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Chapter 6 Gauss' Law

02_06002

6.1 Flux and Electric Flux

a) In general flux is defined by the differential expression

$$d\underset{\sim}{F} = \underset{\sim}{f}(\underset{\sim}{r}) \cdot d\underset{\sim}{A}$$

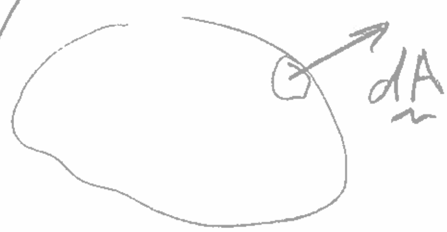
differential flux

$\underset{\sim}{f}(\underset{\sim}{r})$ is any vector field e.g., velocity of a fluid in which case the term flux makes intuitive sense, but the field can be the E-field, B-field, gravitational field etc.

$d\underset{\sim}{A}$ is a differential surface area vector

It points outward which for an open surface can be chosen for convenience

But for a closed surface is outward



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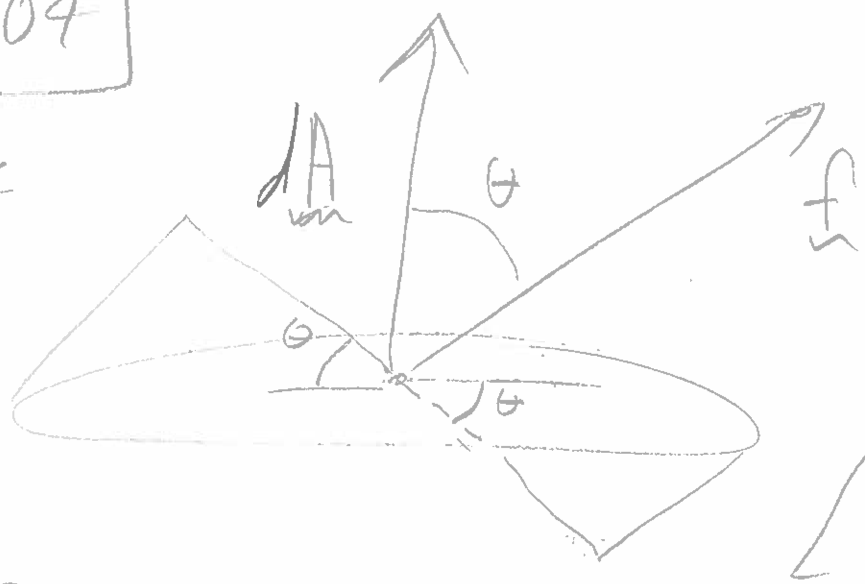
) Note



So $\underset{\sim}{f} \cdot d\underset{\sim}{A} = f \, dA \cos\theta$

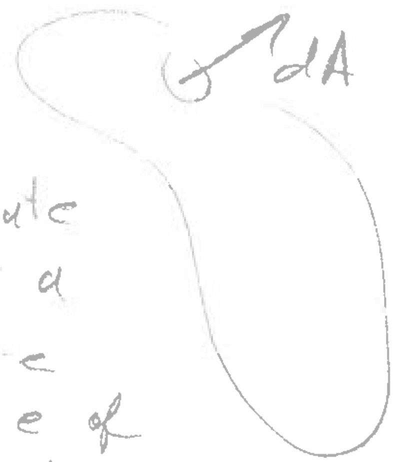
$= f \, dA_\perp$

Dropping vector squiggle indicates only magnitude is meant

Finite flux is

$$F = \int \underset{\sim}{f} \cdot d\underset{\sim}{A}$$

is where you integrate over a finite piece of surface area.



$dA_\perp$ is my own symbol for area perpendicular to the field vector direction and it is given by

$$dA_\perp = dA \cos\theta$$

> 0 for $\theta < 90^\circ$

$= 0$ for $\theta = 90^\circ$

< 0 for $\theta > 90^\circ$

$F = \oint \underset{\sim}{f} \cdot d\underset{\sim}{A}$ in my notes means closed surface area integral, but also contour integral. Wik tells me $\oiint$ is formally closed surface area integral.

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Electric flux is defined

$$d\Phi = \underset{\sim}{E} \cdot d\underset{\sim}{A}$$

Capital Greek phi and NOT a universal symbol I think.

$$d\Phi_B = \underset{\sim}{B} \cdot d\underset{\sim}{A}$$ magnetic flux

$$d\Phi_G = \underset{\sim}{g} \cdot dA$$ gravitational field flux

$\underset{\sim}{g}$ is the gravitational field

little $g = 9.8\ m/s^2 = 9.8\ N/kg$ is the fiducial magnitude of the Earth's near surface gravity

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Note closed surfaces to which Gauss' law are applied are called Gaussian surfaces (wik)

6.2 Gauss' Law

a) Formula for $\underset{\sim}{E}$-fields

$$\underbrace{\oint \underset{\sim}{E} \cdot d\underset{\sim}{A}} = 4\pi k Q_{enclosed} = \frac{Q_{enclosed}}{\varepsilon_0}$$

Integral of Electric Flux over a closed surface due the charge enclosed (The E-Field due the enclosed charge you are <u>counting</u>. There can be other E-fields due to charge enclosed and/or NOT enclosed in which case $\underset{\sim}{E}$ will <u>NOT</u> be the total E-field

Coulomb constant

$k = \frac{1}{4\pi\varepsilon_0} = 8.987\,551\,792(14) \times 10^9$ MKS

$\doteq 10^{10}$ MKS

$\varepsilon_0 = 8.854\,8188(14) \times 10^{-12}$ MKS

$\doteq 10^{-11}$ MKS

$Q_{enclosed}$ is all the charge enclosed in the closed surface (that you are counting)

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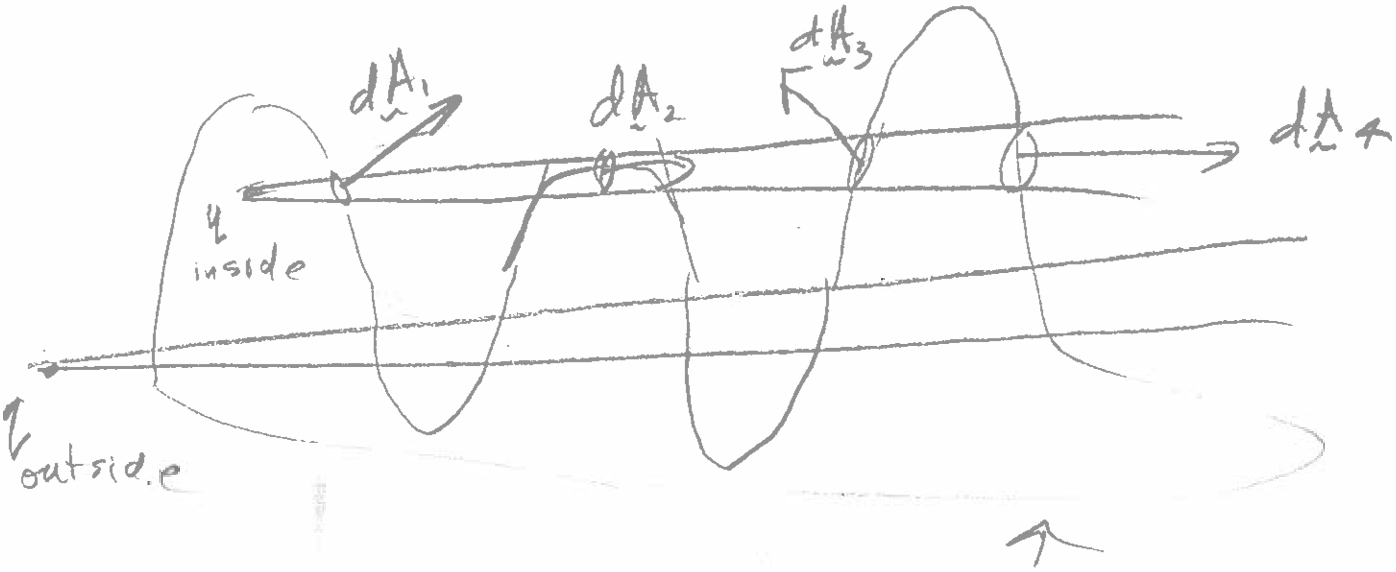
b) Proof of Gauss' Law

Consider point charge q



general closed surface

$$\underset{\sim}{E} = \frac{kq}{r^2}\hat{r}$$

Very key point: this result depends on the inverse-square law nature of the field.

$$d\Phi = \begin{cases} kq\,d\Omega & \text{for } q \text{ inside} \\ 0 & \text{for } q \text{ outside} \end{cases} = \sum_i \frac{kq}{r_i^2}\underbrace{\hat{r}\cdot dA_i}_{dA_{\perp i}} = \sum_i kq \underbrace{\frac{dA_{\perp i}}{r_i^2}}_{\pm d\Omega}$$

$d\Omega$ has the same absolute value for all crossings

$\pm d\Omega$ solid angle upper/lower case for outward/inward

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If q inside, one more exit than entry and the sum over all crossings is $d\Omega$

If q outside, exits and entries are equal in number and the sum of crossing is 0.

<u>Now</u>

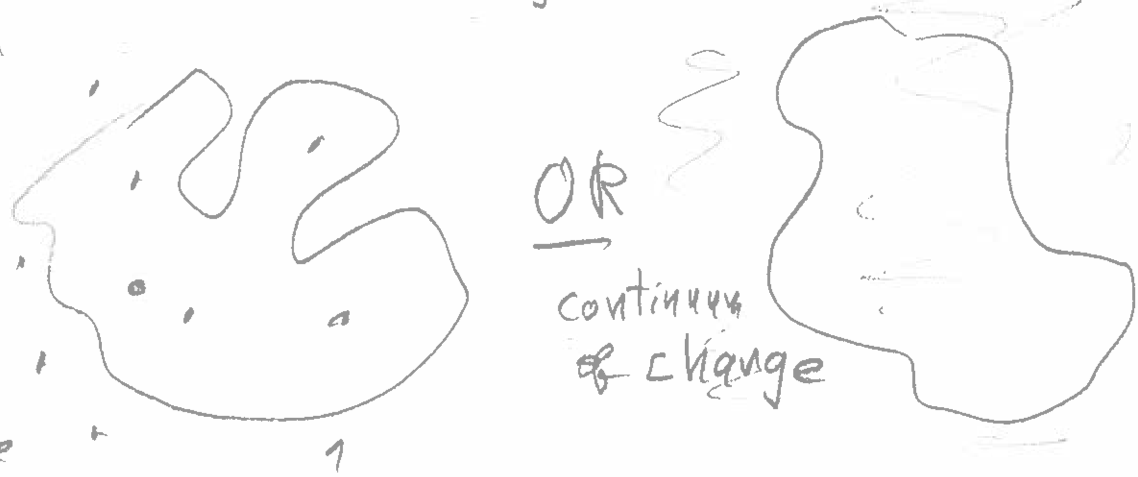
$$d\Phi = k\,q_{enclosed}\,d\Omega$$

$$\therefore \oint d\Phi = \oint \underset{\sim}{E}\cdot d\underset{\sim}{A} = 4\pi k\,q_{enclosed}$$

$$= \frac{q_{enclosed}}{\varepsilon_0}$$

Note, it does <u>NOT</u> matter where the point charge is,

Thus for a set of point charge inside and outside



$\therefore$ QED

Thus $\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = 4\pi k\, Q_{enclosed} = \dfrac{Q_{enclosed}}{\varepsilon_0}$

where $Q_{enclosed} = \begin{cases} \sum_i q_{i\,enclosed} \\ \oint \rho(\underset{\sim}{r})\, dV \end{cases}$ differential of Volume

QED

Note $Q_{enclosed}$ will be 0 if positive and negative charge cancel exactly

Emphasis: Gauss' law depends absolutely on the field obeying an inverse-square law behavior from point charges q or differential charge $\rho(\underset{\sim}{r})dV$

c) A Vector calculus Proof (Omit in class)

for those who know vector calculus and those who will know it.

The differential Gauss' Law is the Maxwell's equation Gauss' law (p. 02-05003)

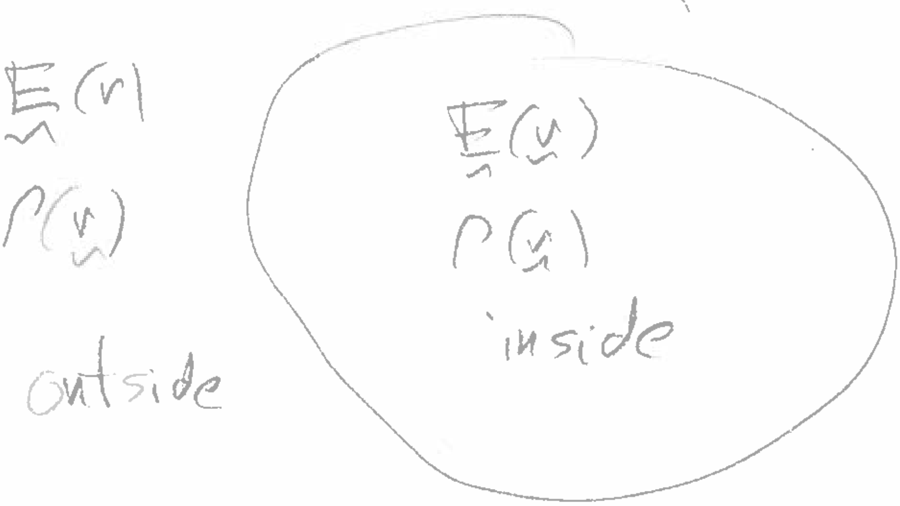
divergence of $\underset{\sim}{E}$ which is a scalar

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\varepsilon_0}$$

charge density divided by vacuum permitivity

This is an axiom of classical E+E

Integrate over some volume V



$$\oint \nabla\cdot\mathbf{E}\,dV = \oint_V \frac{\rho}{\varepsilon_0}\,dV$$

by divergence theorem (Just accept it)

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{Q_{enclosed}}{\varepsilon_0} \quad \text{QED}$$

The differential form of Gauss' law is generally considered the most basic/useful form

Useful for solving for general fields and for describing electromagnetic radiation

But at our level the integral form is as we will soon show

d) Gauss' Law for B-field

Another Maxwell equation (p. 02-05003)

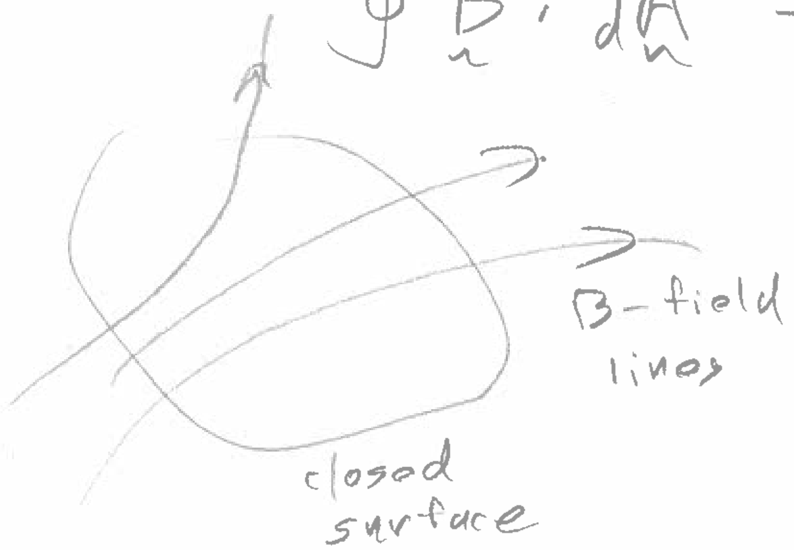
$$\nabla \cdot \underset{\sim}{B} = 0$$

differential Gauss' law for B-fields

and just recapitulating steps on p. 02-06009 — 02-06010),

we get

$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{A} = 0$$



Not such useful formula for us

Why is the right-hand side zero?

In our observable universe magnetic monopoles do NOT exist

<u>or</u> they are so rare we don't see them.

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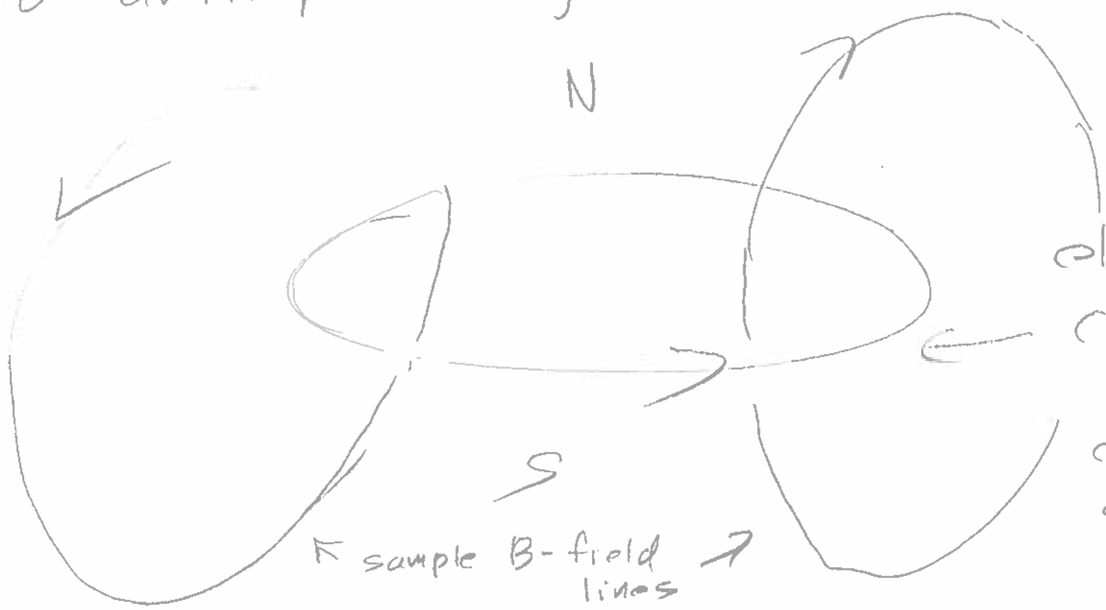
What is a magnetic monopole?

A particle which is a pure north/south pole.

B-fields from monopole are expect to obey an inverse square law since magnetic dipoles exist and have a limiting behavior consistent with closely spaced north and south monopoles obeying ~~an~~ inverse-square law.

B-field lines for magnetic monopole

To anticipate magnetism



electric current loop creates a dipole magnetic field

A north/south pole is where B-field lines diverge/converge from/to.

Some quantum field theories predict magnetic monopoles and it's rather sad that we don't observe any.

e) Gauss Law for gravity

The gravitational field is an inverse-square law

$$\underset{\sim}{g} = -\frac{Gm}{r^2}\hat{r}$$ for a point mass

gravitational field lines around a point mass

The minus sign and $\hat{r}$ means the field always points toward the point mass

<u>But</u> there is a profound difference from the E&M case,

Mass comes in only <u>ONE</u> 'charge' and likes attract.

By the same steps as on p. 02-06007,

$$\oint \underset{\sim}{g} \cdot d\underline{A} = 4\pi G(-M_{enclosed})$$

or

$$\oint \underset{\sim}{g} \cdot d\underset{\sim}{A} = -4\pi G M_{enclosed}$$

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f) Are there Gauss' laws for forces other than inverse-square law forces from point charges?

Yes. For the linear force:

$$\underset{\sim}{f} = q\,\underset{\sim}{r}$$ where q is the "charge"

If $q < 0$, then the force is the 3-dimensional Hooke's law force

It's actually very easy to prove that only the inverse-square law and the linear-force law for point "charges" have Gauss' laws (and Gauss must have known this the moment he thought in the 18th century)

But we don't go into this

The linear force Gauss' law doesn't have any applications, except in cosmology where it's actually a bit important.

6.3 Gauss' Law (Integral Form)

Applications

a) Gauss' Law (integrat form) can be used to solve for the electric field exactly in ideal cases:

i) Spherically symmetry which is described by the shell theorem. (p. 02-06017)

ii) Cylindrical symmetry (p. 02-06025)

iii) Planar symmetry (p. 02-06027)

These are ideal cases, but important practical cases (e.g., capacitor discussed in Chapter 8) are approximated by the ideal cases, and so the ideal cases give a lot of insight.

Gauss' Law (integral form) is usual for insight in other cases of interest (e.g., surface charge on conductors: see p. 02-06040ff).

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b) The differential form of Gauss' law ($\nabla \cdot \underset{\sim}{E} = \rho/\varepsilon_0$) is important in solving E-fields in general cases (by solving DEs by advanced analytic means or numerically) and in other E+M developments (all importantly for proving electromagnetic radiation (EMR) exists and explicating visible light as EMR).

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6.4 The Shell Theorem

Gauss' Law applied to the case of spherical symmetry

a) A very important result derived easily from Gauss' law (which I've known since circa 1977, but only knew it had a name (i.e., shell theorem) since circa 2017 thanks to Wikipedia and the textbook doesn't know the name either!)

Newton had to work very hard to prove the shell theorem in the 17th century because he did NOT know Gauss' law — even an incomparable genius can't know everything all at once.

He needed the shell theorem for celestial mechanics in order treat spherical bodies (moons, planets, stars) as point masses.

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b) Shell Theorem Proof

If you have a spherically symmetric charge distribution (mass distribution for gravity, but we won't do that here)

$$\underset{\sim}{E} = \frac{k\, q(r)}{r^2}\,\hat{r}$$

where $q(r)$ is the charge enclosed by a shell of radius r

Proof from Gauss' law
(and so dependent on inverse-square law nature of field)

By spherical symmetry

$$\underset{\sim}{E}(\underset{\sim}{r}) = E(r)\,\hat{r}$$ {If $E<0$, the field points inward

$$\therefore \oint \underset{\sim}{E}\cdot d\underset{\sim}{A} = 4\pi k\, Q_{enclosed}$$

$$E(r)(4\pi r^2) = 4\pi k\, q(r)$$

$$E(r) = \frac{k\,q(r)}{r^2}$$

$$\therefore \underset{\sim}{E}(\underset{\sim}{r}) = \frac{k\,q(r)}{r^2}\,\hat{r}$$

QED

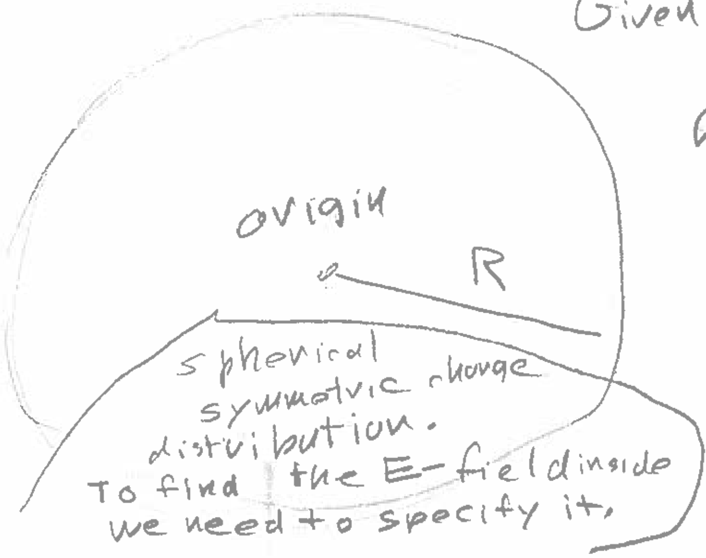
There are 3 immediate interesting corollaries of the shell theorem

c) Point Charge Equivalence Corollary

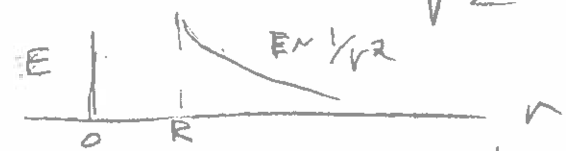
Given $q(r > R) = 0$

and $q(r = R) = q_{total}$,

then



$$\underset{\sim}{E}(r \geq R) = \frac{k\, Q_{total}}{r^2}\, \hat{r}$$



The E-field outside charge distribution is exactly the E-field of a (classical) point charge of charge Q_{total} at the origin.

In fact, the classical point charge is the limit $\lim_{R \to 0} \underset{\sim}{E}(r > R)$

with Q fixed.

Actually, recall our brief discussion of multipole expansion (p. 02-05066) which shows why it is valid to treat quantum mechanical point particles as classical point particles in many cases.

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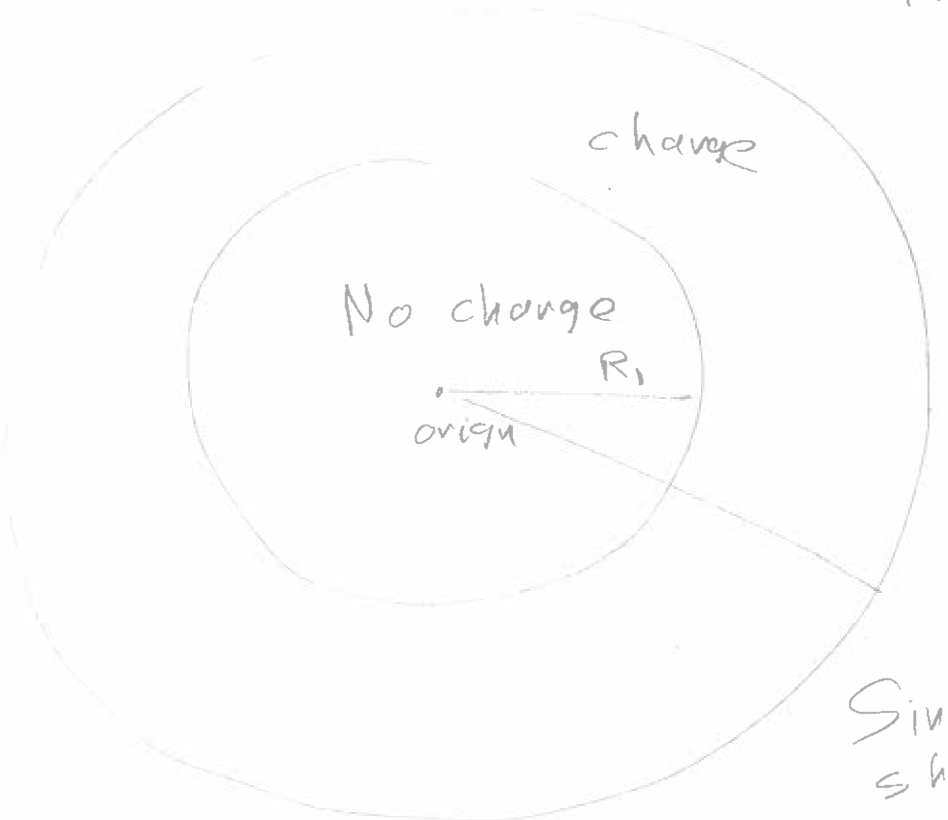
Also actually in the gravitational context, the point mass equivalence is what is needed for celestial mechanics starting from Newton.

d) The Spherical Cavity Corollary



The spherically symmetric charge distribution has a spherically cavity with NO charge in it.

Since by shell theorem

$$\vec{E}(\vec{r}) = \begin{cases} \dfrac{k\,q(r)}{r^2}\hat{r} \\ \emptyset & r < R_1 \\ \dfrac{k\,q_{total}}{r^2}\hat{r} & r \geq R_2 \end{cases}$$

This really is a somewhat surprising result.

And once again, it is based on the inverse-square law nature of the field of classical point charges.

But since classical point charges do NOT exist, it is more fundamentally true to say based on differential Gauss' law of Maxwell's equations (see p. 02-05003) rather classical point particle but it seems less physically intuitive to say that.

e) The Force Between Spherically Symmetric Charge Distributions Treatable as (Classical) Point Charges is the Same as Between (Classical) Point Particles Corollary.

We are assuming the spherically symmetry is fixed and not perturbed by the interaction.

It seems physically intuitively clear it should be so since their E-fields are those of point charges, but it still takes a proof to be definite.

Proof short form

$$\underset{\sim}{F}_{1,2} \underset{\text{shell theorem}}{=} \underset{\sim}{F}_{1\,\text{point},\,2} \underset{\text{3rd law}}{=} -\underset{\sim}{F}_{2,\,1\,\text{point}} \underset{\text{shell theorem}}{=} -\underset{\sim}{F}_{2\,\text{point},\,1\,\text{point}} \underset{\text{3rd law}}{=} \underset{\sim}{F}_{1\,\text{point},\,2\,\text{point}}$$

QED

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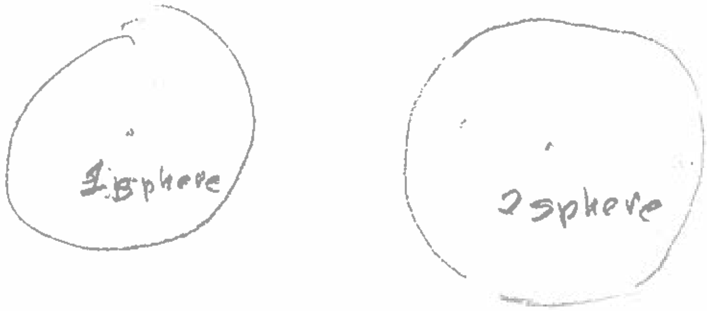
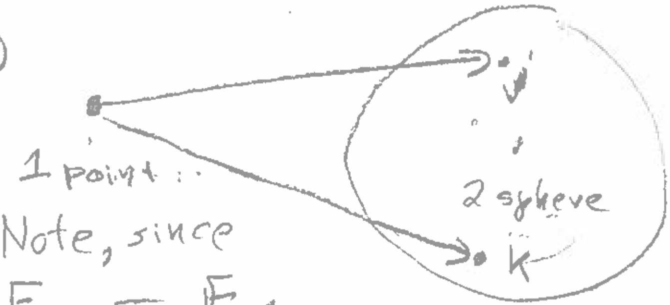
Proof Explicated

$$\underset{\sim}{F}_{12} = \underset{\sim}{F}_{1point,\,2}$$

by shell theorem





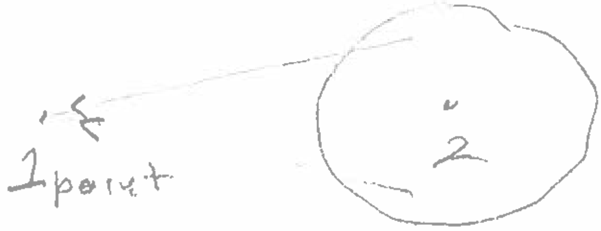
Note, since

$\underset{\sim}{E}_{2point} = \underset{\sim}{E}_1$

by shell, the forces on each bit of 2 sphere are exactly where they should be,

by 3rd law

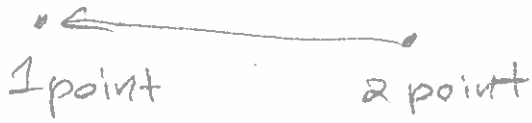
$$= -\underset{\sim}{F}_{2,\,1point}$$



Note, forces of sphere2 on 1point are **NOT** where they are on 1sphere, but the equation still true

by shell theorem

$$= -\underset{\sim}{F}_{2point,\,1point}$$



↓ by 3rd law

$$= \underset{\sim}{F}_{1point,\,2point}$$

To summarize

$$\underset{\sim}{F}_{1,2} = \underset{\sim}{F}_{1\,\text{point},\,2\,\text{point}}$$

But this does NOT mean the two spherically symmetric charge distributions interact dynamically like two point charges.

Recall

$$\underset{\sim}{F}_{\text{net}} = m\,\underset{\sim}{a}_{\text{center of mass}}$$

The net external force on a body determines the acceleration of the body's center of mass.

Proof

2nd law on particle i

$$\underset{\sim}{F}_{i\,\text{ext}} + \sum_{j \neq i} \underset{\sim}{F}_{j\,i} = m_i \underset{\sim}{a}_i$$

general body with center of mass CM

Sum over all particles

$$\sum_i \underset{\sim}{F}_{i\,\text{ext}} + \sum_{\substack{i,j \\ i \neq j}} \underset{\sim}{F}_{j\,i} = \sum_i m_i \underset{\sim}{a}_i$$

cancel by 3rd law

$$\underset{\sim}{F}_{\text{net}} + 0 = m \frac{\sum m_i \underset{\sim}{a}_i}{m}$$

$$\underset{\sim}{F}_{\text{net}} = m\,\underset{\sim}{a}_{cm} \quad \text{QED}$$

Returning to the two spherically symmetric charge distributions.

$$\underset{\sim}{F}_{1,2} = F_{1\,point,\,2\,point}$$

Note points 1 and 2 are centers of charge, NOT in general centers of mass

If $\underset{\sim}{F}_{1,2} = \underset{\sim}{F}_{net\ on\ 2}$ (NOT counting magnetic force See Lorentz force p. 02-05006)

the $\underset{\sim}{F}_{1,2} = \underset{\sim}{F}_{1\,point,\,2\,point} = m\,\underset{\sim}{a}_{2\,CM}$

<u>but</u> the center of charge and the center of mass do <u>NOT</u> have to be the same point in Coulomb force interactions

In the <u>gravity</u> case, they are the same point since mass = "charge"

Therefore, point 2 does NOT in general follow the path CM$_2$ in general.

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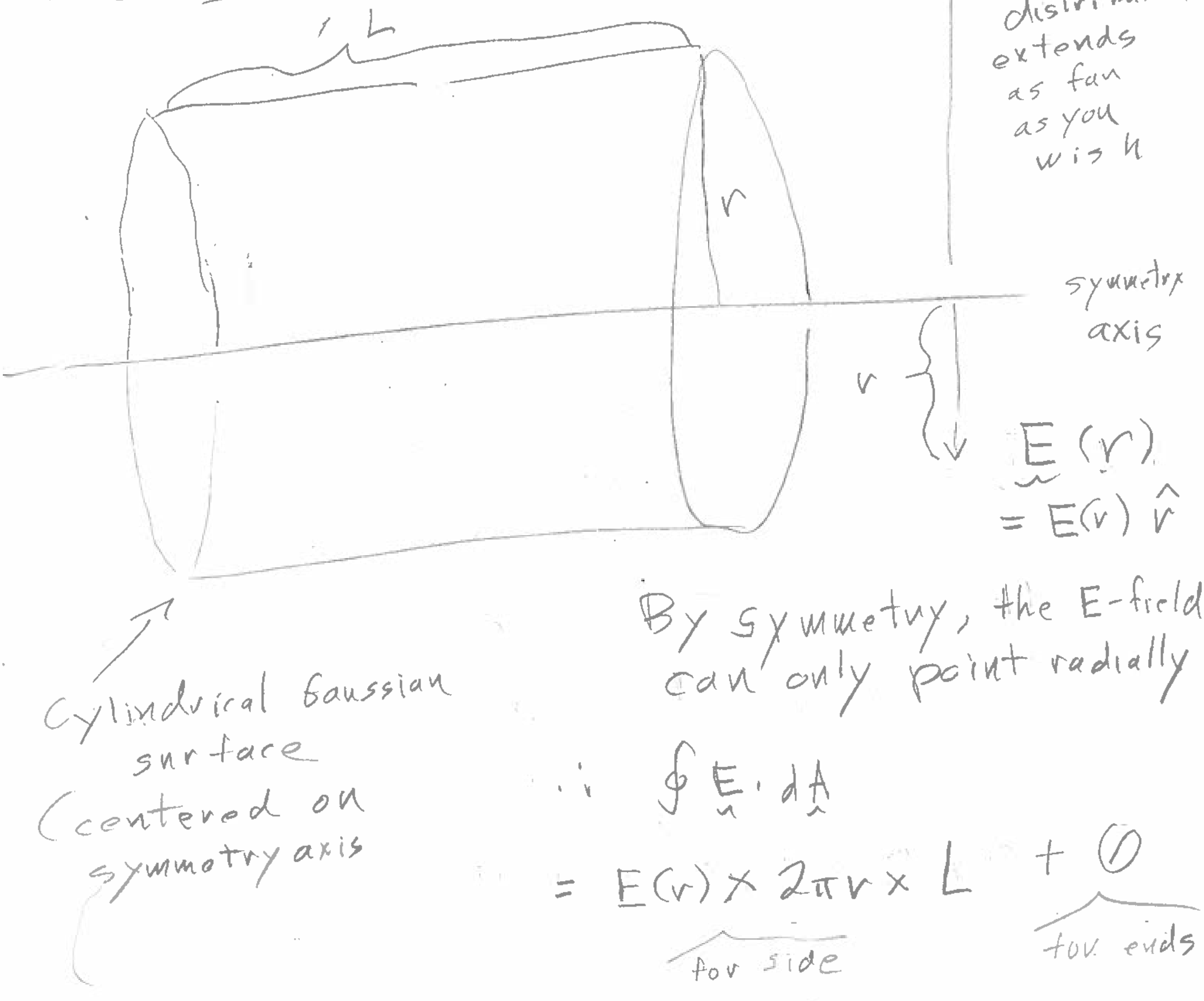
6.5 Gauss law Applied to Cylindrical Symmetry

Consider an infinite axis cylindrically symmetric charge distribution.



By symmetry, the E-field can only point radially

$$\therefore \oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \underbrace{E(r) \times 2\pi r \times L}_{\text{for side}} + \underbrace{0}_{\text{for ends}}$$

$$\therefore \oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = E\, 2\pi r L = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\underset{\sim}{E} = \frac{Q_{enclosed}}{2\pi r \varepsilon_0 L}\hat{r} = \frac{\lambda(r)}{2\pi r \varepsilon_0}\hat{r}$$

and to recapitulate

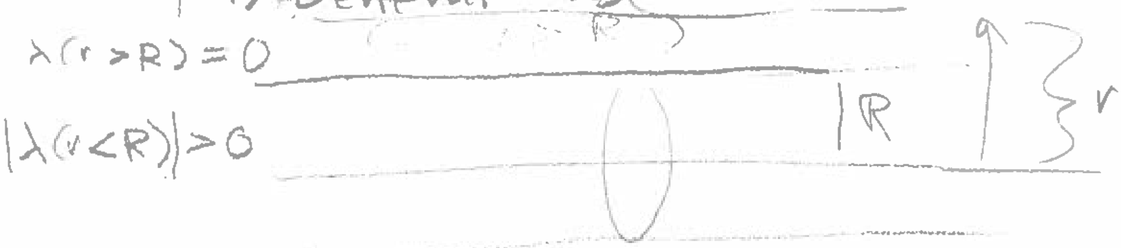
$$\underset{\sim}{E} = \frac{\lambda(r)}{2\pi r \varepsilon_0}\,\hat{r}$$

where $\lambda(r) = \dfrac{Q_{enclosed}(r, L)}{L}$

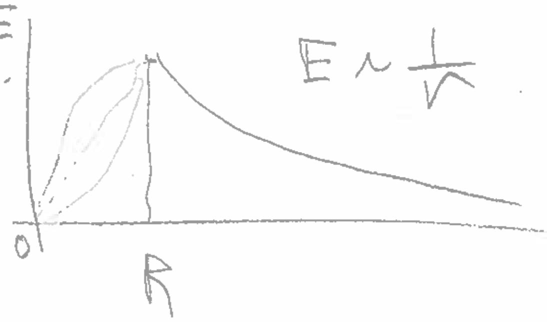
is the line charge density (G-64)
(MKS units C/m)

Note, analogus to the corollaries of the shell theorem (see p. 02-06019, p 02-0620)

i) General Case



$$\underset{\sim}{E} = \frac{\lambda(r)\,\hat{r}}{2\pi r \varepsilon_0} = \begin{cases} \dfrac{\hat{r}}{2\pi\varepsilon_0}\displaystyle\int_0^r \frac{\rho(r')\,2\pi\, dr'}{r^2} & \text{for } r<R \\[2ex] \dfrac{\lambda_{total}}{2\pi\varepsilon_0}\,\dfrac{1}{r}\,\hat{r} & r \geq R \end{cases}$$



ii) The Cylinder to R is a Conductor in Electrostatics (see p. 02-06037 for conductors)

$$\underset{\sim}{E}(r) = \frac{\lambda(r)\,\hat{r}}{2\pi r \varepsilon_0} = \begin{cases} \emptyset & \text{for } r<R \\[2ex] \dfrac{\lambda_{total}}{2\pi r \varepsilon_0} & \text{for } r>R \end{cases}$$

But what is $\underset{\sim}{E}(r=R)$?

It's sort of undefined. Note the limiting behavior.

$$\lim_{r\to R} \underset{\sim}{E}(r<R) = 0$$

$$\lim_{r\to R} \underset{\sim}{E}(r>R) = \frac{\lambda_{total}}{2\pi r \varepsilon_0}\hat{r}$$

This means the derivative at $r=R$ is infinite or undefined if you prefer.

In fact, in math jargon
a 2-d surface has volume zero (Google AI)
relative to 3-d space
and a point like $r=R$ has volume zero
relative to 1-d space.

If you have to integrate over $r=R$
or up to $r=R$ (as you do to
find electric potential which we
cover in chapter 7), then there
is no problem since $r=R$
contributes nothing (it has volume zero).

However, you can reasonably
divide $\underset{\sim}{E}$ infinitesimally close to $r=R$
into local and a long-range contributions.
We show how this is done
on p. 02-06041
and 02-06042.

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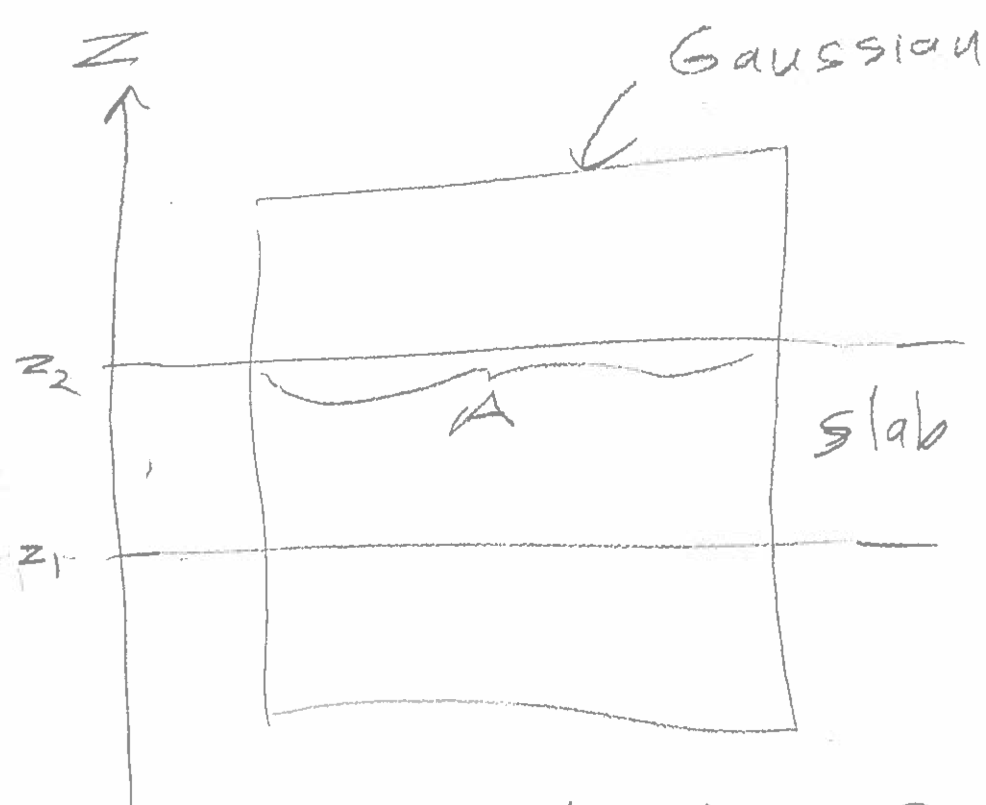
6.6 Gauss' Law Applied to Planar Symmetry

a) Consider a infinite slab with planar symmetry



For slab $\sigma = \frac{Q}{A}$ is the surface area density

Note, $\rho(z)$ can have any variation with z

$$\sigma = \int_{\Delta z} \rho(z)\,dz$$

By symmetry $\underset{\sim}{E}$ can only be aligned with the z-direction and can only point away from the slab.

Applying Gauss' law

$$\oint \underset{\sim}{E} \cdot dA = E\hat{z} \cdot A\hat{z} + E(-\hat{z}) \cdot A(-\hat{z}) = \sigma A / \epsilon_0$$

$$E = \frac{\sigma}{2\epsilon_0} \quad \text{and} \quad \underset{\sim}{E} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

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To recapitulate

$E = \frac{\sigma}{2\varepsilon_0}$ and $\underset{\sim}{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$

where $\hat{n}$ point normal to the surface

Note
E is constant

$\underset{\sim}{E}$ is constant aside from the direction switch.

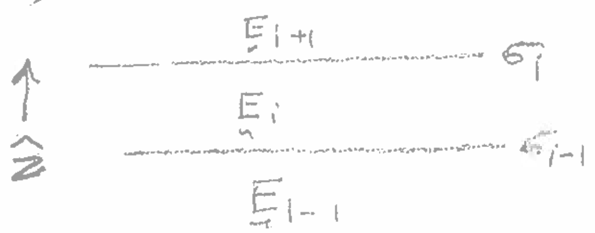
It does NOT matter how far from the slab when you are outside it.

This constancy of $\underset{\sim}{E}$ traces back to the infinite planar extent of the slab

Special Cases

Questions

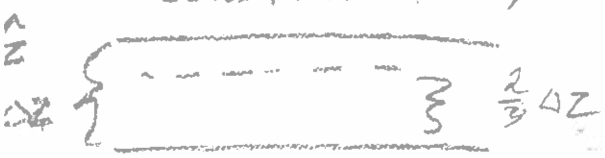
i) Two infinite thin slabs



What are E_{i+1}, E_i and E_{i-1}?

ii) What are they if $\sigma_{i-1} = -\sigma_i$

iii) Thick slab of constant $\rho(z)$



What is $\underset{\sim}{E}$ at $2/3\,\Delta z$?

Answers (cover in class)

i) Answer

$\underset{\sim}{E}_{i+1} = \frac{\sigma_i + \sigma_{i-1}}{2\varepsilon_0}\hat{z}$

$E_i = -\frac{\sigma_i + \sigma_{i-1}}{2\varepsilon_0}\hat{z}$

$E_{i-1} = \frac{-\sigma_i - \sigma_{i-1}}{2\varepsilon_0}\hat{z}$

ii) $E_{i+1} = 0$

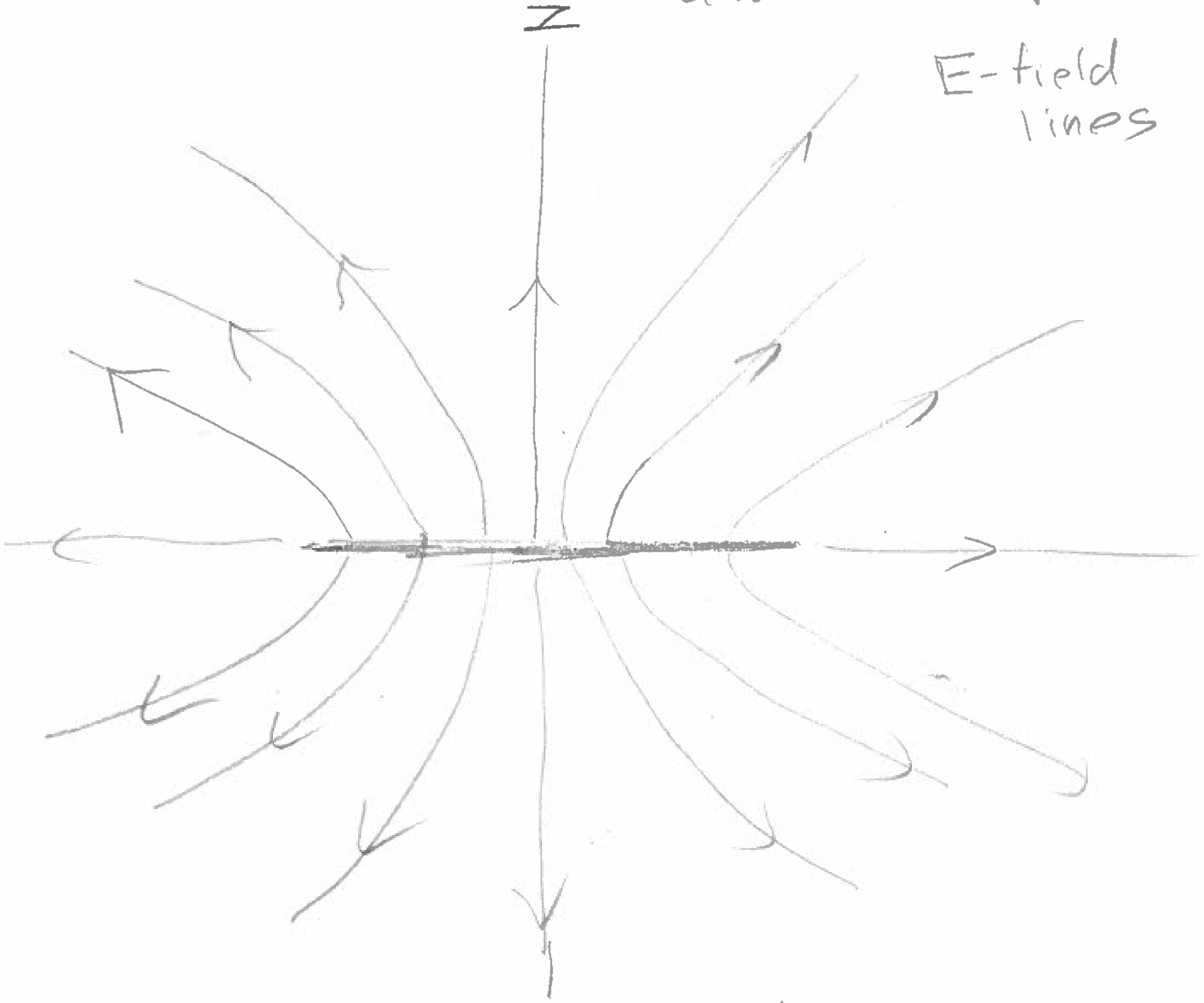
$E_i = -\frac{2\sigma_i}{2\varepsilon_0}\hat{z} = -\frac{\sigma_i}{\varepsilon_0}\hat{z}$

$E_{i-1} = 0$

iii) $\underset{\sim}{E} = \frac{\frac{1}{3}\Delta z\rho}{2\varepsilon_0}(-\hat{z}) + \frac{\frac{2}{3}\Delta z\rho}{2\varepsilon_0}\hat{z}$

$= \frac{1}{3}\frac{\Delta\rho}{2\varepsilon_0}\hat{z}$

c) What if you have a finite thin disk of uniform σ and total charge Q



Very close to center

$$\underset{\sim}{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

In the asymptotic far field limit

$$\underset{\sim}{E} = \frac{kQ}{r^2}\hat{r}$$ which is the monopole term of a multipole expansion (see p. 02-5066).

12-06032

6.7 Conductors in Electrostatics I

a) We consider idealized conductors but many real conductors (G-97ff) (e.g., solid sample of metals) approximate them.

We are following the path of scientific idealization usual in science and in intro physics courses: Understand the ideal system first and then later add the secondary effects (often numerous and tricky) afterwards.

The ideal conductors have an unlimited amount of free charge and infinite potential energy jumps at their surfaces

In metals, the free charge (Wiki: charge carriers) one or two valence electrons

that are NOT bound to ions,
but are bound inside
the sample by surface.

Absence of electrons
can be considered
as mobile positive charge

In fluids, one can
positive and negative ions
as well as free electrons
(as in liquid mercury: Google AI).

But in this discussion, we
are largely thinking
of solid metal samples

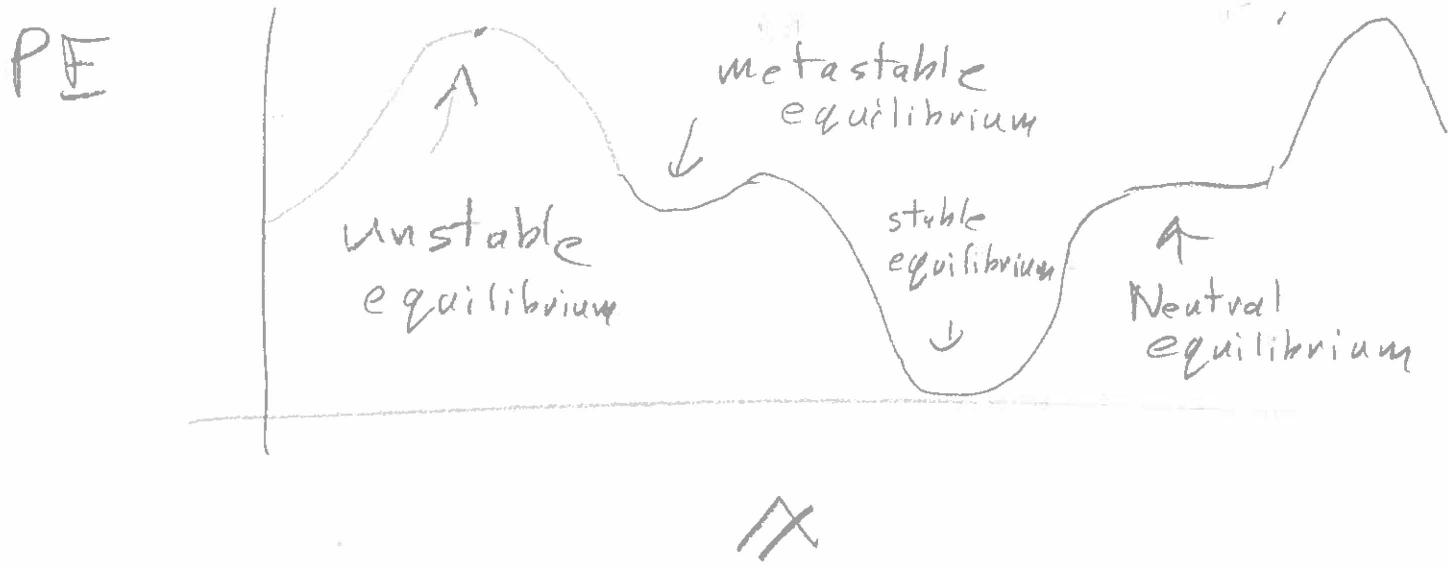
b) Equilibria

We deal with electrical potential
in Chapter 7

However, in a first course of
intro physics you encountered
potential energy (capital U or PE)
and maybe the idea of
a potential energy landscape
(by some name)

02_06034 | The Potential Energy of a force as a function of position.

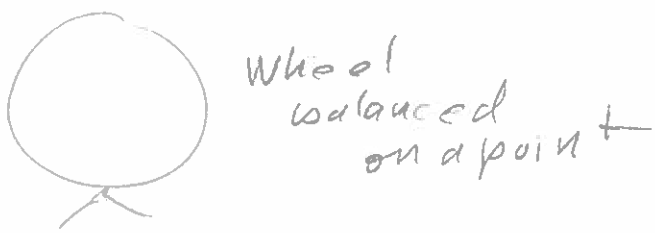
$$\text{Force} = \frac{\Delta PE}{\Delta x} = \text{slope of } PE$$



The force is zero where slope is zero

i) Unstable equilibrium
- any perturbation causes the body to move away and NOT return due to a local restoring force

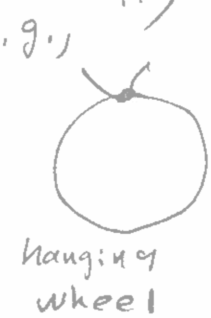
e.g.



ii) Stable equilibrium

There is a restoring force and so a perturbation causes the body to oscillate around the equilibrium point perpetually if no dissipation of kinetic energy somehow for a finite time if there is dissipation

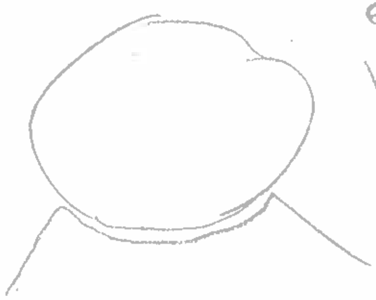
e.g.,



iii) Meta stable Equilibrium

Stable against small perturbations, but not large ones

e.g.,



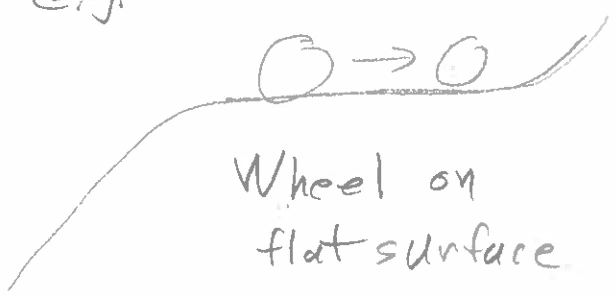
Wheel sitting shallow cup

Actually, all equilibria are metastable, but some are more metastable than others

iv) Neutral Stability

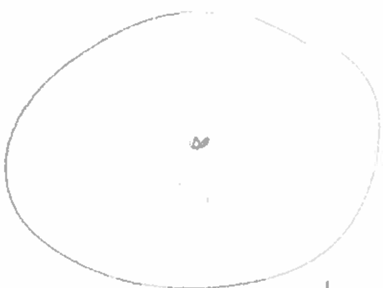
If you displace an object initially at rest, it stays at rest at the displace position

e.g.



Wheel on flat surface

or



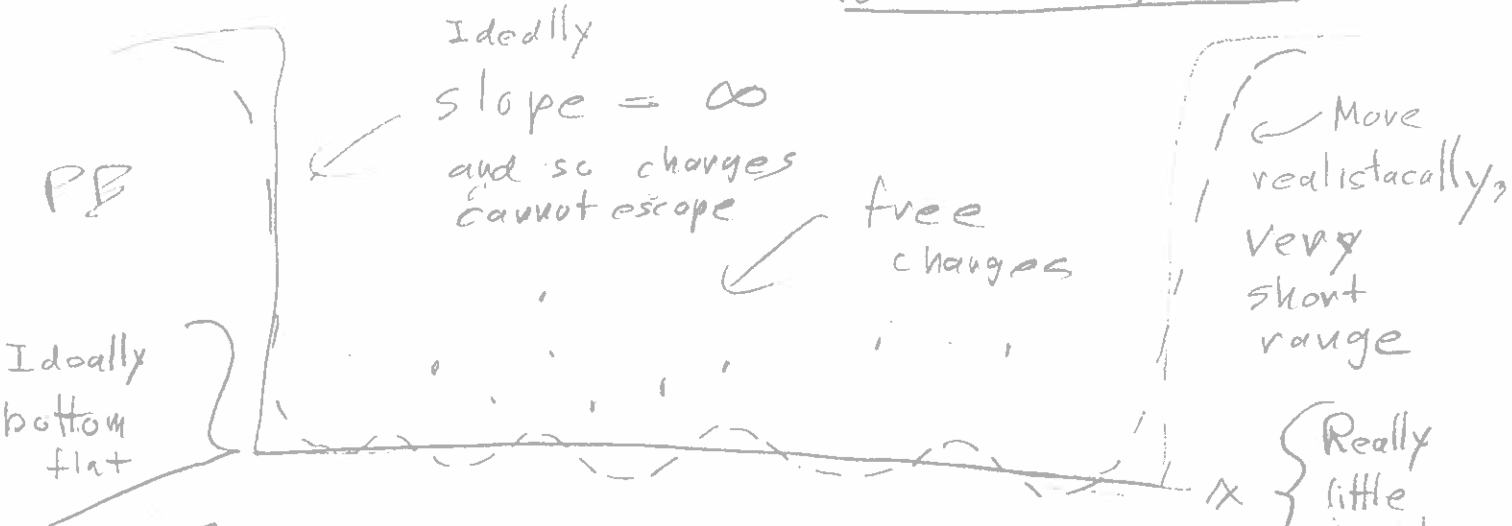
Wheel hung at its center of mass

it stays at rest at any position you rotate it to without giving it kinetic energy

02_06036)

Our Ideal conductors Bound the free charge in infinite Potential (energy) Wells whose sides are the surfaces

Potential (Energy) Well



Really little variations due ions in metal crystal

The potential well is created by the rigidly fixed positive ions in metals

The Potential energy walls are, of course, really electrostatic forces too but they are effectively very short range and in the ideal limit zero range, like the normal force

The free charges inside move under external applied E-fields and under their mutual interactions before they come to rest.

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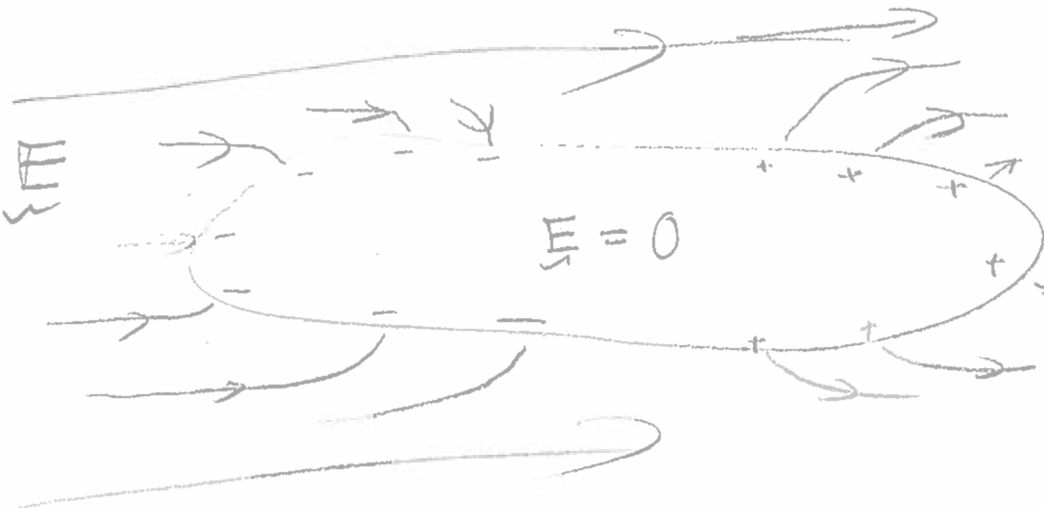
6.8 Conductors in Electrostatics II

a) Where Free Charge is in Electrostatic case and What is The Internal E-Field

i) Neutral in applied external E-field



Final electrostatic state after applying an external E-field,

The conductor becomes polarized overall

At first the charge moves under the external E-field, but the conductor does have electrical resistance and so the loses kinetic energy (KE) and comes to rest, but that can't be in the interior where it is free to move.

It reaches a lower energy state

The KE dissipates to waste heat

We consider consider the surface charge density of a conduction with steady-state current in Chapter 9 P,02_02036ff

face Charge Density
d Surface E-Field
r An Electrostatic Conductor

The local and long-range Coulomb E-fields can be cleanly distinguished in an ideal sense

er a piece of surface
e mesoscopic scale
ow macroscopic scale
above microscopic scale (i.e., atomic scale))

ly, smooth with charge
ng an infinitely thin layer,

rest
e
side,
he
es

r a sufficiently small Gaussian pillbox
el
the surface, so small the surface approximates an infinite plane
le;

lines
rdicular
ce,

ise
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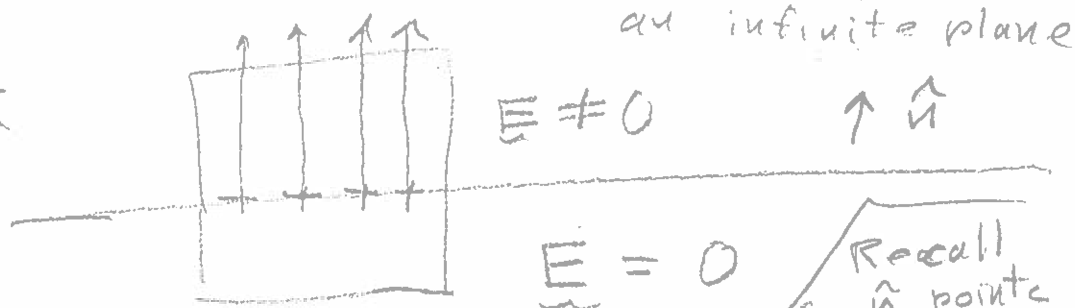


Recall $\hat{n}$ points normally outward from surface

∅

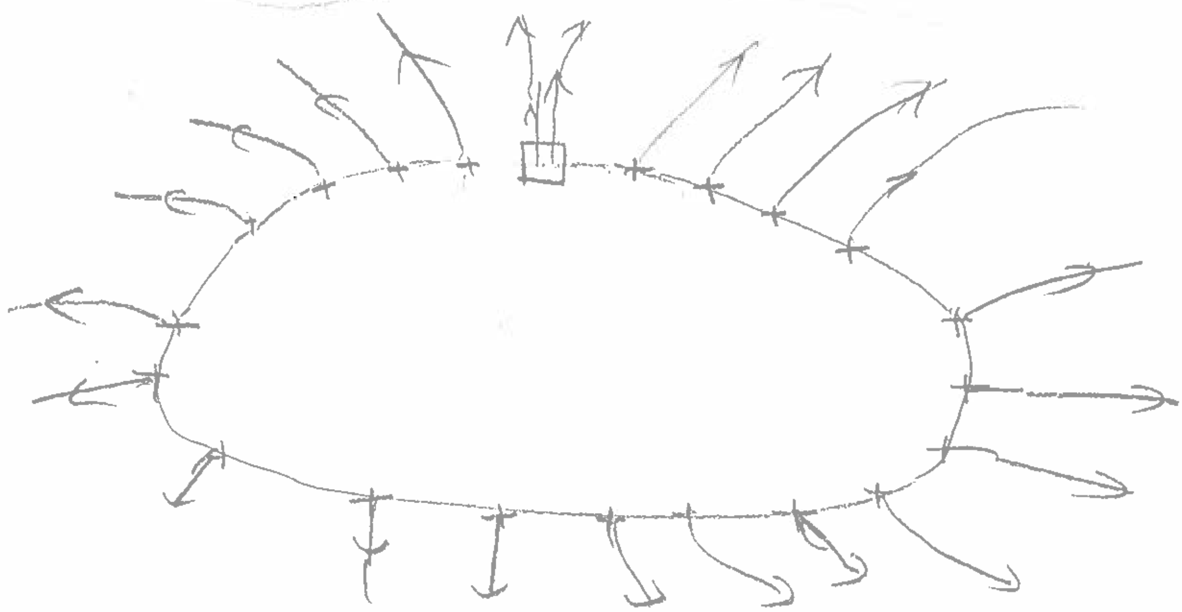
$$\underset{\sim}{E}_{local} = \frac{\sigma}{2\varepsilon_0}(\pm\hat{n})$$

(from the tiny Gaussian pillbox

see p. 02_0627 for the infinite planar slab result, but here compressed to infinitely thin layer.

But $\underset{\sim}{E}_{inside} = 0$

$$\therefore \quad = \underbrace{\underset{\sim}{E}_{\substack{local\\ inside}}}_{} + \underset{\sim}{E}_{long\ range}$$



Recall, the Coulomb force is an inverse-square law force

$$\underset{\sim}{E} = \frac{kq}{r^2}\hat{r}$$

For a point charge. It only goes to zero at infinitely formally.

So the Coulomb force is a long-range force intrinsically, but charge cancelling effects can make it effectively short-range in many cases

$$\therefore \ \underset{\sim}{E}_{\text{longrange}} = -\underset{\sim}{E}_{\text{local,inside}}$$

$$= -\frac{\sigma}{2\varepsilon_0}(-\hat{n})$$

$$= \frac{\sigma}{2\varepsilon_0}\hat{n}$$

lote the local charge does NOT push on itself (in own local sense), so is the long-range coulomb force that pushes the local charge outward and that force is cancelled by the infinite-potential-energy wall force of the conductor surface (see p. 02-6043-02-6044)

$$\therefore \ \underset{\sim}{E}_{\text{outside}} = \underset{\sim}{E}_{\substack{\text{local}\\ \text{outside}}} + \underset{\sim}{E}_{\text{longrange}}$$

$$= \frac{\sigma}{2\varepsilon_0}\hat{n} + \frac{\sigma}{2\varepsilon_0}\hat{n} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

To summarize

call n p. 02-0627 e mentioned we would how how to divide the -field at a conductor surface d we have done that here.

$$\underset{\sim}{E}_{\text{local}} = \frac{\sigma}{2\varepsilon_0}(\pm\hat{n})$$

+ve for outside

-ve for inside

$$E_{\text{long-range}} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

$$\underset{\sim}{E}_{\text{inside}} = \emptyset$$

$$\underset{\sim}{E}_{\text{outside}} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

c) But what is $\sigma(\underset{\sim}{r}_{\text{surface}})$?

In general, it is going to have a very complex variation with position.

However, the charge is in a stable equilibrium everywhere (See p. 02-06032)

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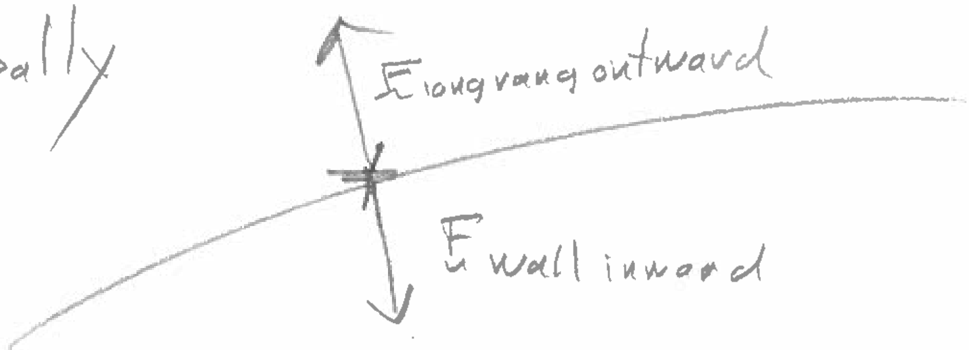
with the short range surface force inward and the long range $\underset{\sim}{E}$ outward providing the restoring force

Ideally



Less idealized

surface layers seen mesoscopically and microscopically are usually rough even if smooth macroscopically and they also usually have at least a thin layer of metal oxide (e.g., rust, tarnish)

A pure metal surface exposed to air usually develops this layer quickly

<u>Gold</u> strongly resists oxidation, but this is bulk property. Gold atoms are <u>not</u> as inert as noble gas atoms (Nature, 2026may29)

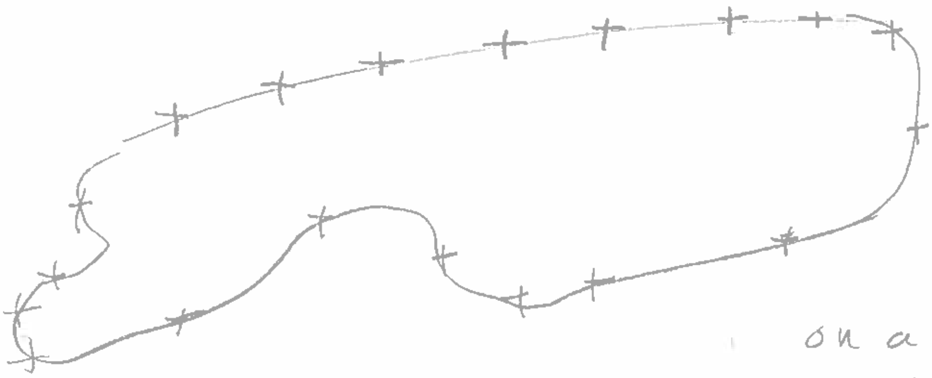
02_06044

And the potential well wall is NOT infinite slope

So the charge layer is NOT infinitely thin microscopically

However, Google AI tells me that some free electrons (free relative individual s, not conductor) can float barely bound above the ion surface. They are in image-potential states, but to know more I'd have read detailed articles. Wikipedia fails me

) However revert to the ideal case, it is still hard to determine the stable equilibrium $\sigma(\underline{r})$ distribution on a conductor in electrostatics.



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<u>But</u> the solution is unique according to uniqueness theorem of electrostatics. (Google AI)

Some exact analytic solutions $\sigma(\underset{\sim}{r})$ exist for special conductor shapes, but for many shapes probably only numerical or empirical solutions exist

$\underbrace{\text{numerical}}_{\text{the computer}}$ $\underbrace{\text{empirical}}_{\text{the experiment}}$

e) One Exact Solution for $\sigma(\underset{\sim}{r})$ Is Easy for a Finite Conductor

What is the shape of that finite conductor

<u>Ans</u>, A sphere.

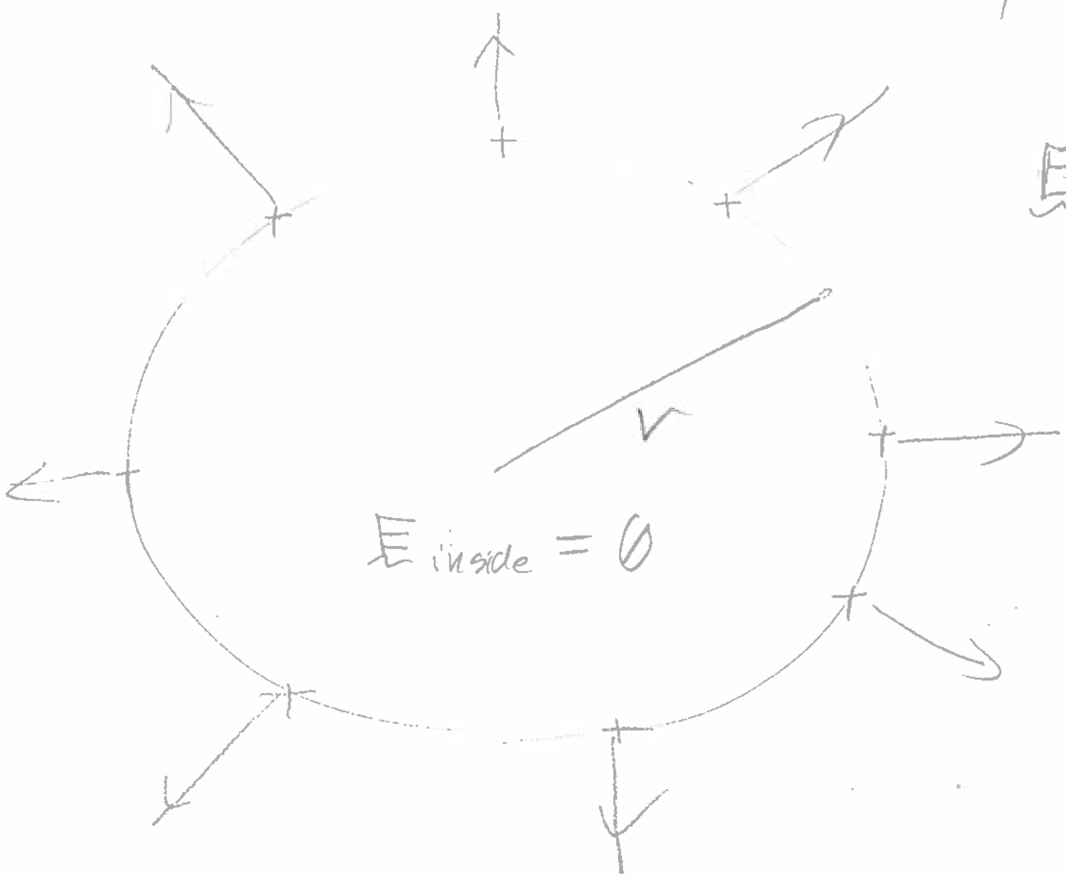
Why is it's solution easy?

<u>Ans.</u> The very high symmetry of a sphere.

What is $\sigma(\underset{\sim}{r})$ for a spherical conductor with total charge Q?

Ans.

$$\sigma(r) = \frac{Q}{4\pi r^2}$$



$$\underset{\sim}{E}_{outside} = \frac{kQ}{r^2}\hat{r}$$

by the shell theorem (p. 06-06012)

f) Cavities in Conductors in Electrostatics and Faraday Cages

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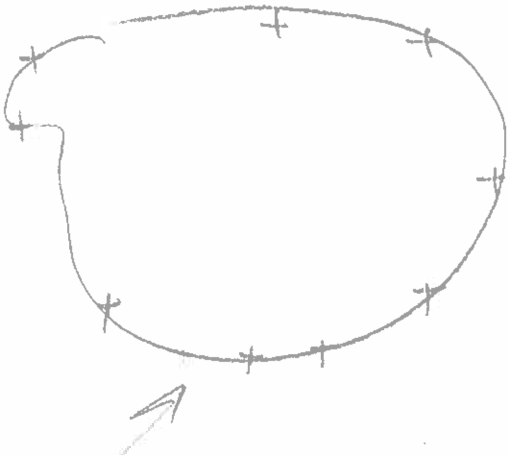
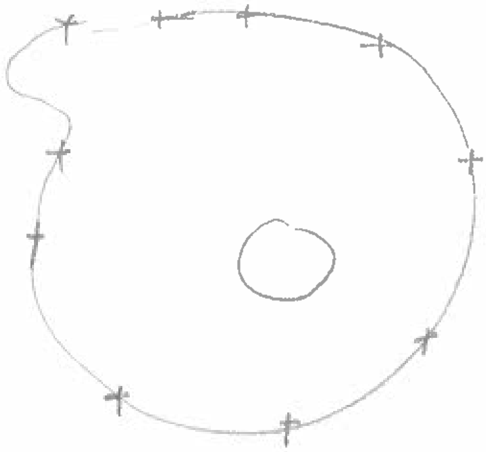
What is $\underset{\sim}{E}$ in the cavity?

What is $\sigma(\underset{\sim}{r})$ on the cavity surface?

The answers are easy



No cavity



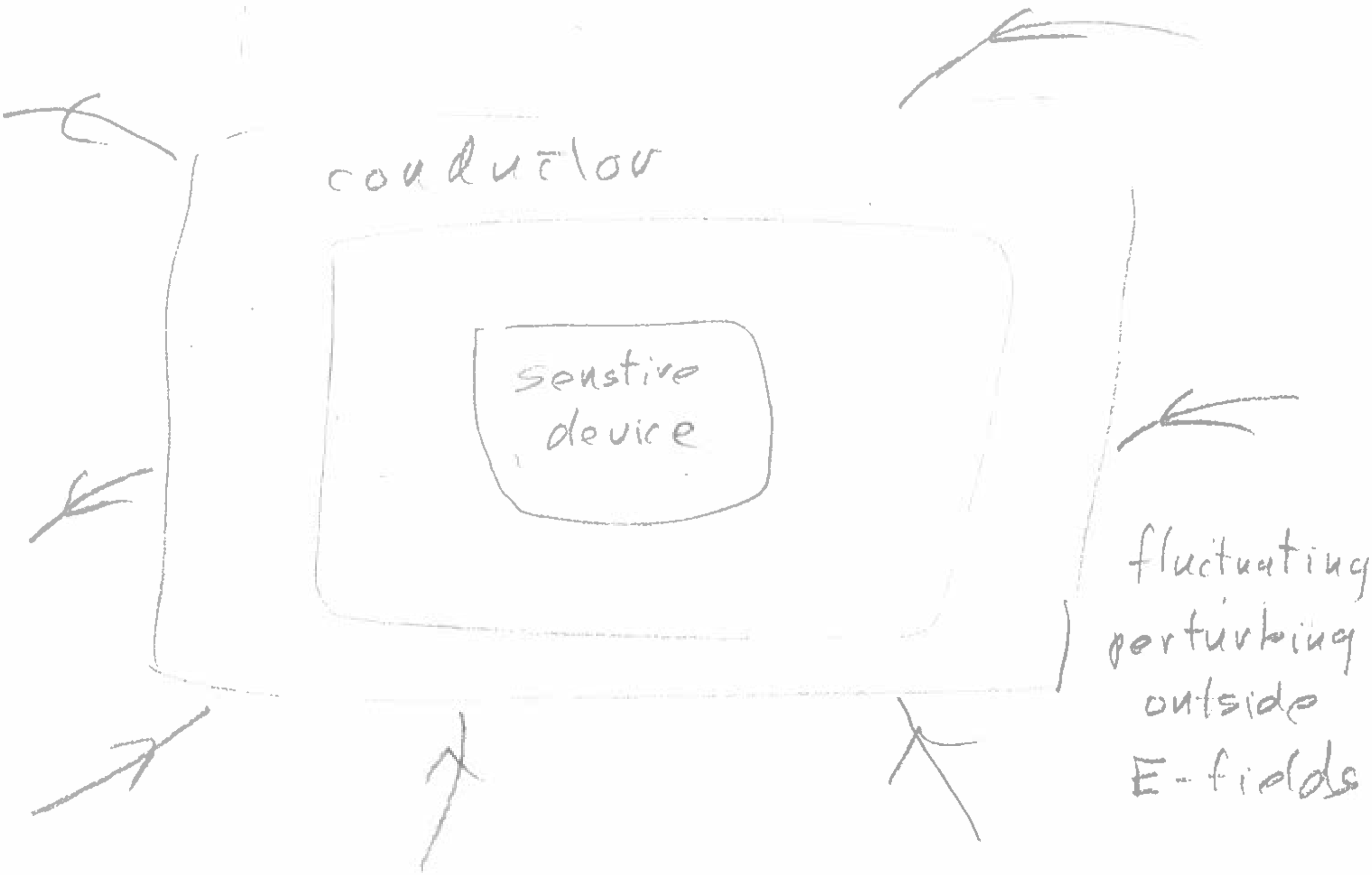
Magically create a cavity.

There was **No** E-field or charge before the cavity, there can be none after.

There are more mathematical proofs (G-102), but this suffices.

)2-06048

The shielding from outside electrical fields feature of conductors is technologically very import. Many devices are affected badly by outside E-fields, but if you enclose them in a conductor they are protected.



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Perfect shielding requires complete enclosure in the conductor and static outside E-fields, but in practice even a wire mesh shield is often good enough and responds quickly enough to fluctuating E-fields (6-102).

The shielding device is called a Faraday cage after inventor Michael Faraday (1791-1867).

e) Point Effect

The surface charge density $\sigma(\underset{\sim}{r})$ on conductors in electrostatics tends to be highest at protuberances, ridges, and sharp points on the conductor surface.

0206050 | At least Google AI calls this
effect the point effect.

Empirically, it is extremely well known
and there are some
exact analytic solutions
exemplifying it.

Richard Feynman (1918–1988)
in his famous lectures
(Lect. 6.11) give a mathematical
proof using electric potential.
We will give this proof on p. 02_07056
in Chapter 7.

In fact, the point effect is
a tendency usually manifested,
but it can be overruled
in some cases
unless you define pointiness
in a non-obvious way.

I think there is no fully convincing
word proof (or some one would
have given it), but some
plausibility arguments can be given

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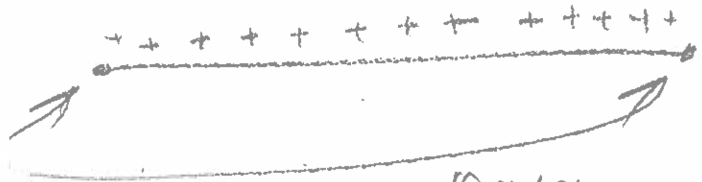
02_060-5 1

Case 1 An infinitely thin finite line segment conductor. Give it uniform $\lambda(x)$ at time zero.

Note conduction is all surface



the end point a zero range infinite potential wall force

The mutual repulsion cause of the

long range Coulomb force inceases towards the ends

Relaxation to equilibrium

So initially, the Coulomb force tries to concentrate the charge at the end points.

After electrical resistance dissipates all KE to waste heat one imagines the final distribution is probably

or as plot

NOT a realistic case since the wall for at the end has to infin

$$E(end) = \int_a^L \frac{k\lambda(x)dx}{x^2}$$

$$\approx k\lambda\left[\frac{1}{a} - \right.$$

which goes to infinity as $a \to 0$

2_06052

The short range end forces forbid charge to escape, but there is <u>no</u> long range Coulomb force from beyond the <u>ends</u> to counter the long range Coulomb force between the ends.

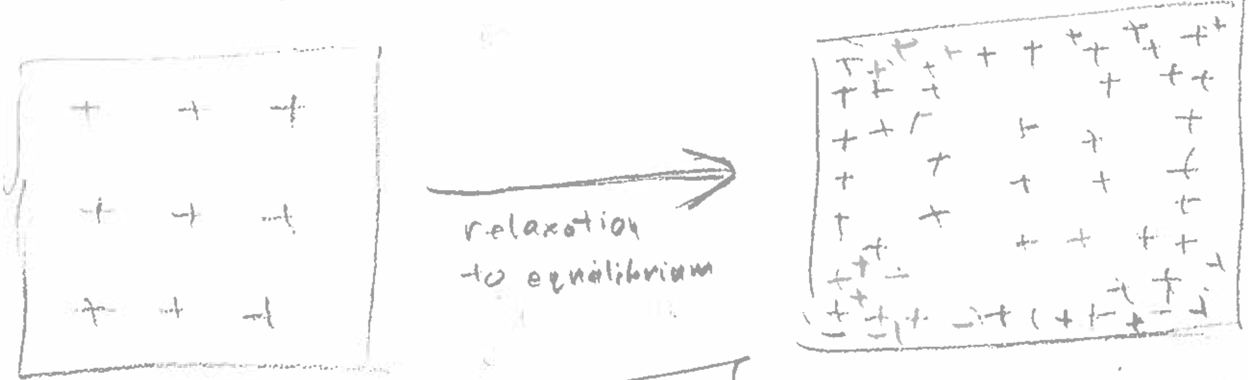
[the] long range force concentrates the charge at the ends

So the point effect arises

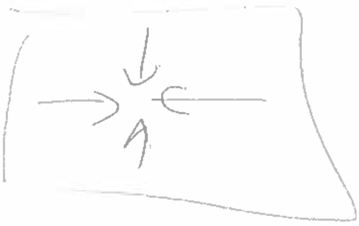
<u>but</u> I can't convince myself that the inverse-square law Coulomb force may lead to some non-physically intuitive distribution

Case 2 An infinitely thin square segment of a plane

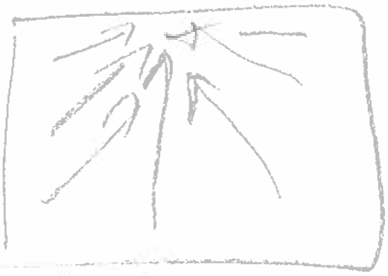
Start with uniform σ



Since equilibrium the sum of <u>all</u> forces (long range and short range) is <u>Zero</u>.



at center
all long range
force points
to center
(where they
cancel)



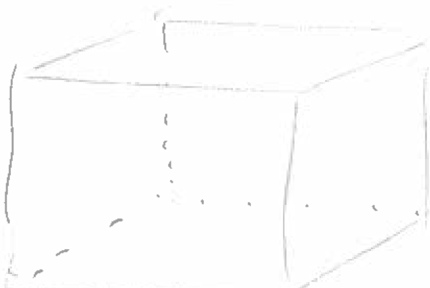
At mid side
there is a
concentration
of long range
force to
edge, but
still lots
of parallel
to edge
concentration

– The short range
edge force
keeps charge
contained

But the charge
is more
compressed
at the mid edge
than at center

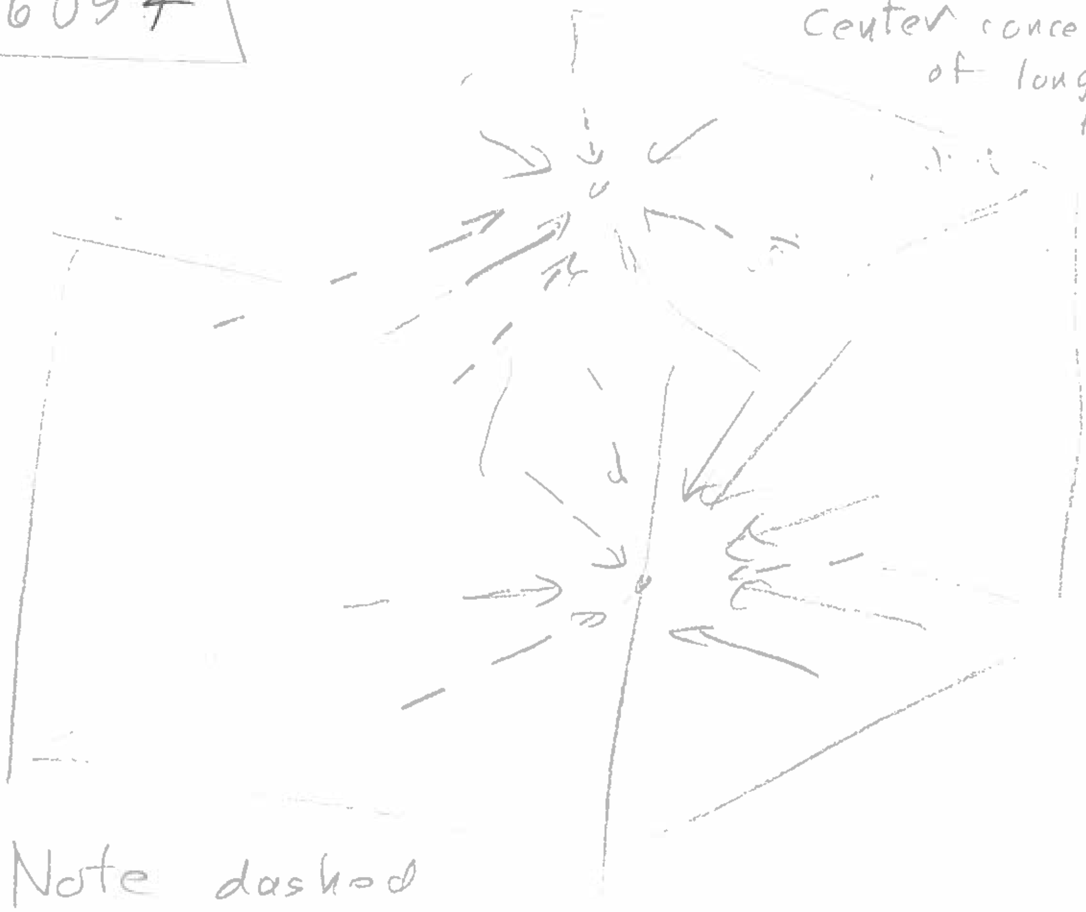
At a corner
there is a
greater
concentration
of long range
force,
and so
a greater
concentration
of
charge

Case 3 Consider a cube conductor



Start with a uniform
surface density
σ and then
allow relaxation
to equilibrium.

)2_06054



Center concentration
of long range
force and
charge

mid
edge
concentration
of
long range
force
and charge,

Note dashed
line indicate long range force
from across the interior
Where the E-field sums to
zero magically, but
sums to a finite outward
E-field at the surface

Even more
concentration
of long range
force and
charge toward
corners.

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Case 4 You can generalize Case 3

to Platonic Solids : i.e.,

regular polyhedrons :

tetrahedron 4 faces

cube 6 faces

Octohedron 8 faces

Dodecahedron 12 faces

Icosehedron 20 faces

then generalize to

all irregular

3-dimensional conductor samples

But recall the "proof"

is just plausibility.

f) Electrical Breakdown (Wik)

Concentration of high surface charge density

and therefore high surface E-field

can be a problem.

Insulators will suffer electrical breakdown

and become conductors for

sufficiently high E-field.

02_06056

Then you get electrical discharge (Wik)

Corona discharge (Wik)
is special name
for electric discharge
in air (or other fluid)
around a conductor

An electric spark
is an abrupt discharge (Wik)

Lightning (not a point effect case)
is an extreme electric spark.