2026aug01 really

D2_19001

# Chapter 19 Inductance

02_14002

U2_14003

# 14.1 Recapitulation Faraday's Law of induction and The Inductance of a Solenoid

a) Faraday's Law of Induction

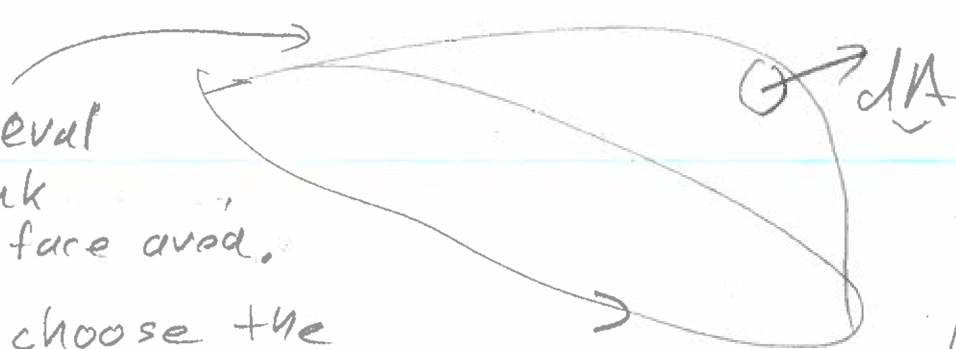
$$\mathcal{E} = \oint \underset{\sim}{E} \cdot d\underset{\sim}{l} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int_{\text{Linked}} \underset{\sim}{B} \cdot d\underset{\sim}{A}$$

Can be called emf even there is no free charge to move

The line integral can also be called the circulation of the induced E-field around closed loop

Faraday Law Induced E-field NOT due t

$\Phi_B$ is the magnetic flux through a linked area




general link surface area.

You choose the convenient one for a calculation.

They all give the same answer by "Flux conservation".

the sense of the differential area vector is determined by curling right-hand fingers in fiducial current direction and the thumb gives the sense.

02-14004

The statements embody Lenz's law which allows us to determine the sense without being careful about signs

The minus sign tells us that the induced emf $\mathcal{E}$ opposes the change in B-field that caused the emf $\mathcal{E}$.

Corollary: The B-field is caused by a current $I$, then the emf $\mathcal{E}$ opposes changing current: the sense of the emf is

$$\mathcal{E} < 0 \quad \text{if} \quad \frac{dI}{dt} > 0$$

$$\mathcal{E} > 0 \quad \text{if} \quad \frac{dI}{dt} < 0$$

b) Say the B-field is due to a current confined to a thin wire (ideally an infinitely thin wire), then

$$\int \underset{\sim}{B} \cdot d\underset{\sim}{A} = \underbrace{L \, I}$$

Where $L$ is the inductance and depends on geometry and, in general, material properties of any medium containing the B-field.

$\mu_0$ is the vacuum permeability. In general, permeability of magnetization of the medium. We leave this topic sine die.

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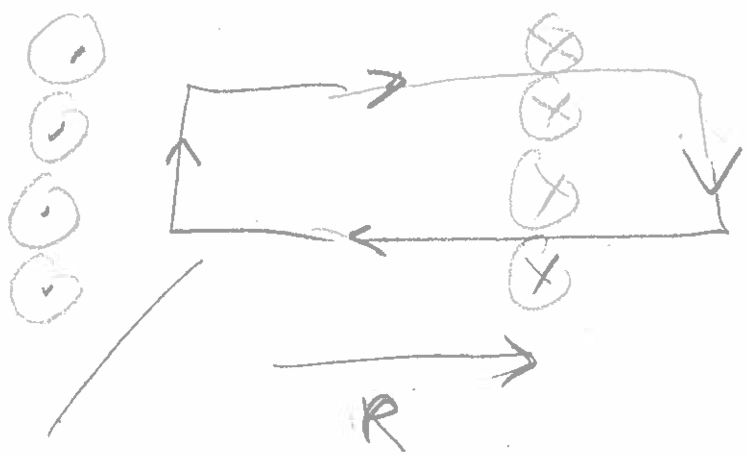
For the inductance case

$$\mathcal{E} = -L\frac{dI}{dt}$$

General rule for devices called inductors

d) For ideal infinite solenoid




Given that symmetries and other arguments tell us

$$\underset{\sim}{B} = B(r)\hat{z} \begin{cases} >0, & r<R \\ =0, & r>R \end{cases}$$

Amperian loop

$$\oint \underset{\sim}{B}\cdot d\underset{\sim}{l} = \mu_0 I_{total}$$

$$\underbrace{B\,l}_{\text{length of z-direction side of the Amperian loop}} = \mu_0 \underbrace{N}_{\text{turns inside Amperian loop}} I$$

$I$ ← current in an individual wire

$$\therefore \underset{\sim}{B}(r) = \begin{cases} \mu_0 n I\,\hat{z} & \text{inside with } n=\frac{N}{L} \\ 0 & \text{outside} \end{cases}$$

From p. Wb2_13008 $\mathcal{E} = -\mu_0 n^2 \mathcal{V}\frac{dI}{dt} = -L\frac{dI}{dt}$ for solenoid, and so $L = \mu_0 n^2 \mathcal{V}$ where $\mathcal{V}$ = volume of [illegible]

# 14.2 The R.C Circuit without a Driver

A warm-up for the LC circuit

A pure RC circuit is (i.e., a power source)

C capacitor

R resistor

Recall

$U = \frac{1}{2}\frac{Q^2}{C}$

$= \frac{1}{2}CV^2$

for a capacitor

## Initial and Final Conditions

i) For the capacitor, $Q(t=0) = Q_0$ since we assume it is charge to $Q$.

And $Q(t \to \infty) = 0$ since the capacitor is fully discharge and its energy is dissipated to heat in the resistor.

ii) For the resistor, we apply Kirchhoff's voltage law $\frac{Q}{C} = IR$ (Rise, Fall) to time zero and get

$$I_0 = \frac{Q_0}{RC} = \frac{Q_0}{\tau}$$

where $\tau$ is the time constant or e-folding time as before on p. 02_09050 ff.

Note, also as before, there is an instantaneous start to current in the quasi-steady approximation.

The time-varying E-field establishing the guiding charge is just so fast.

And $I(t \to \infty) = 0$ when all capacitor charge is discharged and we have neutrallity everywhere.

Solution of

$$\frac{Q}{C} = IR$$

which is an integro-differential equation.

To Solve take the derivative which gives

$$\frac{1}{C}\frac{dQ}{dt} = \frac{dI}{dt}R ,$$

but there is a difference from p. 02_04050.

Here $-\frac{dQ}{dt} = I$ since $\frac{dQ}{dt}$ is decreasing

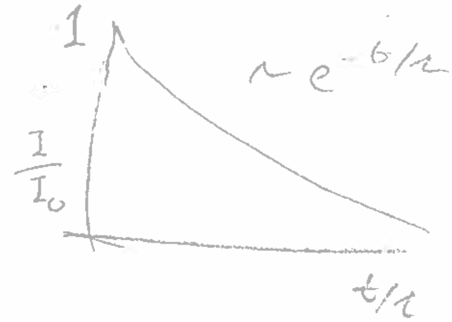
$$\therefore \quad \frac{dI}{dt} = -\frac{1}{RC} I = -\frac{1}{\tau} I$$

where $\tau = RC$ is again the time constant or e-folding time.

02-19008

$$\frac{dI}{I} = -\frac{1}{\tau}\,dt$$

$$\ln[I/I_0] = -\frac{t}{\tau}$$




$$I = I_0 e^{-t/\tau}$$

which is NOT surprising since it just to p. 02_09051.

However $$Q = \int_0^t (-I)\,dt + Q_0$$

$$Q = -\tau I_0(1 - e^{-t/\tau}) + Q_0$$

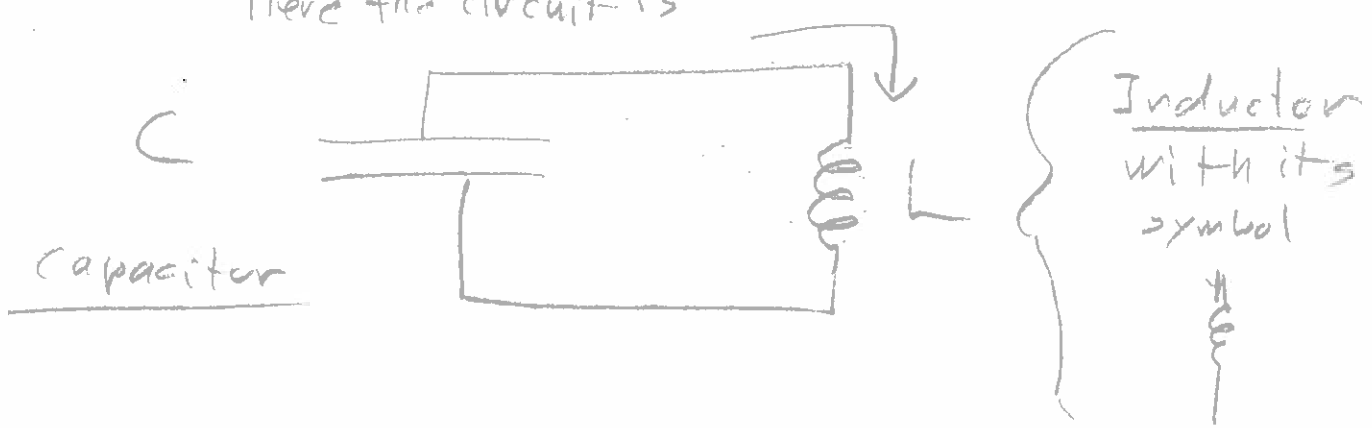
$$Q = -Q_0(1 - e^{-t/\tau}) + Q_0$$

$$Q = Q_0 e^{-t/\tau}$$

and so Q declines, NOT increases as on p 02_04043.

## 4.3 The LC circuit without a Driver

Here the circuit is



# Initial and Final conditions

For Q-capacitor, $Q(t=0) = Q_0$

and $Q(t \to \infty) = ?$

For the current, $I(t=0) = I_0 = ?$

In fact, $I_0 = 0$. In this case, the inductor prevents an instantaneous jump from zero to finite current.

What of final conditions? There is no way to dissipate energy. So one could expect some perpetual oscillation in a potential energy landscape

Capacitor stores energy in potential of charge and inductor stores energy as magnetic field energy


$E_B = \frac{1}{2\mu_0} B^2$

energy density

Kirchhoff's voltage law

$$\frac{Q}{C} = L I'$$

$$\frac{1}{C} \frac{dQ}{dt} = L I''$$

and again differentiate to turn into a differential eqn.

and as on p.02_19007

$$-\frac{dQ}{dt} = I$$

$$\therefore \quad L I'' = \frac{-1}{C} I$$

A 2nd order ordinary linear DE. Since 2nd order there are 2 independent solutions

$$I'' = -\frac{1}{LC} I$$

which is the simple harmonic oscillator DE

and the resonance (angular) frequency

$$w = \frac{1}{\sqrt{LC}},$$ Relabeled $w_0$ on p. 02-14013

A resonance frequency because it is the natural frequency and NOT a frequency from a driver; i.e., an AC source

$$I'' = -w^2 I$$ A 2nd order ordinary linear DE

with general solution

Note 2nd order ordinary linear DEs have two independent solutions

$$I = A\cos wt + B\sin wt$$

where A and B are constants to be set by the initial conditions

Recall $I(t=0) = 0$, and so $A = 0$

$$I = I_a \sin wt,$$

where $I_a$ is the amplitude of the oscillation

Q must oscillate back and forth between $Q_0(2\pi n)$ and $-Q_0(2\pi n+\pi)$, and so $-Q_0 = \frac{I_a}{w}[-1-1] + Q_0$

$\frac{I_a}{w} = Q_0$

$I_a = wQ_0$

$$Q = \int_0^t (-I)\,dt' + Q_0 = \frac{I_a}{w}\cos wt'\Big|_0^t + Q_0$$

$$Q = \frac{I_a}{w}[\cos wt - 1] + Q_0 = Q_0\cos wt$$

$$Q = Q_0\cos(wt)$$

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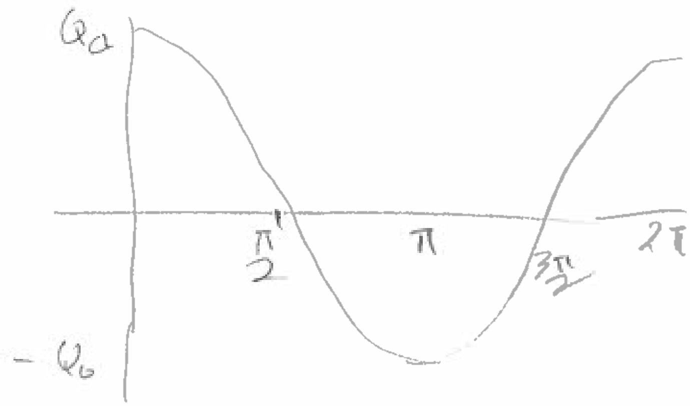
$I = \omega Q_0 \sin(\omega t)$ , $Q = Q_0 \cos(\omega t)$


# 14.4 The LRC Circuit without a Driven

a) In fact, the case of real interest is with an AC driven and the solution of this case is trickier than I recall (if ever saw it or did it which may be never, and the Text-611 has a wrong answer which I learnt when I gave up trying to reproduce and I just asked Google AI.

Zen Kōans for Our Time

AI is the answer that is not the answer.

A discovery is well known.

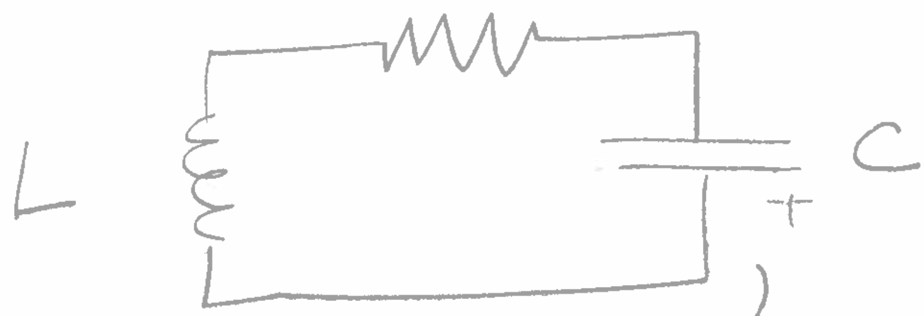
Kōans come in threes.

2026aug02 (The most useful product of my weekend, alas)

02-14012

b) An LRC circuit without a driver (which is a AC power source usually and which we consider in Chapter 13 probably)

R




Applying Kirchoff Voltage law

$$\frac{Q}{C} = LI' + IR$$

and differentiating the Integro-DE with $\frac{dQ}{dt} = -I$ gives

$$0 = LI'' + I'R + \frac{I}{Q}$$

{ A 2nd order linear ordinary DE

This DE is an exact analog of the damped harmonic oscillator DE, where R acts as a damping force to dissipate energy to heat,

Ansatz the solution $I = e^{-gt}$ (because we know this the path) Reduced the differential eqn. to an algebraic equation.

$$\therefore \quad 0 = Lg^2 - Rg + \frac{1}{C}$$

which is quadratic eqn! Recall

$0 = ax^2 + bx + c$

gives $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

2026aug02 | 02_14015

$$\therefore\ I = -\frac{\omega_0^2 Q_0}{2\beta}\left[e^{-(\alpha+\beta)t} - e^{-(\alpha-\beta)t}\right]$$

$$I = \frac{\omega_0^2 Q_0}{2\beta}\left[\underbrace{e^{-(\alpha-\beta)t}}_{\alpha-\beta>0} - e^{-(\alpha+\beta)t}\right]$$

Google AI agrees, except it had an overall minus for a different fiducial current direction

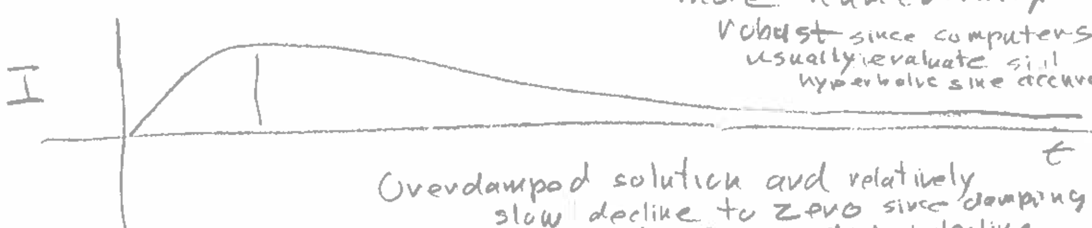
$\alpha - \beta > 0$

and this gives slow exponential decline

$\alpha + \beta > \alpha - \beta$

and this gives fast exponential decline

also $I = \frac{\omega_0^2 Q_0}{2\beta} e^{-\alpha t} \sinh(\beta t)$ which doesn't show the behavior so well, but is more numerically robust since computers usually evaluate sinh hyperbolic sine accurately.




Overdamped solution and relatively slow decline to zero since damping itself prevents fast decline.

For maximum

$$\frac{dI}{dt} = 0 = -(\alpha-\beta)e^{-(\alpha-\beta)t} + (\alpha+\beta)e^{-(\alpha+\beta)t}$$

$$0 = -(\alpha-\beta)e^{\beta t} + (\alpha+\beta)e^{-\beta t}$$

$$0 = -(\alpha-\beta)e^{2\beta t} + (\alpha+\beta)$$

$$\therefore\ (\alpha-\beta)e^{2\beta t} = (\alpha+\beta)$$

$$2\beta t = \ln\left(\frac{\alpha+\beta}{\alpha-\beta}\right) > 0$$

$$t_{max} = \frac{1}{2\beta}\ln\left(\frac{\alpha+\beta}{\alpha-\beta}\right)$$

$$I_{max} = \frac{\omega_0^2 Q_0}{2\beta}\left[\left(\frac{\alpha+\beta}{\alpha-\beta}\right)^{-(\alpha-\beta)/(2\beta)} - \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^{-(\alpha+\beta)/2\beta}\right]$$

Critical damping
Fastest decline to zero
What one wants
2-way swinging restaurant doors, but oscillating doors are fun

e) Case 2 From p. 02_14014

$$q = \alpha \quad \text{since } \alpha^2 - \omega_0^2 = 0$$

and seems only one solution

Our Ansatz solution does NOT conform to 2nd solution in this case

$$I \propto e^{-\alpha t}$$ which cannot match initial condition $I(t=0)=0$.

But 2nd order ordinary linear DE always has a 2nd solution (Arken-469).

There is, of course, a general procedure for finding a 2nd solution, but is there something simpler?

Try $I = \underbrace{f(t)}_{\text{This is a correction factor}} \underbrace{e^{-\alpha t}}_{\text{This is the main damping behavior}}$

$$0 = L I'' + I'R + I/C \quad \text{from p. 02\_14012}$$

$$0 = L\underbrace{[f'' - 2\alpha f' + \alpha^2 f]}_{\substack{\text{Using Leibniz rule} \\ \text{Arfken-667, 985} \\ \text{a generalization} \\ \text{of the product rule}}} + [f' - \alpha f]R + f/C$$

$$0 = L[f'' - 2\alpha f'] + f'R + \underbrace{\{L[\alpha^2 f] - \alpha f R + f/C\}}_{[L\alpha^2 - \alpha R + 1/C]f \,=\, 0 \text{ since } \alpha \text{ satisfies}} = 0$$

$$0 = L[f'' - 2\alpha f'] + f'R$$

original quadratic equ. p. 02_14012

At this point we guess $f = t$

$\therefore \quad f'' = 0$

$f' = 1$

and $\quad 0 = L[-2\alpha] + R$

leads to $\quad \alpha = \dfrac{R}{2L}$

which is consistent with the p. 02_14013 definition of $\alpha$.

$$\therefore \quad I = A\,t\,e^{-\alpha t}$$

and we need to determine coefficient $A$.

$t_{max} = \dfrac{1}{\alpha}$ and $I_{max} = \alpha Q_0 e^{-1}$

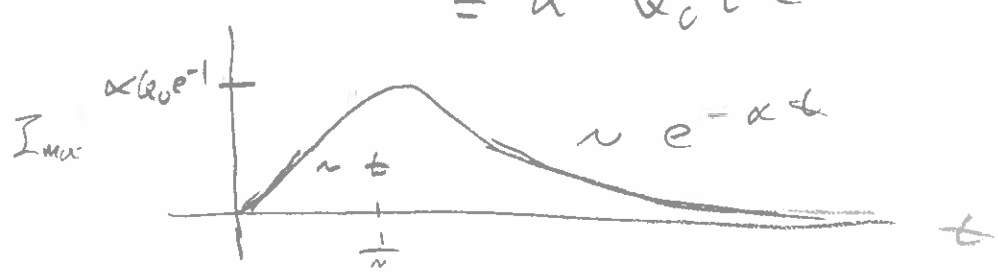
$$I' = \left[A e^{-\alpha t} - \alpha A t e^{-\alpha t}\right]\Big|_{t=0}$$

$= \omega_0^2 Q_0$ from p. 02_14017

$A = \omega_0^2 Q_0$

$\therefore \quad I = \omega_0^2 Q_0\, t\, e^{-\alpha t}$ where $\omega_0 = \frac{1}{\sqrt{LC}} = \alpha$

$= \alpha^2 Q_0\, t\, e^{-\alpha t}$ since $\omega_0^2 = \alpha^2$ for critical damping.



02 14018

(Which is very need for two-way swing restaurant doors. Great for slapstick comedy

f) Case 3 Underdamping which gives oscillatory behavior.

$$g = \alpha \pm i\omega_d$$

$$\alpha^2 - \omega_0^2 < 0$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

So $I = I_{0+} e^{-(\alpha + i\omega_d)t} + I_{0-} e^{-(\alpha - i\omega_d)t}$

Now the complex number solutions are valid solutions of the DE, but don't match the requirement that $I(t)$ be a pure real.

But with some complex number trickery we can find the real solutions

Show of hands: How many of you-all have covered complex number math to some degree?

Some? All?

Especially electrical engineers will need complex number math because it is used for alternating currents. It is a very useful tool for avoiding endless trigonometric identity use.

And I intend to cover a bit of it for Chapter 15.

2026aug01 C02_19019

Trickery : Recall from p. C2_19019

$$I = \frac{\omega_0^2 Q_0}{2\beta}\left[e^{-(\alpha-\beta)t} - e^{-(\alpha+\beta)t}\right]$$

was a solution

Now $\beta \to i\omega_d$

and $\frac{1}{i} = \frac{i}{i^2} = -i$

$i = \sqrt{-1}$ is the imaginary unit

$$I = \frac{\omega_0^2 Q_0}{2\omega_d} e^{-\alpha t}\left[-i\, e^{i\omega_d t} + i\, e^{-i\omega_d t}\right]$$

Now we need Euler's formula

$e^{\pm ix} = \cos x \pm i \sin x$ (Wiki: Euler's formula)

and so

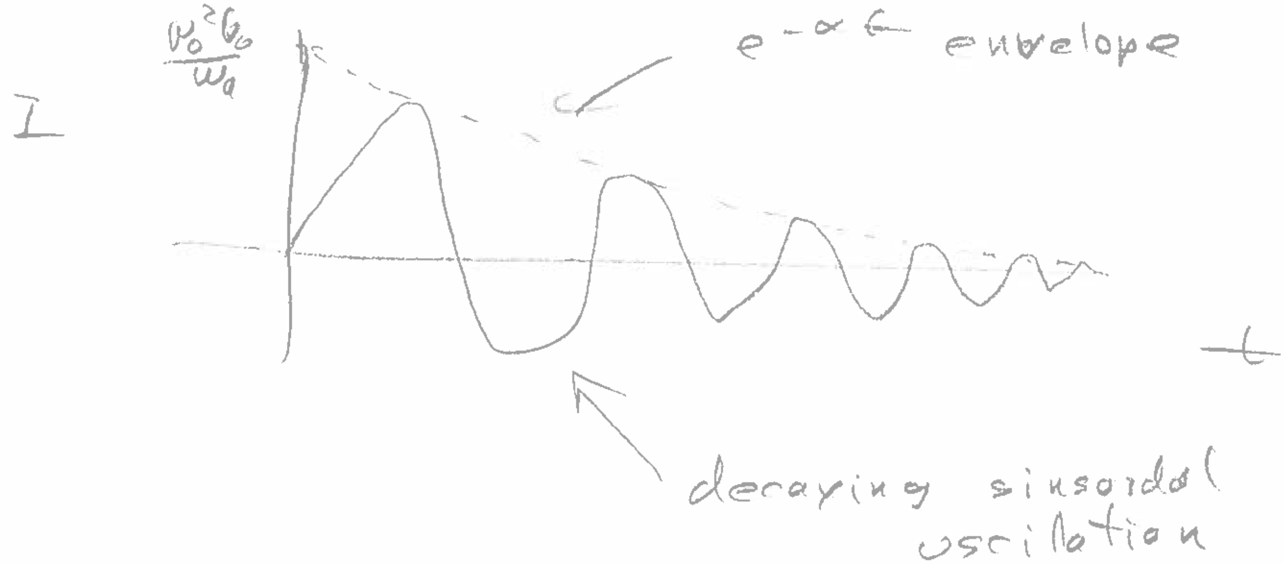
$$I = \frac{\omega_0^2 Q_0}{\omega_d} e^{-\alpha t}\Big[\underbrace{i[-\cos x + \cos x]}_{0} + \underbrace{[-i(i)\sin x + i(-i)\sin x]}_{\sin x}\Big]$$

$$I = \frac{\omega_0^2 Q_0}{\omega_d} e^{-\alpha t} \sin x$$

So, in fact, the solution above is a pure real after all and it satisfies the initial condition

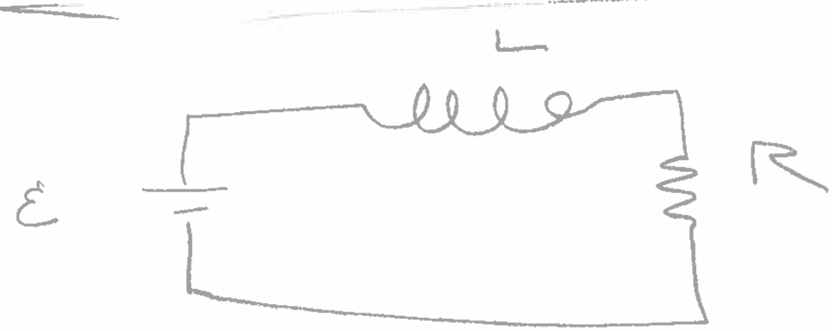
02_14020

$$\therefore \quad I = \frac{\omega_0^2 Q_0}{\omega_d} e^{-\alpha t} \sin \omega t$$




g) We could Now integrate to get $Q(t)$, but we have done enough.

## 14.5 An LC circuit with a constant emf




Apply Kirchhoff's Voltage law

$$\mathcal{E} = LI' + IR$$

Question: Initial condition is on $I$ and $I'$ are?

Answer: $I_0 = 0$ because of inductor.
$I_0' = \mathcal{E}/L$ to satisfy Kirchhoff's voltage law

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Solution

$$LI' = \mathcal{E} - IR$$

$$\frac{LI}{\mathcal{E} - IR} = 1$$

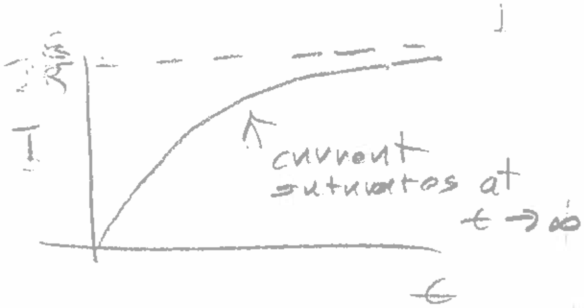
$$\frac{L\,dI}{\mathcal{E} - IR} = dt$$

$$\frac{L}{-R} \ln(\mathcal{E} - IR) \Big|_0^t = t$$

$$\ln \frac{\mathcal{E} - IR}{\mathcal{E}} = -\frac{R}{L} t$$

$$\frac{\mathcal{E} - IR}{\mathcal{E}} = e^{-\frac{R}{L} t}$$

$$\mathcal{E} - IR = \mathcal{E} e^{-\left(\frac{R}{L}\right) t}$$




$$I = \begin{cases} \dfrac{\mathcal{E}(1 - e^{-(R/L)t})}{R} = \dfrac{\mathcal{E}(1 - e^{t/\tau_L})}{R} \\ 0 \quad \text{at} \quad t = 0 \\ \dfrac{\mathcal{E}}{R} \quad \text{at} \quad t = \infty \end{cases}$$

When the current is steady, the inductor no longer causes an emf

$\tau_L = \frac{L}{R}$ is the inductive time constant or inductive e-folding time

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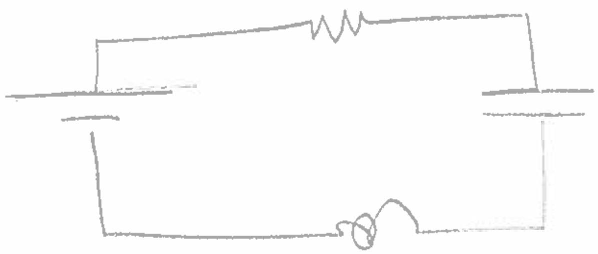
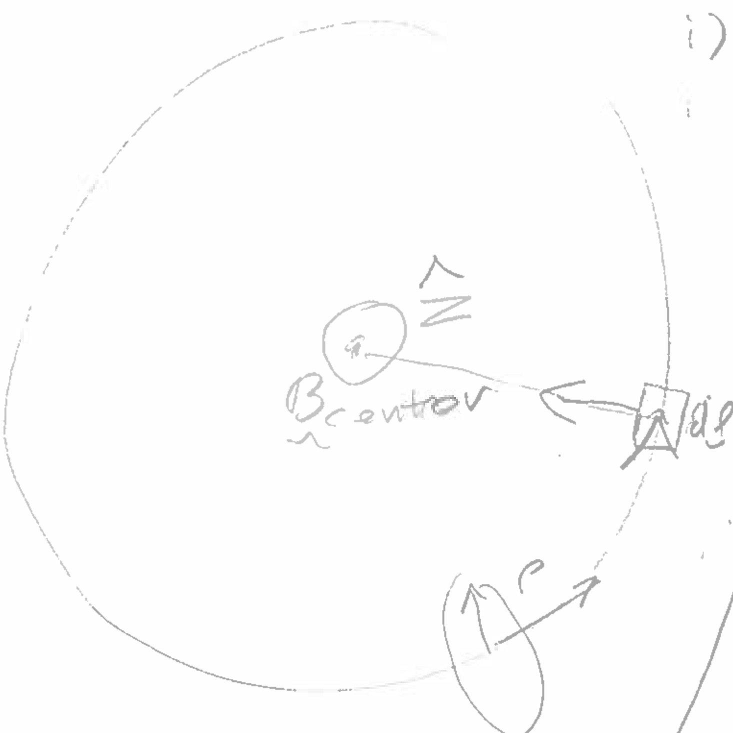
## 14.6 More on Self-Inductance

Ordinary circuit self inductance



Approximate as a circular loop.




Two approximations:

i) Biot Savart law (Text-502)

$$dB = \frac{\mu_0}{4\pi}\frac{I d\ell \times r}{r^2},$$

So at center

$$B = \frac{\mu_0}{4\pi}\frac{I \cdot 2\pi r}{r^2}\hat{z}$$

$$B = \frac{\mu_0}{2r}\hat{z}$$

ii) Ampere's law for an infinite straight wire

$$\oint B \cdot d\ell = \mu_0 I_{linked}$$

(Text-515)

$$B \cdot 2\pi\rho = \mu_0 I$$

$$B(\rho) = \frac{\mu_0 I}{2\pi\rho}\hat{z}$$ in plane of loop

$$B(\rho = r) = \frac{\mu_0 I}{2\pi r}\hat{z}$$

These two approximations suggest that in ordinary circuits

$$B \simeq \frac{\mu_0 I}{r}$$

where $r$ is of order loop size

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02_14023

i. $$\mathcal{E} = -\frac{d}{dt}\int \underset{\sim}{B}\cdot d\underset{\sim}{A}$$

$$= -\frac{d}{dt}\frac{\mu_0 I}{r_{loop}} A_{loop}$$

$$= -\frac{\mu_0 A_{loop}}{r_{loop}}\frac{dI}{dt} = -L_{loop}\frac{dI}{dt}$$

$$L_{loop} = \frac{\mu_0 A_{loop}}{r_{loop}}$$

Now recall from p. 02_13005

$$\mathcal{E}_{ideal\ solenoid} = -\mu_0 n^2 V \frac{dI}{dt}$$

ii. $$L_{solenoid} = \mu_0 n^2 V = \mu_0 \frac{N^2}{l^2} A l$$

$$= \frac{\mu_0 N^2 A_{solenoid}}{l_{solenoid}}$$

$$\text{Ratio} = \frac{L_{solenoid}}{L_{loop}} = \frac{N^2 A_s / l_s}{A_{loop}/r_{loop}}$$

For typical values for solenoid from Google AI for small inductors

$$= \frac{(10^3)^2 \frac{(30\,mm)^2}{30\,mm}}{1\,m^2 / 1\,m} = \frac{10^6 \cdot 0.03\,m}{1\,m}$$

$$= \frac{10^6}{30} = 3\times 10^4$$

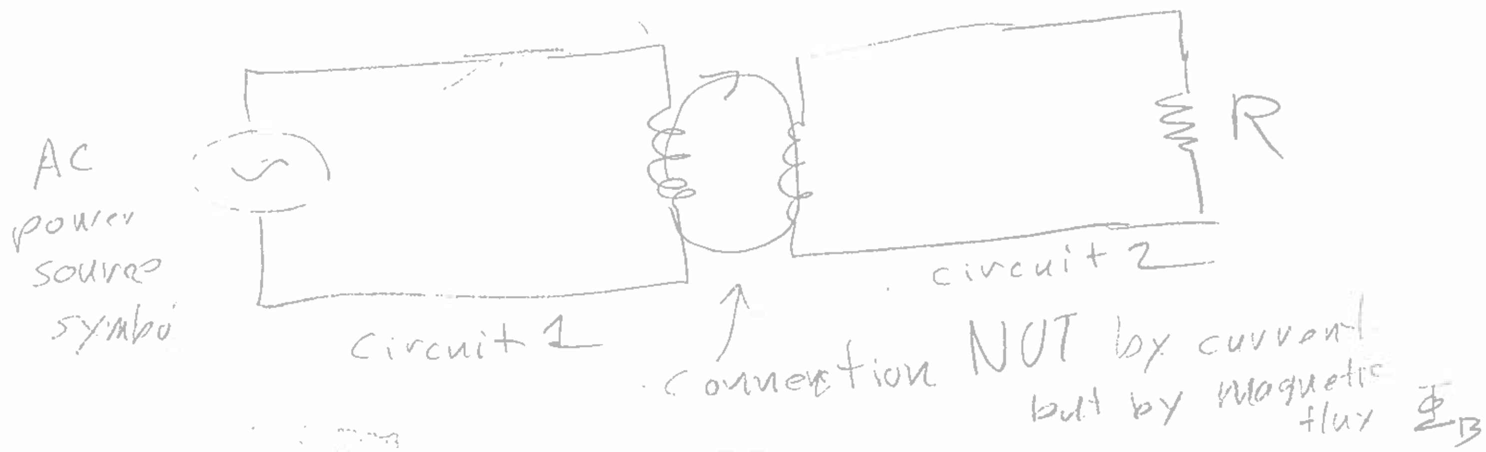
02_14024

This ratio suggests to me that the self-inductance of wire loops in a circuit is negligible compared to ordinary inductors, and so probably negligible in general.

But if you are doing high precision circuitry, you may have to worry about the self-inductance of wires.

# 14.7 Mutual Inductance M

Clearly, a B-field produced by one circuit can induce an emf in a second circuit



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Now for self-inductance

$$\mathcal{E} = -\frac{d}{dt}\Phi_B = -\frac{d}{dt}\int \underline{B}\cdot d\underline{A}$$

$$= -L\frac{di}{dt}$$

Flux due in a circuit due to current in that circuit

ands

and $\Phi_B = Li$

Apparently, the convention to use small $i = i(t)$ for alternating current

But one can also have

$$\mathcal{E}_2 = -\frac{d\Phi_{21}}{dt}$$

$\Phi_{21}$ is Flux in 2 due 1. Apparently, the conventional order for indices

$$\Phi_{21\,total} = M_{21}\, i_1$$

Where $\Phi_{21\,total} = N_2\, \Phi_{21}$

flux in a single turn of a circuit 2 solenoid (or coil) which we are assuming is the device of interest

$$\therefore\ N_2 \Phi_{21} = M_{21}\, i_1$$

There is also flux in circuit 1 due to circuit 2.

So $N_1 \Phi_{12} = M_{12}\, i_2$

D2-14026

Something beyond our scope to prove (and Google AI gives a proof using the vector potential of magnetism $A$) is that $M_{12} = M_{12}$,

and so here we can just use $M$ for both.

$$\mathcal{E}_2 = -\frac{d}{dt}(M i_1)$$

$$\mathcal{E}_1 = -\frac{d}{dt}(M i_2)$$

and we will leave this story (before it gets interesting) until Chapter 15 on alternating current (AC).

So far we have been mostly thinking of DC (direct current) except for our treatments of oscillating $LC$ and $LRC$ circuits without drivers.