit hard to control if strong ~~fric~~ sideways friction is demanded of turned wheels. ? I sure think so.

So road designers would rather not rely on friction at all and one can bank the road so that this is the case.

Consider a banked curve and ~~the~~ let's write

$$\underset{\sim}{F}_{contact} = \underset{\sim}{F}_f + \underset{\sim}{F}_N$$

So we we only consider the net road force.



Now if you are in fact making the bend, there must be a centripetal force (which is the net force of F=ma) ~~only~~ only $\underset{\sim}{F}_{contact}$ has a component in the center direction.

$$\therefore \quad m\frac{v^2}{r} = F_{con} \sin\theta_{con}$$

6-16

There is no acceleration in the ~~y~~ direction.

Thus $ma_y = F_{Ron}\cos\theta_{Ron} - mg$

or $mg = F_{Ron}\cos\theta_{Ron}$

to find θ_{Ron} first divide

$$\frac{v^2}{rg} = \tan(\theta_{Ron})$$

F_{Ron} & m nicely cancel out

or $\theta_{Ron} = \tan^{-1}\left(\frac{v^2}{rg}\right)$

~~or~~ using the 2nd, we have found what θ_{Ron} has to be if the car is in fact making the ~~turn~~ bend.

Now we are road engineers (civil engineers ~~or~~ who are ~~even~~ very very polite engineers) and we adjust or set $\underbrace{\theta}_{\text{bank angle}} = \theta_{Rcon}$

$\therefore \theta = \theta_{Rcon} = \tan^{-1}\left(\frac{v^2}{rg}\right)$

In this case $\underset{\sim}{F}_{Rcon}$ is perpendicular or normal to the road.

Friction which is always parallel to the road can make <u>no</u> contribution to $\underset{\sim}{F}_{Rcon}$

and $\underset{\sim}{F}_{Rcon} = \underset{\sim}{F}_N$.

Friction is never called upon and so can never fail if $\theta = \theta_{Rcon}$

So even is the road were frictionless, the car would still make the bend.

But one must go at v <u>specified</u>.

6-18

~~Now one can't actually have the banking angle~~

Too slow and ~~you~~ some friction is needed to prevent sliding in.
uphill friction

Too fast and some friction is needed to prevent sliding out
downhill friction

Remember the ~~form~~ normal force and the static friction forces are both forces of reaction.

— they are as strong as they need to be — until they fail.

— static friction fails relatively easily — and becomes kinetic friction

— the normal force of a asphalt road doesn't fail easily
the road doesn't deform easily

⟹ it takes a lot of applied force to seriously deform a road.

Our analysis actually applies to all ~~the~~ bits and ~~other~~ occupants of the car.

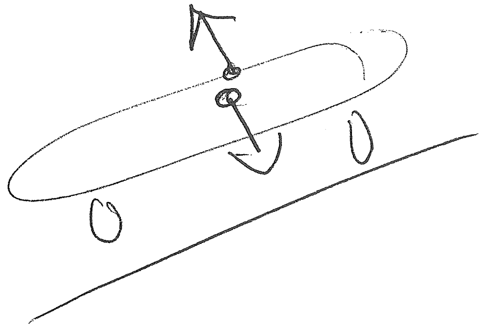
— If you go at the ramp speed (in our ideal case) you don't have any tendency to slide sideways in your seat.

making a bend just pushes you ~~down~~ directly into the seat.

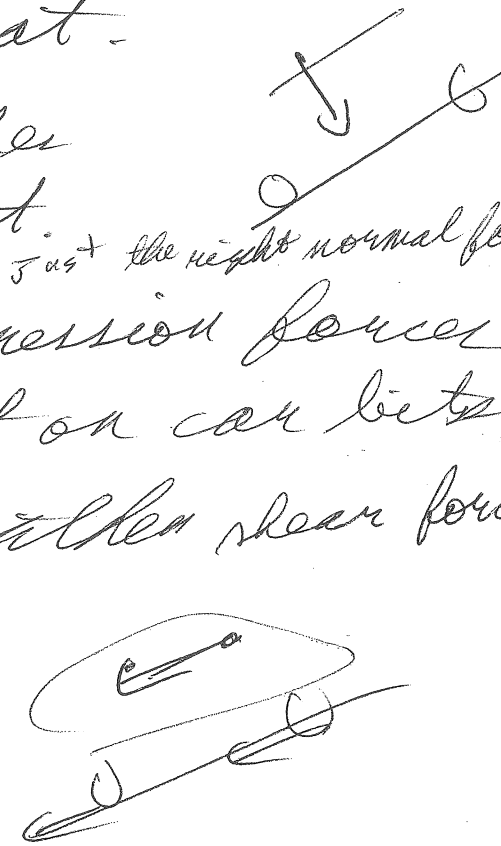
So the seat pushes directly out. ~~giving~~ ~~only~~ just the right normal force

Also only compression forces act on car bits neither shear forces.

at ramp speed



going too fast when shear forces must work.



6-20 — ~~using~~

— shear forces tend to be hard on a car.

Now there really can be only one ramp speed for practical driver reasons.

— but the banking angle θ must vary from 0 to θ_{max} back to 0 in a quasi-continuous way

— So road engineers must vary v from ∞ to v_{min} to ∞

$$\theta = \theta_{Con} = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

in order for the road force to always be normal.

All the above is somewhat idealized

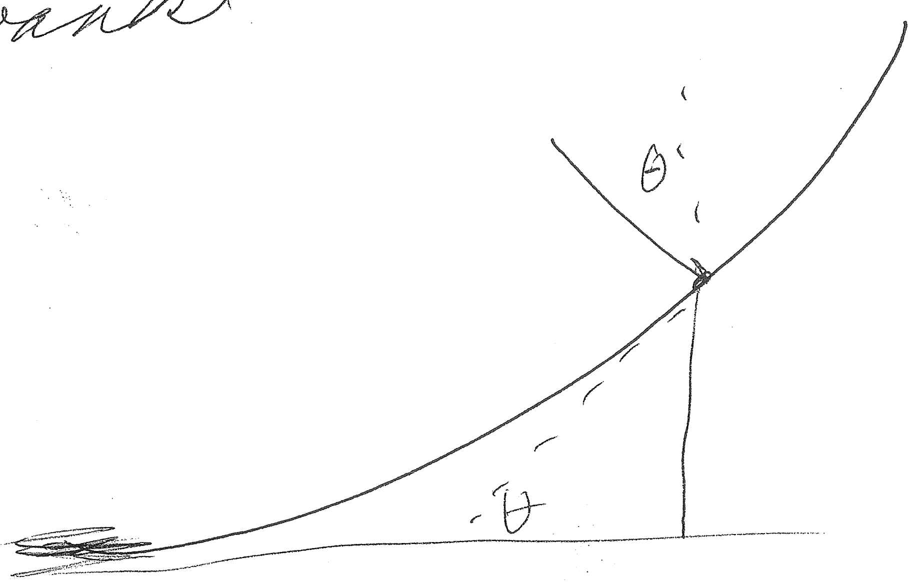
— road engineers must have other considerations but to first order they must use $\theta = \theta_{Con} = \tan^{-1}\left(\frac{v^2}{rg}\right)$ for banking.

There are nearly frictionless end cases actually — not for cars, of course, but ~~for~~ for bobsleds.

— and for bobsleds there is no ~~specified~~ specified speed.

— the bobsledders want to go as fast as possible — or so I guess — maybe the slowest sled wins? I've no idea.

So ~~the~~ <u>curves</u> are continuously bank



— Since there is no friction (ideally), ~~the~~ the sled will slid sideways up or down

6-22

~~until~~ until the road force needed to make the bed is all normal force.

$\therefore \quad \theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$

— higher ~~v~~ v (for fixed r), larger θ, higher up the slope the sled rises.

All this you qualitatively understand when you think ~~of~~ of it. — same for skate boarders, but ~~except that~~ in a wheeled car.

Some values from our earlier example $v = 13.4\,m/s$

$r = 35\,m \qquad g \cong 10$

$$\theta = \theta_{Ban} = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

$$= \tan^{-1}\left(\frac{180}{350}\right) \cong \tan^{-1}\left(\frac{1}{2}\right)$$

$\cong \frac{1}{2}$ radians using the small angle approx.

$= \frac{1}{2}$ rad $60 \left(\frac{deg}{rad}\right)$

$\cong 30°$ Ans $27.6°$

§ 6.2 Non-Uniform Circular Motion

In polar coordinates

$$\underset{\sim}{a} = \left(\frac{d^2 r}{dt} - r\omega^2\right)\hat{r} + \left(2\frac{dr}{dt}\omega + r\frac{d\omega}{dt}\right)\hat{\theta}$$

For circular motion but not uniform

$$\underset{\sim}{a} = \underbrace{(-r\omega^2)}_{\underset{\sim}{a}_c}\hat{r} + \underbrace{r\frac{d\omega}{dt}}_{a_{tan}}\hat{\theta}$$

$a_c = r\omega^2$ but $\omega = \frac{v}{r}$

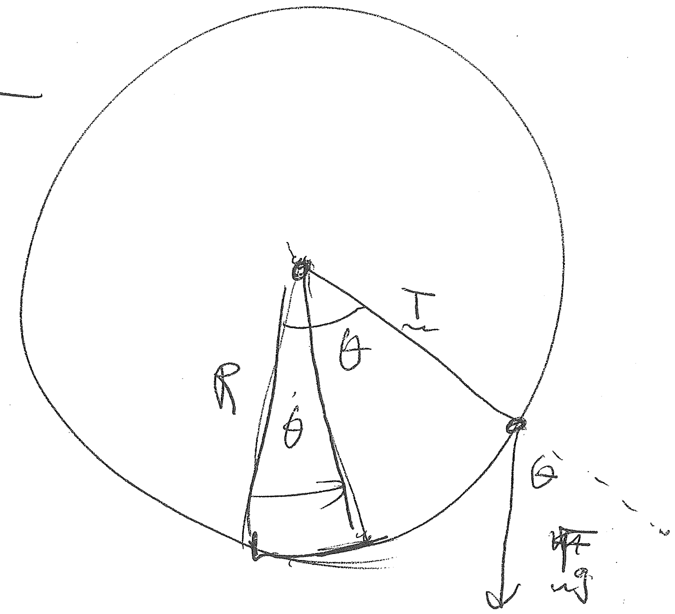
$= \frac{v^2}{r}$ our usual magnitude formula.

6-24

Just one example

Ex 6.6

Swing a bob in a vertical loop with a string.



Measure θ positive from vertical

$$ma = m\frac{v^2}{R} = F_c = T - mg\cos\theta$$

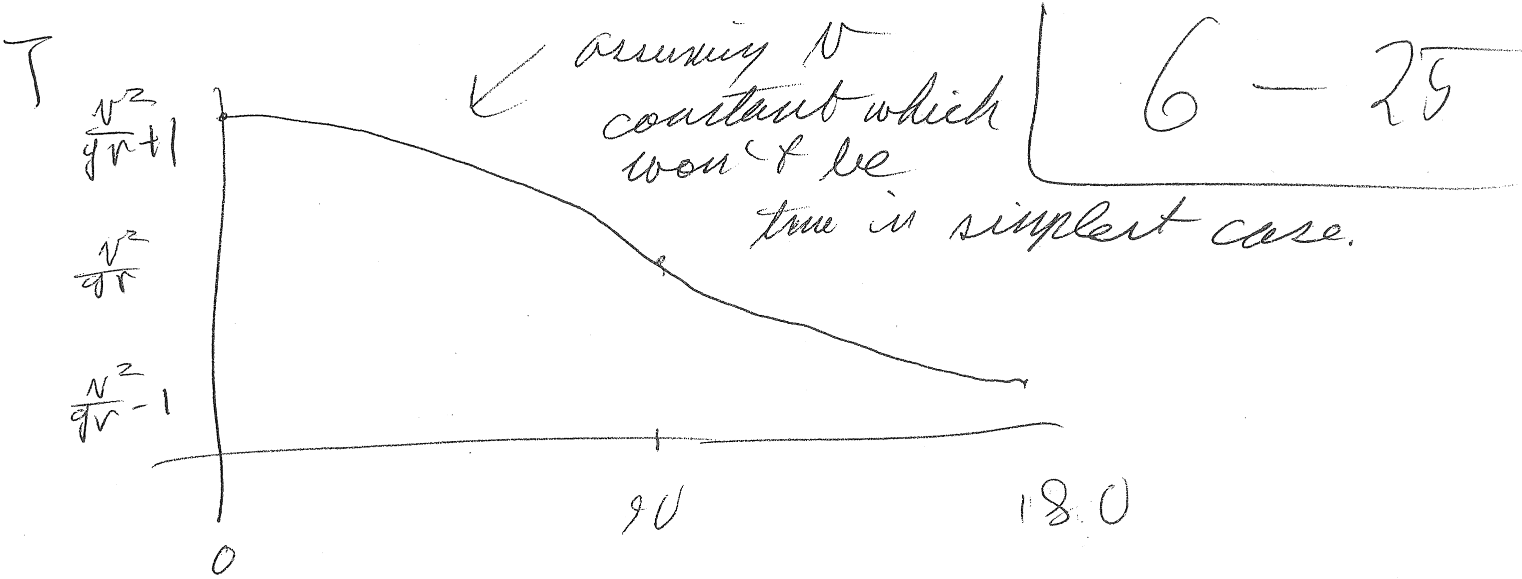
~~taking~~ positive inward for consistency with text.

$$T = mg\cos\theta + \frac{mv^2}{r}$$

$$= mg\left(\frac{v^2}{gr} + \cos\theta\right)$$

Tension force from Newton's 2nd law and gravity law.

v varies in this case



No problem if $T \geq 0$ always.

But what if $T < 0$?

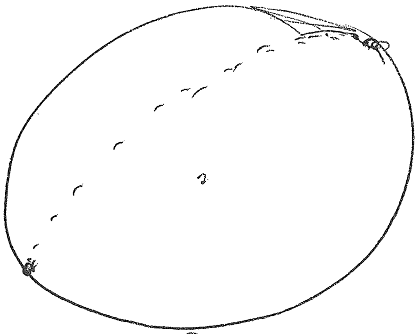
Well in deriving our tension rule we assume tension would always be adequate to circular motion.
— this is true for stretching force for our ideal string.

But ~~one~~ can't push on a string (at least an ideal one).

When T goes to zero the bob will go into <u>parabolic flight</u>.

6-26

Something like this



~~symmetrical~~

~~v_0 constant~~

~~$v_y^2 = v_i^2 + 2a\Delta y$~~

~~$v_y = v_i$~~

~~for $\Delta y = 0$~~

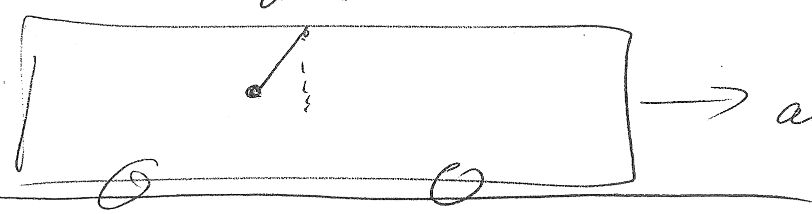
Leaving and rejoining the circular path is NOT symmetrical in general

Proof is tricky
See SJ-144

§ 6.3 Non-Inertial Frames & Inertial forces

We already considered the elevator case in Ch. 5 notes (I went a bit beyond SJ there.)

Ex 6.7 Train case or elevator on it's side.



→ accelerating train

$\underline{F}_{net} = m\underline{a}$

always true but referenced to an inertial frame.

In an accelerating train –

– the y direction is normal

6-27

First assume $\underset{\sim}{a}_{train} = 0$

Then for an object $\rightarrow$ $\underset{\sim}{v}''_{relat} = \underset{\sim}{v}_{ground} - \underset{\sim}{v}_{train}$

$\underset{\sim}{v}'_{rel} = \underset{\sim}{v}_{ground} - \underset{\sim}{v}_{train}$

$\underset{\sim}{a}'$ relative to train $=$ $\underset{\sim}{a}$ relative to ground

Not derivatives here, but prime for other frame.

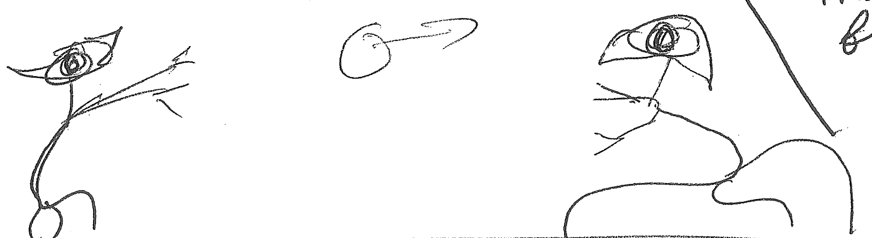
same

$\underset{\sim}{F}_{net} = m\,\underset{\sim}{a}_{rel}$

is fine either relative to train or ground.

These means all forces give the same acceleration in ~~both~~ ground and train frame.

Say you play toss



$\underset{\sim}{v}'_{rel} = \underset{\sim}{v}_{ground} - \underset{\sim}{v}_{train}$

for all objects.

calculate all kinematics in train frame.

Everything is perfectly normal.

and $\underset{\sim}{a}'_{rel} = \underset{\sim}{a}_{train}$

$\underset{\sim}{v}'_{rel} = \underset{\sim}{v}_{ground} - \underset{\sim}{v}_{train}$

or $\underset{\sim}{v}_{ground} = \underset{\sim}{v}'_{rel} + \underset{\sim}{v}_{train}$.

6-28

So all mechanical experiments like toss are the same in train as relative to ground. F=ma predicts the same results, same kinematics except for additive constants of integration.

To get ground ~~~~ velocities, just add $\underset{\sim}{v}_{train}$.

Review your whole life experience in trains, planes, and cars and boats

when these ~~accels~~ don't accelerate relative to the ground everything is normal.

Remember play toss with your little in the back seat thing.

But when the train ~~one~~ accelerates things are a bit different.

Say it is all horizontal x acceleration — so nothing funny in y-direction.

a_{xt} and nothing forbids

as from ~~~~, $a_x = a'_x + a_{xt}$

or $a'_x = a_x - a_{xt}$

Relative acceleration.

Now $F_{x\,net} = m a_x = m(a'_x + a_{xt})$

$$\underbrace{F_{x\,net} - m a_{xt}} = m a'_x$$

define as a new field force $F_{inertial} = -m a_{xt}$ — like a sideways gravity field force.

$$F'_{net} = m a'_x$$

A new inertial or fictitious force.

just $F = ma$ relative to train with the new fictitious or inertial force $-m a_{xt}$

Say $F_{net} = 0$ or $F'_{net} = -m a_{xt}$ only.

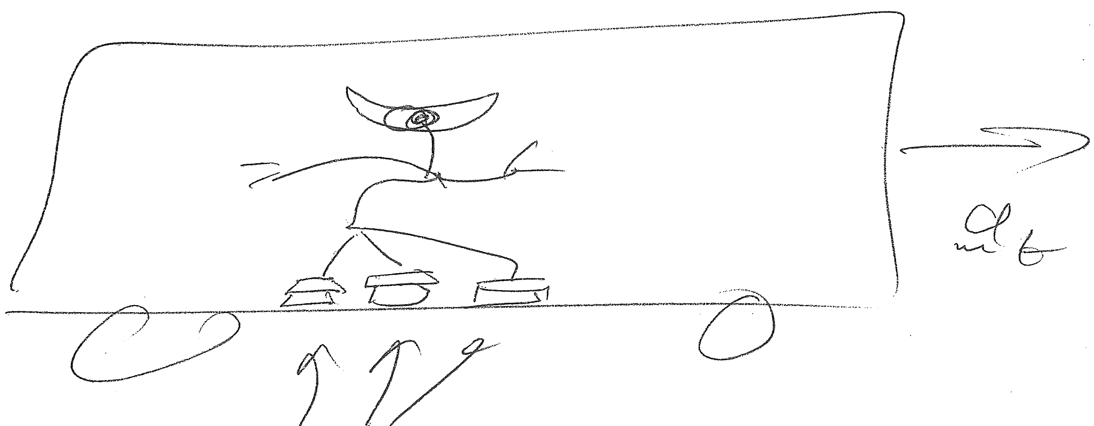
Then $-m a_{xt} = m a'_x$

$$a'_x = -a_{xt}$$

So an object without any real forces would accelerate

6-30) in the train frame
even though unaccelerated
in the ground frame.

— but it's a field force
~~not a b~~ or a body force.
— it "pulls" atom by
atom



frictionless ice

— the alien would slide back
freely ~~with~~ $a_x' = -a_{xb}$

— no stress or strain as long
as he doesn't try to resist.

— if he does it's like fighting
sideways gravity.

6-31

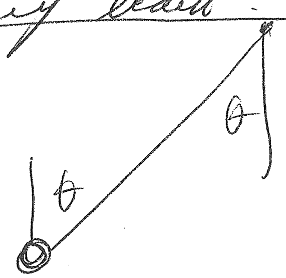
You know the sideways gravity effect from being in ~~cars~~ cars when they accelerate.

— you don't feel the inertial force → there is no force to feel

What you feel is the stress/strain of accelerating yourself to go with the train.

Hanging object at rest in an accelerating train.

done problem before on a truck.



In train frame

$$\underset{\sim}{a}' = 0$$

prime relative to frame

$$\sum F_x' = T\sin\theta - m a_{xt}' = 0$$

$$\sum F_y' = T\cos\theta - mg = 0$$

Just divide to get.

$$\tan\theta = \frac{a_t'}{g}$$

6-32

$$\theta = \tan^{-1}\left(\frac{a'_{xt}}{g}\right)$$

ou
ould
ve
ne
his
nalysis
irtually
he same
n the
ground
frame.

→ as in a
hmk problem

Say $a_{xt} \cong 2\,m/s$

$g = 9.8\,m/s^2 \cong 10$

$\therefore \theta \cong \tan^{-1}\left(\frac{2}{10}\right) = \tan^{-1}(.2)$

$\cong .2\text{ rad}$ in small angle approximation

$\cong 12°$

Rotating Frames are trickier

Top view.

Non-Inertial Frames

To make $F = ma$ work in a ~~rotating~~ frame requires two ~~[illegible]~~ inertial forces.

One is easy to derive

$$\underset{\sim}{a}'_{rel} = \underset{\sim}{a} - \underset{\sim}{a}_c$$

centripetal acceleration

$$\underset{\sim}{F}_{net} = m\underset{\sim}{a} = m\underset{\sim}{a}_{rel} + m\underset{\sim}{a}_c$$

Define $\underset{\sim}{F}'_{net} \equiv \underset{\sim}{F}_{net} - m\underset{\sim}{a}_c = m\underset{\sim}{a}'_{rel}$

$$\underset{\sim}{a}_c = -\frac{v^2}{r}\hat{r}$$

Not a general formula Really just for confined to a pla

$\therefore$ Define $\underset{\sim}{F}'_{inertial} = -m\underset{\sim}{a}_c = m\frac{v^2}{r}\hat{r}$

This is the <u>centrifugal force</u>.

– the force ~~that~~ tries to throw you off merry-go-rounds and <u>carnival centrifuges</u>.

If you are at rest in the rotating frame, forces are balanced and $\underset{\sim}{F}'_{net} = \underset{\sim}{F}'_{centripetal} + \underset{\sim}{F}'_{centrifugal}$
$= 0.$

6-34

Actually our derivation above is ~~slightly faulty~~ due to subtle reasons.

omit?

$$\underset{\sim}{a}'_{rel} = \underset{\sim}{a} - \underset{\sim}{a}_c$$

is the relative acceleration

but it is not the acceleration measured in the rotating frame

in a sense it is what one thinks of when one defines relative acceleration

— rotating frames are tricky.

When you do the analysis carefully (which we won't do and most books seem to muff it — it takes a few pages, Symon 3rd, 2?, does ... — get the right result but ~~logic~~ obvious reasoning is faulty it seems to me. French 571 muffs it.) you find

$$\underset{\sim}{a} = \underset{\sim}{a}' + \underset{\sim}{a}_{centripetal} + \underset{\sim}{a}_{coriolis}$$

relative to inertial frame

acceleration measured in rotating frame

centripetal acceleration

Coriolis acceleration that depends on $\underset{\sim}{v}'$ the velocity in the rotating frame.

If the body is at rest in the rotating frame,

$$\underline{a}_{coriolis} = 0$$

$$F_{centrifugal} = -m a_{centripetal} = m \frac{v^2}{r} \hat{r}$$

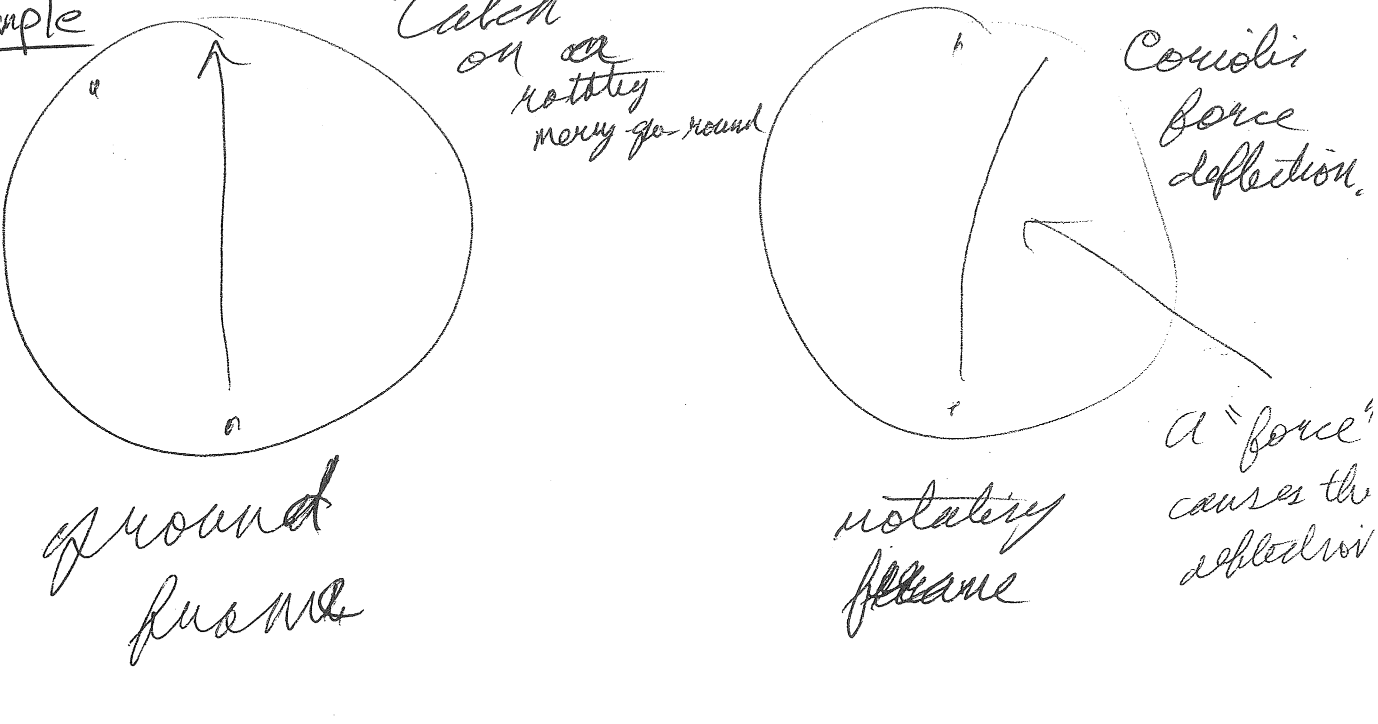
$$F_{coriolis} = -m \underline{a}_{coriolis}$$

I won't ~~rep~~ write it out — it's not so bad, but it's beyond our scope.

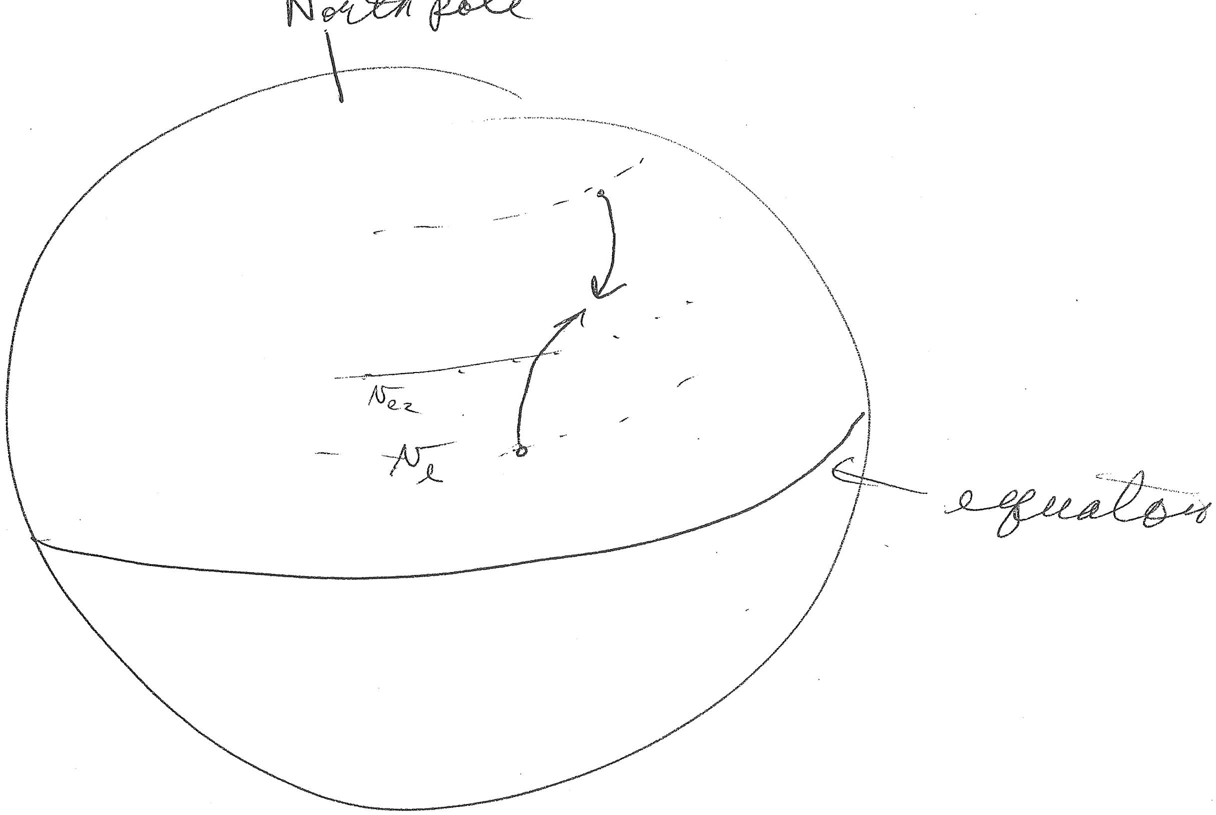
But the Coriolis force in action is easy to notice and understand in simple cases.

Example

Playing Catch on a rotating merry-go-round



6-36) The Coriolis force is important in climate and weather



$v_{e2} < v_{e1}$

ω is the same at all latitudes, but

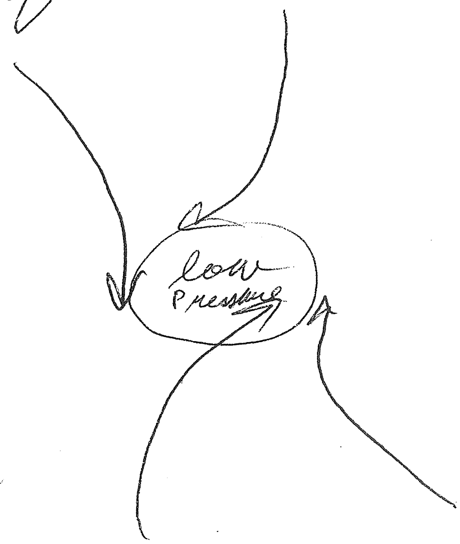
so a wind blowing north tends to deflect to the east and one blowing south deflects to the west.

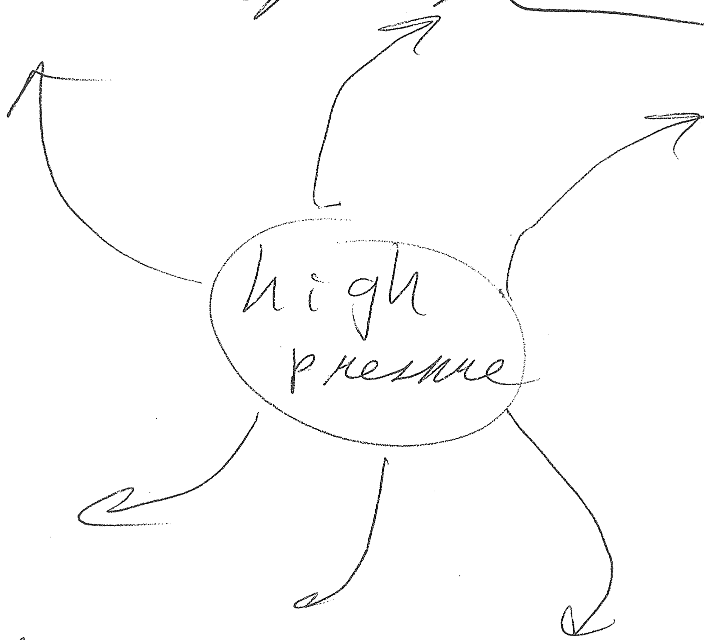
This gives rise to cyclones and and anticyclones in weather.

Northern hemisphere

Cyclones

anticyclones





swirl counterclockwise

clockwise swirl.

most powerful are hurricanes/typhoons

In the southern hemisphere everything is mirrored. (or mirror image).

6-38

§6.4 Motion in resistive Media or Drag

i.e., fluids which are comprised of both liquids and gases

The resistive ~~forces~~ oppose motion and are usually velocity dependent.

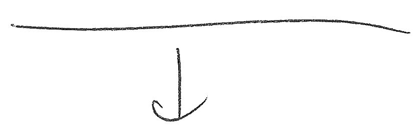
velocity relative to fluid. Object moving or fluid moving relative to outside world

in detail in a very complex fashion that depends on velocity, shape, orientation, texture, rigidity of moving body.

For orientation effect consider a sheet of paper dropped

this way

or this way.



But there are a couple of very useful approximate laws —— that in some cases are quite exact.

They are simple enough to allow~~ed~~ some analysis of how ~~they~~ resistance affects motion

6-40

(French 210 ff

Case 1: A linear resistive drag ~~force~~

(Stokes Drag)

$$\underset{\sim}{F}_R = -b\,\underset{\sim}{v}$$

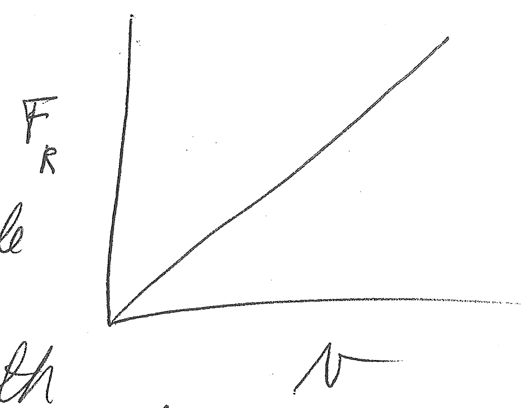
velocity of body

points opposite to motion

a constant depending on shape, orientation, texture, ... of object

$b > 0$.

The resistive force magnitude increases linear with ~~vel~~ speed.



This force law is most appropriate for relatively low speeds of small objects.

To analyze how this force

Consider a body of mass m sinking under the gravitational force.

Say y down is positive

~~roller~~

– no forces in the x-direction.

We will neglect the buoyancy force which for simplicity $F_b = m_{dis} g$ upward Serway -396 which just effectively changes ~~gravity~~ g, mass $(m - m_{dis})g$ & g actual

$$\sum_i F_{y_i} = mg - bv = ma$$

or

Note b is mass independent but $D = \frac{b}{m}$

In fact $b \sim$ Area cross-sectional
$b = \rho A_{c} l_{m}$ ⊥ to motion direction
$b \propto A_c$

For objects of similar … e.g. … $\propto V$
$D = \frac{b}{m} \propto \frac{A}{m} = \frac{1}{\rho\, l_m}$
resistive effect $\propto \frac{1}{\rho}$
bigger density, the smaller it

$$ma + bv = mg$$

$$\frac{dv}{dt} + \frac{b}{m} v = g$$

D has units of inverse time so that $\frac{dv}{dt}$ and Dv have same units.

B … does … change … inverse mass

This for some of you may be your first example of a differential equation (DE)

which is … nontrivial

6-42

An equation whose solution is <u>not</u> a number, but a function (e.g. $v(t)$) of an independent variable (e.g., t)

and where the function and ~~hup~~ various orders of derivatives appear in the equation.

DEs are in fact everywhere in physics and in life — In fact much of the numerical science and engineering amounts to solving DEs.

→ including economics (not that you'd ever know it)

This is a 1^{st} order DE — since the highest derivative of v is $\frac{dv}{dt}$ the 1^{st} order derivative.

— it is also linear

since v and $\frac{dv}{dt}$

occur linearly in the DE.

— It has a driven term g which is not multiplied by v or a derivative

This DE is analytically solvable.

$$\frac{dv}{dt} + Dv = g .$$

We'll do that solution (and go just a bit beyond the text.)

But first a few simple results.

~~W~~We will assume the object starts from rest.

~~so $v=0$~~ $v(t=0)=0$

and thus $\frac{dv}{dt} \cong g$ for t small somehow.

The g causes the motion that the resistive force opposes.

If $D=0$, the DE would reduce

$\frac{dv}{dt} = g$

$v = gt$ assuming the object starts from rest.

a trivial DE

$\therefore \; v \cong gt$

for t small

just ~~gravity~~ acceleration under gravity.

But as the body accelerates v grows and so does

$$Dv.$$

At some point in time one reaches a terminal velocity v_T such that

one might guess where $\frac{dv}{dt} = 0$

$$Dv_T = g$$

$$v_T = \frac{g}{D} \qquad \frac{1}{D} = \frac{v_T}{g}$$

~~Then $\frac{dv}{dt} = 0$ in the DE.~~

~~No further acceleration happens.~~

~~One would guess~~

— the

The DE is then solved if v stays constant at v_T

$$\frac{dv}{dt} = 0 \quad \text{and} \quad D v_T = g.$$

The actual solution of the DE verifies if and when this happens.

Omit ?

$\boxed{\text{Ok let's solve the DE}}$

$$v' + Dv = g$$

The trick is to use an integrating factor f which is initially unknown. It's a function of t.

$$f v' + f D v = f g$$

We demand that

$$\underbrace{(f v)'}_{f v' + f' v} = f v' + f D v$$

by chain rule.

6-46

So our demand becomes

$$\cancel{f' v + D}$$

$$f' v = f D v$$

or
$$f' = f D$$

$$\frac{df}{dt} = f D$$

$$\frac{df}{f} = D\,dt$$

and integrate both sides

If you haven't seen this procedure before just accept it as a forerunner

$$\ln f = Dt + \text{Const}$$

$$e^{\ln f} = e^{Dt + \text{Const}}$$

$$f = e^{Dt}$$

where we drop the constant since the overall scale of f is irrelevant

e is the exponential e number or Euler number or base of natural logarithms. It's irrational (bad, mad, and dangerous to know) e=2.71828... never repeats - can't be expressed as fraction

Now we have f

$$(f v)' = f g$$

integrate both sides with respect to t

$$f v \Big|_0^t = g \int_0^t e^{Dt'} dt'$$

$$e^{Dt} v = g \left. \frac{e^{Dt}}{D} \right|_0^t$$

$$= \frac{g}{D}\left(e^{Dt} - 1\right)$$

$$\boxed{v = \underbrace{\frac{g}{D}}_{\text{our } v_T}\left(1 - e^{-Dt}\right)}$$

We can define $\tau = \frac{1}{D} = \frac{v_T}{g}$

time constant
or e-folding time in the
jargon I use.

$$v = v_T\left(1 - e^{-t/\tau}\right)$$

v_T for $t \to \infty$ only

$v_T\left[1 - \underbrace{(1 - t/\tau)}\right]$

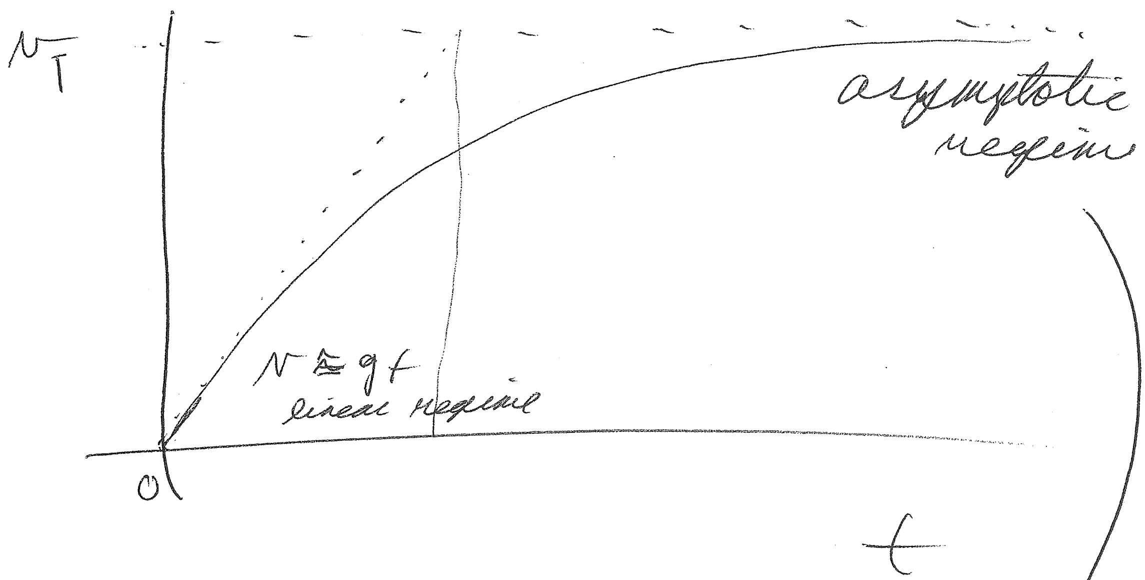
$= v_T\, t/\tau$

$= \frac{g}{D} D t$

$= g t$ which is the small t velocity behavior we anticipated.

1st order Taylor expansion for small t
$t/\tau << 1$

6-48



Characteristic means characterizing. — No precise meaning. — there is no sharp change in behavior between the two regimes

The object formally never reaches v_T in finite time. ~~At~~ v just becomes asymptotic to v_T — always approaching.

The characteristic transition time between the two regimes can be set

at $t = \tau$

where $e^{-t/\tau} = e^{-1}$

Note if one just had linear increase then

at $t=\tau$, $v_{linear} = g\tau = \frac{g}{D} = v_T$

Actual $v(\tau) = v_T[1 - e^{-1}] \cong v_T * .632$

You may wonder what ~~[illegible]~~ v_T actually means ~~that~~ in realistic not so idealized cases.

— In real cases there are always velocity perturbation of some size δv.

Small Greek delta δ — meaning usually small change in

At some point the formal solution will have

$$|v_T - v| \leq \delta v$$

and after this one can say the velocity has reached v_T for practical purposes.

<u>Or</u> if $|v_T - v|$ is smaller ~~than~~ than some value you care to as a standard of ~~of~~ smallest significance,

then for practical purposes velocity has reached $v_{terminal}$ for your purposes.

ually in
few e folding
times.
$-t/\tau \doteq 10^{-.43\, t/\tau}$
o in $t = 10\tau$, down by $10^{-4.3}$
d $t = 100\tau$, down by 10^{-43}
lich
e
niscule
,
most
ny
ceivable
ose.

Ex 6.8

$m = 2g = .002kg$

in oil and reaches $v_T = 5$ cm/s

determine τ and t such that $v = .9 v_T$?

$$v_T = \frac{g}{D} = g\tau \text{ and so } \tau = \frac{v_T}{g}$$

$$= \frac{5 \text{ cm/s}}{980 \text{ cm/s}^2}$$

$$\cong .005 s$$

Ans $.0051$

$$v = v_T\left[1 - e^{-t/\tau}\right]$$

solve for t

$$\frac{v}{v_T} - 1 = -e^{-t/\tau}$$

$$e^{-t/\tau} = 1 - \frac{v}{v_T}$$

$$t/\tau = -\ln\left(1 - \frac{v}{v_T}\right) = -\ln(1 - .9) = -\ln .1$$
$$= \ln 10$$

$$t = \tau \ln 10$$
$$\cong \tau * 2.3$$
$$= 11.5 \times 10^{-3}\, s$$
$$\cong 10^{-2}\, s \quad \text{Ans } 11.7\, ms$$
$$= 10\, ms$$

ctually though
t follows from
ernoulli eqn.
enway-403, Batchelor
539

Case 2

For objects moving at higher speeds in fluids like air — reaching higher v_T in fact the resistive force is more like a quadratic in speed.

Blunt objects
~~and~~ – ball
not Javelin
– air becomes
turbulent around
the object. – breaks
~~HRW~~ up into
RW-104 trailing
swirls

$$F_R = b v^2$$

$$b v^2 = \frac{1}{2} D \rho A v^2$$

density of fluid e.g. Air

drag coefficient property of medium

cross sectional projected area of object perpendicular to motion

$$M \frac{L}{T^2} = [D] \frac{M}{L^3} L^2 \frac{L^2}{T^2}$$
$$= [D]\, M L / T^2$$

So D is dimensionless
HRW give
D typical
values
of .4 to 1
for air

$$D = \frac{b}{m}$$
$$\propto \frac{A}{m} = \frac{A}{\rho A l_{th}}$$
$$\propto \frac{1}{\rho\, l_{th}}$$

for objects
of similar
~~shape~~ l_{th}
the resistive
effect decreases
as $\rho\, l_{th}$
~~and density~~.

6-52 This is just the magnitude — the force opposes the direction of motion

~~So~~ Again we first consider the illustrative case of sinking starting from rest (neglecting the buoyancy force)

$ma = F_{net}$

$\therefore \quad ma = mg - bv^2$

$a + \frac{b}{m} v^2 = g$

$\frac{dv}{dt} + D v^2 = g$

where $D \equiv \frac{b}{m}$ again.

but here $\quad [D v^2] = \frac{L/T}{T}$

$[D]\left(\frac{L}{T}\right)^2 = \frac{L/T}{T}$

$[D] = \frac{1}{L}$

except for v^2 instead of v this DE is like the last one.

But ~~that~~ v^2 makes a huge difference.

This is a non-linear DE because the solution or some derivative of it enters non-linearly.

Most non-linear DEs ~~do~~ NOT have analytic solutions

i.e., exact simple formula solutions.

They can be solved with certain approximations or numerically but there is always some error although it can often be made as minute as you like for practical purposes.

Omit

I've actually worked out an approximate interpolation formula for v which I'd guess is pretty accurate, but it's too much bother to test ~~and use~~ right now.

(See my own notes on SJ-I-18

6-54)

But we can still extract some partial information.

Since the object starts from rest $v = 0$ initially and we again expect

$$\frac{dv}{dt} \simeq g \quad \text{at early times}$$

or

$$v \simeq gt$$

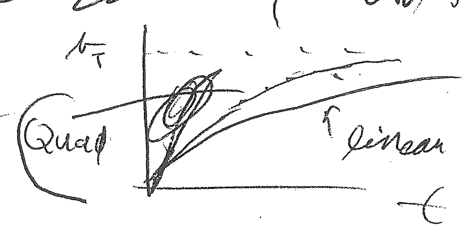
As v grows the resistive force grows and qualitatively we {expect / guess} the same asymptotic approach to a terminal velocity v_T

$$\frac{dv}{dt} = 0\,, \quad D v_T^2 = g$$

$$v_T = \sqrt{\frac{g}{D}}$$

or

It can be shown that this is true, but I'll forbear



When would we guess a transition between the linear regime and asymptotic regime.

Well the linear regime must be over when when linear growth would take the velocity to v_T

$$\therefore \quad v_T = g\, t_{characteristic}$$

$$t_{ch} = \frac{v_T}{g} \simeq \frac{v_T}{10\frac{m}{s^2}} \left\{ \frac{L/T}{L/T^2} = T \right.$$

$$y_{ch} = \tfrac{1}{2} g t^2 = \tfrac{1}{2} g \frac{v_T^2}{g^2} = \tfrac{1}{2} \frac{v_T^2}{g}$$

Omit? or leave as question.

Or another way — the One step numerical solution of a DE.

$$a \simeq \frac{v_T - 0}{t_{ch_2}} = g - D\left(\frac{v_T}{2}\right)^2$$

$$t_{ch_2} = \frac{v_T}{g - D\left(\frac{v_T}{2}\right)^2}$$

$$= \frac{v_T}{g - \frac{g}{v_T^2}\left(\frac{v_T}{2}\right)^2}$$

$$= \frac{t_{ch}}{1 - \frac{1}{4}} = \frac{4}{3} t_{ch}$$

The same number ~~to~~ ~~out~~ roughly speaking.

Terminal velocities in Air

— typical values HRW-105 ↳ ball from vest. " not from high speed

Object	v_T (m/s)	t_{ch} (s)	95% y (m)	y_{ch} (m) $= \frac{1}{2} v_T^2/g$
Shot (from shot put)	145	14.5	2500	
Skydiver (typical	60 = 216 km/h ≙ 140 mi/h	6	430	180
baseball	42	4.2	210	
Tennis ball	31	3.1	115	
ping pong ball	9	.9	10	
raindrop	7	.7	6	2.5
parachutist	5	.5	3	
cat ~~tucked~~ with feet extended to soften	27 = 97 km/h	3		36
cat spread	19 = 68 km/h	2		18

HRW says y for ≈ 18 m = 6 ~~fl~~ 3

HRW-104

As the result of extensive (not to mention sadistic) experiments on cats

— when first thrown off a tall building — actually cats usually jump (they're just so smart you know)

right themselves

cat Righting Reflex.

— they first ~~tuck up~~ & extend feet down to cushion landing
↳ they notice the acceleration
↳ must be from the ~~drag~~ resistance force which is a contact force

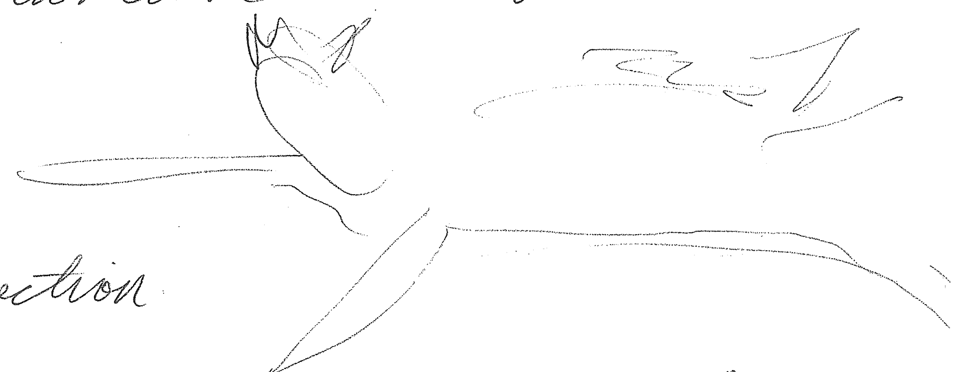
6-57

— then after they reach

v_T ~~tucked~~ pounce $= 27$ m/s after ~18 m or 37 m?

they feel safer and stretch out and reduces v_T to $v_{T\,\text{stretched out}} = 18$ m/s

which is much smaller than human 60 m/s.



must be the cross-section

$$D \propto \frac{A}{m} = \frac{A}{\rho V} = \frac{l^2}{\rho\, l^3_{th}} = \frac{1}{\rho\, l_{th}}$$

a Cat is ~ $\frac{1}{3}$ of a human in length

Cat l_{th} may be a bit smaller than human.

— all that ~~other~~ fur and skin folds may make a difference.

Mik claims it's a natural reflex if they have time which makes more sense to me

cat righting reflex,

6-58

- so cat decelerates to $v_T = 20$ m/s

- as ground approach it sees ~~that~~ and and pulls legs in for a ~~softish~~ landing (to pounce mode)

It doesn't always work, but apparently ~90% of cats who fall to 2-32 stories survive → their injuries are less if they fall more than 6 stories ~ 18 m (Wik)

→ of course this statistic may not count many who simply died and no one bothered to take them to ~~vet~~ — "let's just buried Fluffy in an old shoe box."