2025 jun 01 | 4001

4) Multi-Dimensional Kinematics

4.1, 4.2

Already Covered

These topics in Chapter 3 notes & Lectures

Except we didn't do tedious examples

4.3 Projectile Motion

Motion in orthogonal directions is independent,

Perpendicular directions

In a Newton's 2nd Law Sense.

It's a vector law and true component by component.

$$\underbrace{\vec{F}_{net}}_{\text{Net force on a body}} = m \vec{a}$$

center of mass acceleration

$\therefore$ $F_{i\,net} = m a_i$ ← element representation.

4.002

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Obviously complicated systems can have forces in many directions and those forces can be dependent

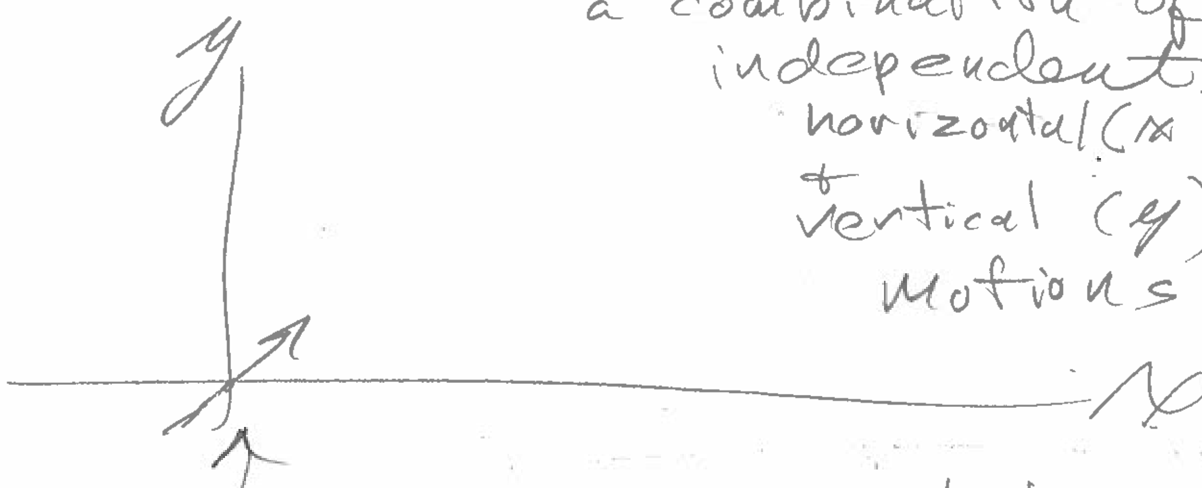
example (ordinary isotropic pressure) Pressure force.

It pushes outward equally in all directions

Mountains slump because pressure pushes horizontally as well as vertically

More on forces later

For the moment just a combination of independent horizontal (x) + vertical (y) motions



Launch a projectile and with $v_0 = v_{x0}\hat{i} + v_{y0}\hat{j}$

No air drag

4004 (2025 jun ~~04~~)

Why? Lack of scientific idealization

All these things had to be discovered

Lack of doctrine of controlled experimentation,

Lack of idea of how to treat accelerated motion mathematically

Galileo (1564-1642) is held high regard because he put all these concepts together and put them in the historical record.

(ca. 287 -212 BCE)

The ancient Greek mathematician Archimedes ~~did~~ too, but he failed establish these ideas in the practice of Nature knowledge

Analysis of projectile motion

a) Shape

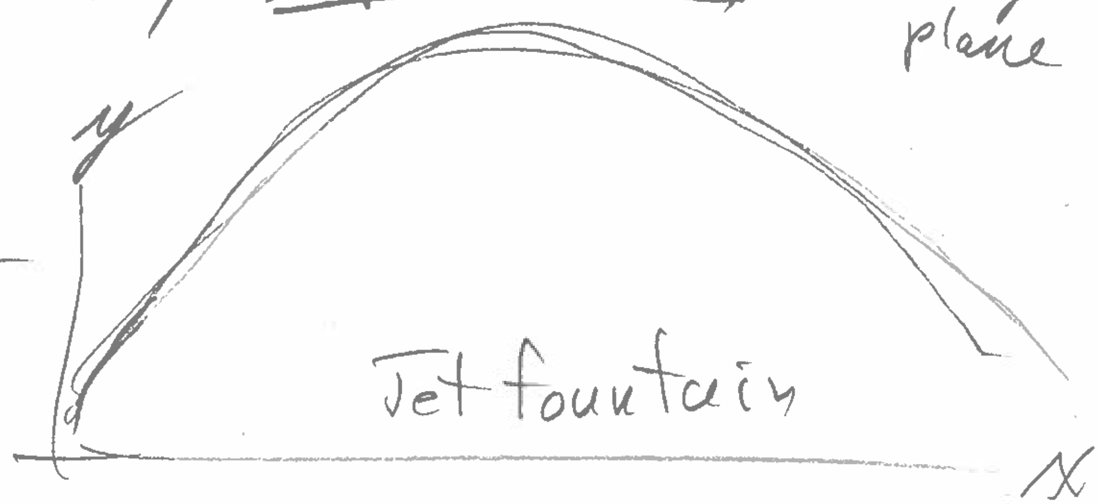
$x = v_{x0} t$

In x direction after launch and before impact, no force acts, ~~and~~ so v_x is constant.

14003

No air drag
is scientific idealization
1000s of years of ballistics
from throwing stones
~~to~~ shooting cannon
and No one until Galileo
found out projectile motion
ideally is parabolic in x-y
plane

You can
even
see it
frozen
in jet
fountains



known since ancient Roman times
and probably a lot
earlier

and the Greco-Roman mathematicians
understood parabolas — though
they understood without formulae
in an ancient Euclidean geometry sense.

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In x direction gravity acts causing free fall

with $a_y = -g$

i.e. equn of kinematics (② in my list which is not a universal ordering)

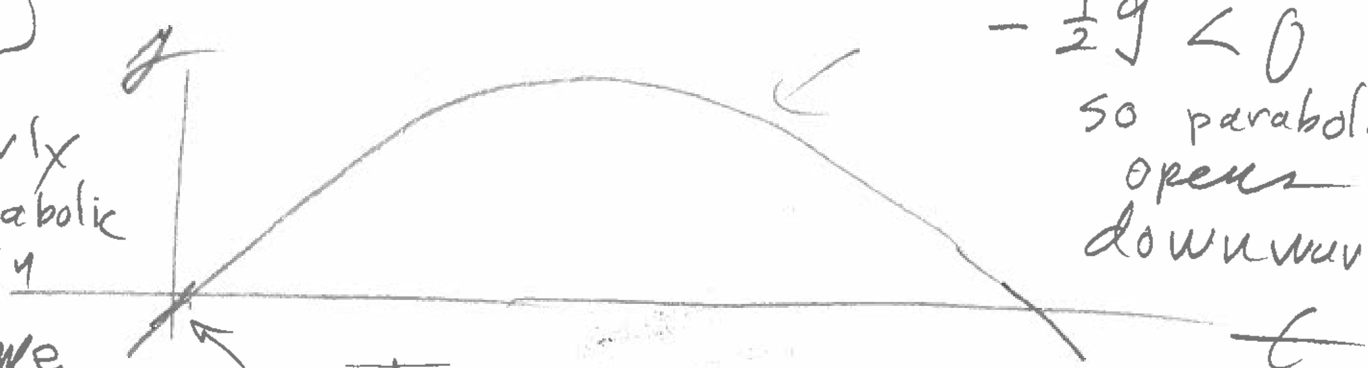
$$y = -\tfrac{1}{2}gt^2 + v_{y_0}t$$

we take ground level ~~as~~ $y = 0$

$X = (x(t), y(t))$ ~~or~~ $x_i(t)$

$-\tfrac{1}{2}g < 0$ so parabola opens downward

clearly parabolic in time



slope at launch $= \left.\dfrac{dy}{dt}\right|_0 = v_{y_0} > 0$

timeless equn (③ in my list of kinematic ~~equation~~)

$$v_y = \pm\sqrt{v_{y_0} + 2(-g)\,y}$$

same speed but

4006 (2025 Jun 01)

<u>But</u> what is <u>path in space</u>?

Well $\underset{\sim}{r}(t) = [x(t), y(t)]$

but that is $= [v_{x0} t,\ y = -\frac{1}{2} g t^2 + v_{y0} t]$

And that's useful,

Time is the great coordinator of the universe

and in the <u>classical limit</u> it's the same in all frames

<u>But</u> to understand the path in space we want $y(x)$

Not $y(t)$

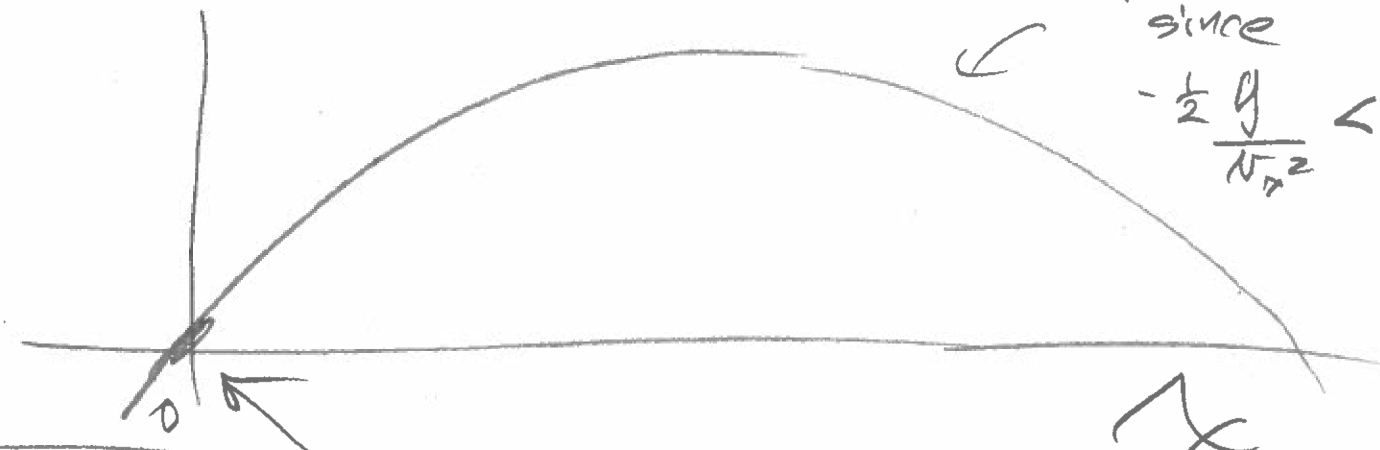
<u>Question</u> Find $y(x)$ by substituting for t.

Go.

<u>Ans.</u> $t = \frac{x}{v_{x0}}$, $y = -\frac{1}{2} g \frac{x^2}{v_{x0}^2} + \frac{v_{y0}}{v_{x0}} x$

A parabola

So Parabolic in x as well as in t



opens downward since $-\frac{1}{2}\frac{g}{v_x^2} < 0$

Not a velocity now

$$\text{Slope} = \left.\frac{dy}{dx}\right|_{x=0} = \left.\frac{d}{dx}\left(-\frac{1}{2}g\frac{x^2}{v_{x0}^2} + \frac{v_{y0}x}{v_{x0}}\right)\right|_{x=0}$$

Question

take the derivative and then set $x = 0$.

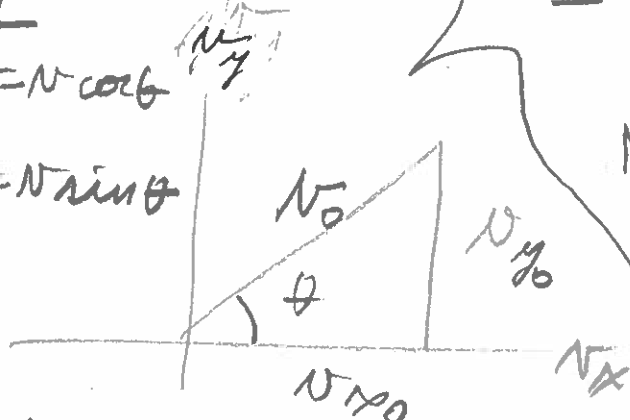
$$\left.\frac{dy}{dx}\right|_{x=0} = \left.\left(-g\frac{x}{v_x^2} + \frac{v_{y0}}{v_{x0}}\right)\right|_{x=0}$$

$$= \frac{v_{y0}}{v_{x0}} \text{ which is a tangent}$$

Question

$v_{x0} = ? = v\cos\theta$

$v_{y0} = ? = v\sin\theta$



Now Recall velocity extents in abstract velocity space, but direction is in space space.

$$\tan\theta = \frac{v_{y0}}{v_{x0}}$$

So θ is the Launch angle.

4008

b) Maximum height

Recall timeless equn

$$v_y^2 - v_{y0}^2 = 2(-g)\,y$$

Question Solve for y_{max}, Go!

$$0 - v_{y0}^2 = 2(-g)\,y_{max}$$

when $v_y = 0$ at top of parabolic path

$$y_{max} = \frac{v_{y0}^2}{g}$$

Question Does Dimensional analysis confirm this result?

Dimensional analysis

$$[y_{max}] = L$$

$$\stackrel{?}{=} \left[\frac{v_{y0}^2}{g}\right] = \left(\frac{L^2/T^2}{L/T^2}\right)$$

$$= L \quad \underline{\text{yes}}.$$

c) Range

Recall $y = -\frac{1}{2} g \frac{x^2}{v_{x0}^2} + \frac{v_{y0}}{v_{x0}} x$

We are on level ground,

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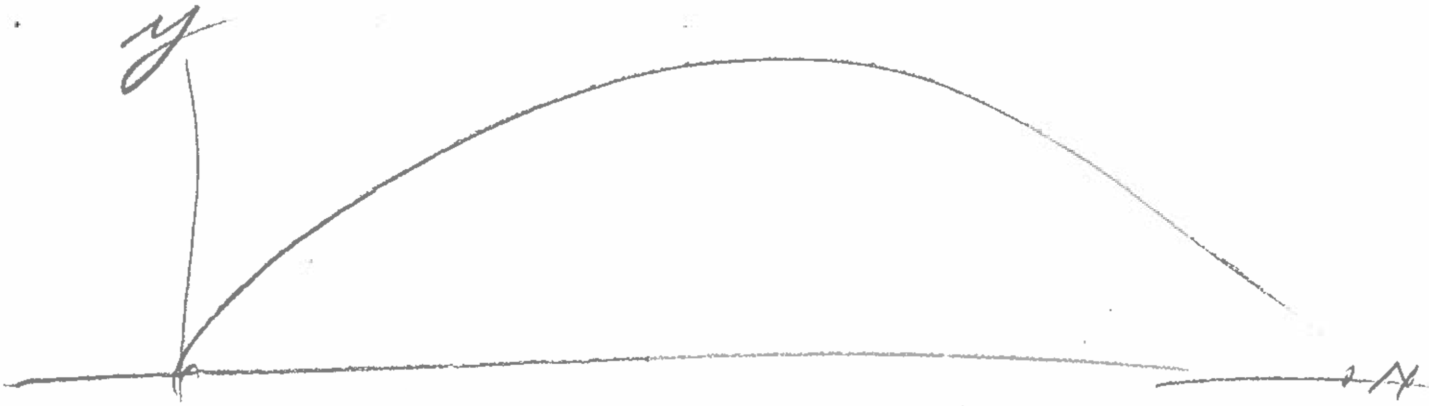

Question Solve for range? Go
$y = 0$ at both ends

$\therefore$ One solution is $x = 0$,
but that is not range

Other solution $0 = -\frac{1}{2} g \frac{x}{v_{x0}^2} + \frac{v_{y0}}{v_{x0}}$

$$x_{range} = \frac{2}{g} v_{x0} v_{y0}$$

Question Recall $v_{x0} = v_0 \cos\theta$

$v_{y0} = v_0 \sin\theta$

Rewrite x_{range} in terms of v_0 and θ

$$x_{range} = \frac{2 v_0^2}{g} \cos\theta \sin\theta$$

Use trig identity
$\sin 2\theta = 2 \sin\theta \cos\theta$
Wik

$$x_{range} = \frac{v_0^2}{g} \sin(2\theta)$$

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Question Given fixed v_0 what is the ~~launch~~ angle for maximum range?

You'll have to use calculus to find the maximum. (So ~~and~~ recall chain rule)

$$x_{ran} = \frac{2v_0^2}{g} \sin 2\theta$$

$$\frac{dx_{ran}}{d\theta} = \frac{2v_0}{g} \cos(2\theta) \cdot 2 = 0$$

for stationary point

$$\therefore \cos(2\theta) = 0$$

$$\therefore 2\theta = \pi/2$$

$$\theta = \frac{\pi}{4} = 45^\circ$$

which seems reasonable. What you might guess.

But air drag especially ~~with~~ spinning objects can cause considerable difference.

Google AI says golf ball distance maximizes for $\theta \triangleq 24^\circ$, but with lots of variation depending on swing speed and other factors

4012

but $|\underset{\sim}{v}|$ is constant for uniform circular motion

$\theta = \omega t$

Greek small omega = universal symbol for angular velocity

units [radians / unit time]

Also called angular frequency

<u>Ordinary frequency</u>

$f = \frac{\Delta N}{\Delta t}$ where ΔN is revolutions or cycles per unit time

$\omega = \frac{\Delta\theta}{\Delta t}$ where $\Delta\theta$ is in radians and so is radians per unit time

This is a special case where we use different symbol and name for <u>quantity</u> just because we use different units

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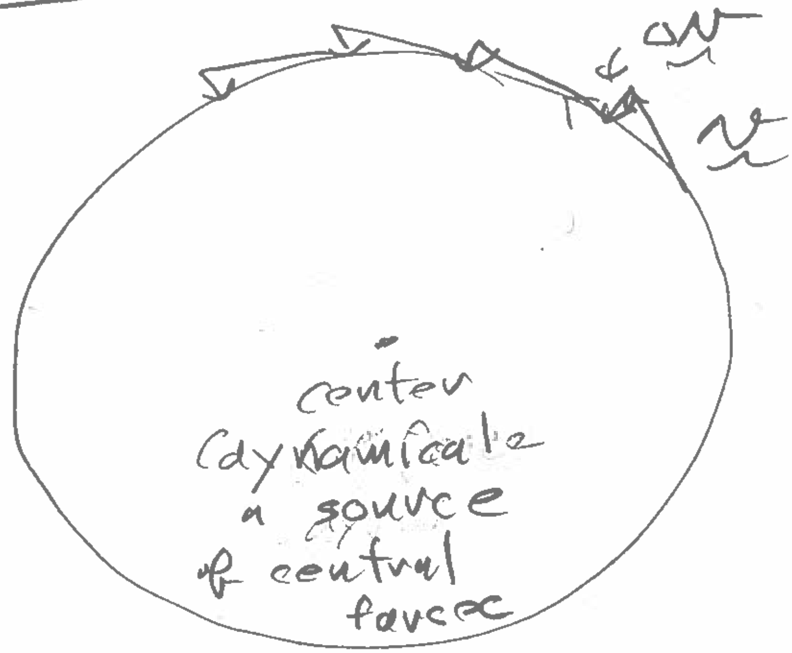
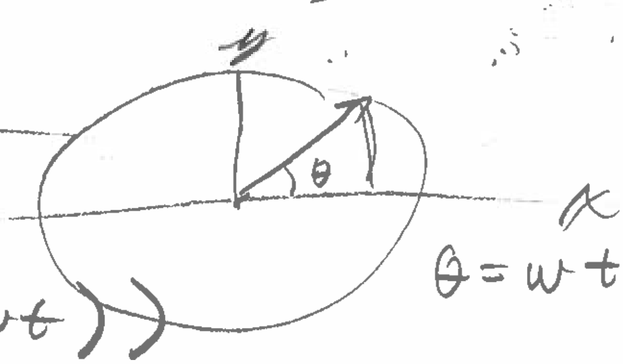
4.4 Uniform Circular Motion



Very qualitatively $\Delta \underset{\sim}{v}$ that pull you back onto the circle points toward the center which dynamically is a center of force.

In fact, the acceleration $\underset{\sim}{a}$ vector points inward radially exactly and it's always changing direction but $|\underset{\sim}{a}|$ (i.e., its magnitude is constant)

Proof



$\theta = \omega t$

$$\underset{\sim}{r} = (x, y) = r(\cos\theta, \sin\theta)$$

$$\underset{\sim}{r} = r(\cos(\omega t), \sin(\omega t))$$

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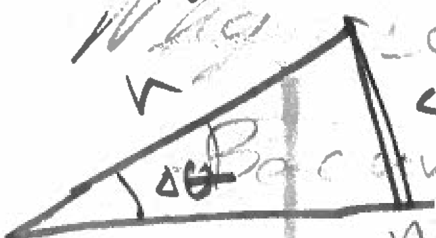
1 rev = 2π radians

$\therefore f * 1 = f\underbrace{\left(\frac{2\pi}{1}\right)}_{\text{factor of units}} = \omega$

$\therefore \omega = 2\pi f$ or $f = \frac{\omega}{2\pi}$

~~Why~~ use ω not f.



$\Delta s = r\Delta\theta$ only if $\Delta\theta$ is in radians

Circumference $= r(2\pi)$

$\therefore$ tangential velocity $v = \frac{ds}{dt} = r\omega$

Also radians are natural units in calculus

Expansions in Taylor series

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$

But only equal if θ is in radian

$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$

To resume

So $\underset{\sim}{r} = r(\cos(\omega t), \sin(\omega t))$

Question

$\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt} = \dot{\underset{\sim}{r}} = ?$

Remember the chain rule

4014 | 2025 jan01

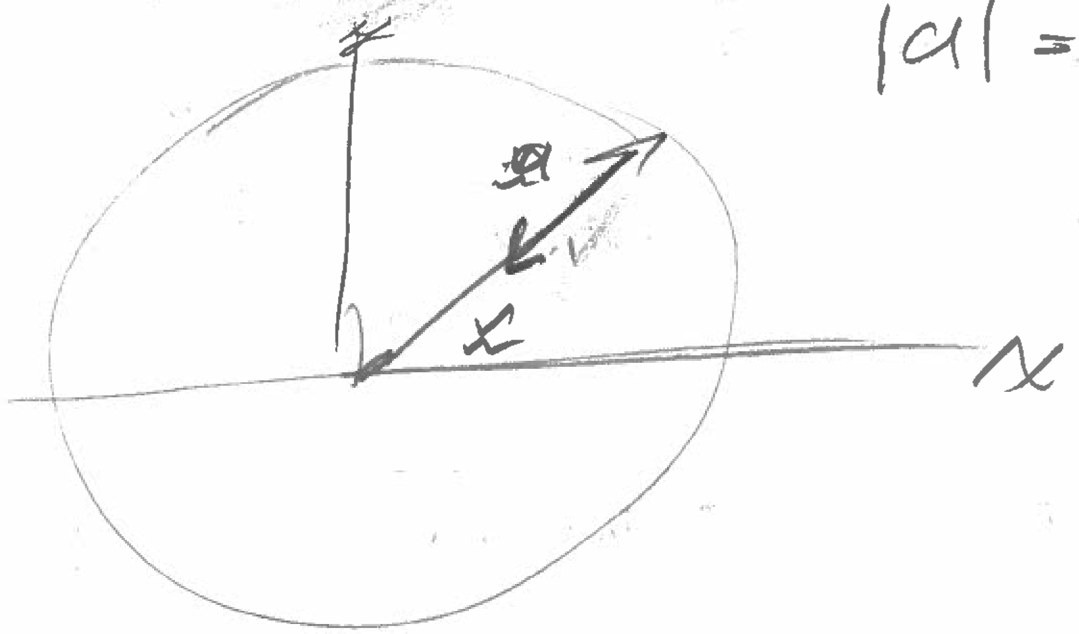
$$\underset{\sim}{v} = \omega r\,(-\sin\omega t,\ \cos\omega t)$$

and $v = |\underset{\sim}{v}| = \omega r$ The tangential velocity

$$\underset{\sim}{a} = \dot{\underset{\sim}{v}} = \omega^2 r\,(-\cos\omega t,\ -\sin\omega t)$$

$$\underset{\sim}{a} = -\omega^2\,\underset{\sim}{r}$$

$$|a| = \omega^2 r$$



$\underset{\sim}{a}_c = -\omega^2\,\underset{\sim}{r}$ is called centripetal acceleration because it points to center (center-pointing acceleration)

or $\underset{\sim}{a}_c = -\dfrac{v^2}{r}\,\hat{r}$

since $v = v_{tangetial} = \omega r$ and so $\omega = \dfrac{v}{r}$.

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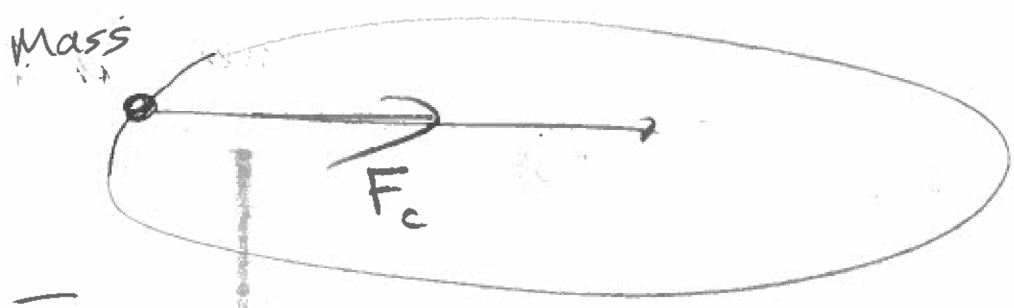
Which means from $\underset{\sim}{F}_{net} = m\underset{\sim}{a}$

there Must be
a centripetal force

Demonstration with sling (undangerous)



$$\underset{\sim}{F}_{centripetal} = m(-\omega^2 \underset{\sim}{r}) = -m\frac{v^2}{r}\hat{r}$$

In ~~some~~ sense,
with a sling, it
is obvious.
The force must
central because
you can't push
with a rope.

Scientific idealization

An ideal rope has
— no friction
~~or~~ air drag
— no mass
— no resistance to shear force

magnitude

$$F_c = m\frac{v^2}{r}$$

Note the
centripetal
force
is a force
named for
what it does
Not for what
it is.
For my sling it's
the tension force
For orbiting planets
it's gravity. it's
it Gravity.

4016 [2025jun01]

4.5 Relative Motion

All motion is relative to something

And inertial frames are the primary ones for physics but we leave them to Ch. 5 which is next.

Classical AKA Galilean transformation between frame S and frame S'

$$x' = x - v_{frame}\, t$$
$$y' = y$$
$$z' = z$$

They coincide at t=0.

A key point to get ahead of ourselves is that if S is an inertial frame so is S′ because v_{frame} is constant.

Any frames that are local to each other (at same place) ('Local' in physics mean either at the same place or nearly the same place depending on context) that are not relatively accelerated with respect to each other are all inertial frames if one is.

But a frame rotating with respect to an inertial frame is NOT inertial frame unless you make it so →

4018 | 2025 June 01 |

using inertial forces
Which you often do
(e.g., the Earth surface)

Now ~~consider~~ an object located in two frames:

$$\underset{\sim}{r}' = \underset{\sim}{r} - \underset{\sim}{v}_{frame} t$$

Question Differentiate

$$\underset{\sim}{v}' = \underset{\sim}{v} - \underset{\sim}{v}_{frame}$$

Question Differentiate again

$$\underset{\sim}{a}' = \underset{\sim}{a}$$

Acceleration is independent of frames that are mutually inertial frames

A key point for Newton's laws.