Introductory Physics: Calculus-Based **Name:**

Homework 15: Oscillations: Homeworks are due as posted on the course web site. Multiple-choice questions will **NOT** be marked, but some of them will appear on exams. One or more full-answer questions may be marked as time allows for the grader. Hand-in the full-answer questions on other sheets of paper: i.e., not crammed onto the downloaded question sheets. Make the full-answer solutions sufficiently detailed that the grader can follow your reasoning, but you do **NOT** be verbose. Solutions will be posted eventually after the due date. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

Multiple Choice Questions:

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016 qmult 00100 1 4 5 easy deducto-memory: oscillation defined

1. "Let's play *Jeopardy*! For \$100, the answer is: It is a motion that repeats itself in equal time periods: i.e., a periodic motion. Perfect repetition is an ideal case that is more or less closely approached in reality. The repeated motion is sometimes called a cycle."

   What is a/an ____________, Alex?

   a) inhalation b) exhalation c) rotation d) accleration e) oscillation

   SUGGESTED ANSWER: (e)

   Wrong answers:
   c) A rotation is not an oscillation, but oscillation can involve rotation in some cases.

   Redaction: Jeffery, 2008jan01

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016 qmult 00110 1 1 4 easy memory: period and frequency

2. Say P is the period of an oscillation. Say you observe N oscillations (or cycles) which take time NP, of course, The frequency of the oscillation (cycles per unit time) is given by:

   a) $f = NP$. b) $f = N/P$. c) $f = P/N$. d) $f = 1/P$. e) $f = N$.

   SUGGESTED ANSWER: (d)

   Behold:

$$f = \frac{N}{NP} = \frac{1}{P} \ .$$

   Wrong answers:
   a) As Lurch would say AAAaaaargh.

   Redaction: Jeffery, 2008jan01

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016 qmult 00120 1 1 1 easy memory: unit of frequency

3. The MKS unit of frequency is the hertz (Hz). It is derived from the second:

   a) $1\,\text{Hz} = 1\,\text{s}^{-1}$. b) $1\,\text{Hz} = 1\,\text{s}$. c) $1\,\text{Hz} = 1\,\text{s}^{-2}$. d) $1\,\text{Hz} = 1\,\text{s}^{2}$. e) $1\,\text{Hz} = 1\,\text{s}^{-1/2}$.

   SUGGESTED ANSWER: (a)

   Wrong answers:
   b) As Lurch would say AAAaaaargh.

   Redaction: Jeffery, 2008jan01

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016 qmult 00200 1 1 2 easy memory: linear force law

4. The linear force law (AKA linear restoring force law, spring force law, Hooke's law force, and simple harmonic oscillator force law) applies approximately to a wide variety of systems. In fact, almost any stable equilibrium system from molecules to bridges and beyond obeys the linear force law for sufficiently small perturbations of any component around the component's stable equilibrium configuration. Components moving under the linear force law alone are in simple harmonic motion. Usually some damping force affects the system. The linear force law in one-dimension with the stable equilibrium point at the origin is given by:

a) $F = -kx^{1/2}$. b) $F = -kx$. c) $F = -kx^{3/2}$. d) $F = -kx^2$. e) $F = kx$.

SUGGESTED ANSWER: (b)

Wrong answers:

e) This gives the magnitude, but the sign is very wrong: this law gives an unstable equilibrium point at the origin.

Redaction: Jeffery, 2001jan01

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016 qmult 00210 1 1 3 easy memory: linear force law ubiquity

5. To be in stable equilibrium, an object must be subject to a restoring force. The restoring force is zero at equilibrium and for any sufficiently small displacement from equilibrium tries to push the object back toward equilibrium. If the kinetic energy built up by the force pushing an object back to equilibrium is too large, then the object can **OVERSHOOT** the equilibrium and an oscillation results. Whether an oscillation results or not depends on the system. In real systems in the absence of any driver force, a damping force will usually dissipate mechanical energy above that of rest at the equilibrium point and cause the oscillation to damp out and the object to come to rest at the equilibrium point again. If the displacement is sufficiently small from a stable equilibrium, then the restoring force will be in most cases linear in the displacement: i.e., for a one-dimensional case

$$F = -kx \ ,$$

where k is a constant and x is the displacement from the equilibrium point. The restoring force in the small-displacement limit is often linear because:

a) of no good reason. b) nature likes discontinuities. c) nature dislikes discontinuities.
d) nature is indifferent to discontinuities. e) it's lies, all lies.

SUGGESTED ANSWER: (c)

If the restoring force is continuous through the equilibrium point (which may be always true if one looks closely enough), then a Taylor's series expansion can be done at that point. Say $F(x)$ is the restoring force and $x = 0$ is the equilibrium. The Taylor's expansion is

$$F(x) = F(0) + x\frac{dF}{dx}\bigg|_0 + \frac{x^2}{2!}\frac{d^2F}{dx^2}\bigg|_0 + \frac{x^3}{3!}\frac{dF^3}{dx^3}\bigg|_0 + \ldots \ .$$

To be a restoring force $F(0) = 0$, $F(x < 0) > 0$, and $F(x > 0) < 0$. Thus, we see for small perturbations, the restoring force must be linear. Unless, $(dF/dx)|_0 = 0$. If $(dF/dx)|_0 = 0$, then we require $(d^2F/dx^2)|_0 = 0$ and $(dF^3/dx^3)|_0 < 0$. Such a 3rd-power restoring force seems to be very rare in nature—I know of no examples. One could design a system with a 3rd-power restoring force if one wanted to.

Wrong answers:

e) A nonsense answer.

Redaction: Jeffery, 2008jan01

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016 qmult 00220 1 1 3 easy memory: simple harmonic oscillator (SHO) 1

6. If the linear force alone acts on an object, the object is a simple harmonic oscillator (SHO) and undergoes simple harmonic motion (SHM). Apply Newton's 2nd law to 1-dimensional SHO with linear force $F = -kx$.

a) $m\frac{d^2x}{dt^2} = kx$. b) $m\frac{dx}{dt} = -kx$. c) $m\frac{d^2x}{dt^2} = -kx$. d) $m\frac{d^2x}{dt^2} = -\frac{1}{kx}$.
e) $m\frac{d^2x}{dt^2} = -x$.

SUGGESTED ANSWER: (c)

Wrong answers:

a) The minus sign is vital.

Redaction: Jeffery, 2008jan01

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016 qmult 00230 1 1 1 easy memory: SHO solution 1

7. The general solution to the SHO differential equation

$$m\frac{d^2x}{dt^2} = -kx$$

is ____________ where amplitude A and phase constant ϕ are set by initial conditions and angular frequency ____________ is determined by the physics of the system itself. The motion of the solution is called simple harmonic motion (SHM).

a) $x = A\cos(\omega t + \phi)$; $\omega = \sqrt{k/m}$ b) $x = A\cos(\omega t + \phi)$; $\omega = \sqrt{m/k}$
c) $x = A(\omega t + \phi)$; $\omega = \sqrt{k/m}$ d) $x = A(\omega t + \phi)$; $\omega = \sqrt{m/k}$ e) $x = A(\omega t + \phi)$; $\omega = \sqrt{km}$

SUGGESTED ANSWER: (a)

For answer (a), we find

$$\text{LHS} = m(-\omega^2)x = -kx = \text{RHS}$$

which verifies that the answer is the solution for $\omega = \sqrt{k/m}$.

Wrong answers:

b) Exactly wrong for omega.

Redaction: Jeffery, 2008jan01

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016 qmult 00232 1 1 2 easy memory: SHO solution 2: SHO angular frequency formula

8. For a one-dimensional simple harmonic oscillator with mass m, force law $F = -kx$, and solution obeying $a = -\omega^2 x$, the formula for the angular frequency ω is given by:

a) $\omega = m/k$. b) $\omega = \sqrt{k/m}$. c) $\omega = \sqrt{m/k}$. d) $\omega = k/m$. e) $\omega = \sqrt{km}$.

SUGGESTED ANSWER: (b)

From Newton's 2nd law we have

$$ma = -kx$$

and from the solution result we have

$$-m\omega^2 = -kx\ ,$$

and so we find

$$\omega = \sqrt{\frac{k}{m}}\ .$$

Wrong answers:

a) A nonsense answer.

Redaction: Jeffery, 2008jan01

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016 qmult 00234 1 1 4 easy memory: SHO solution 3

9. A solution to Newton's 2nd law for the simple harmonic oscillator is simple harmonic motion—which is a melodious, uncomplicated motion. The solutions for displacement, velocity, and acceleration for a case when the motion of the oscillating objects starts at time $t = 0$ at its maximum displacement A are, respectively:

a) $x = A\sin(\omega t)$; $v = \omega A\cos(\omega t)$; $x = -\omega^2 A\sin(\omega t)$.
b) $x = A$; $v = \omega A$; $a = \omega^2 A$.
c) $x = A\cos(\omega t)$; $v = -\omega A\cos(\omega t)$; $x = -\omega^2 A\cos(\omega t)$.
d) $x = A\cos(\omega t)$; $v = -\omega A\sin(\omega t)$; $x = -\omega^2 A\cos(\omega t)$.
e) $x = A\cos(\omega t)$; $v = \omega A\cos(\omega t)$; $x = \omega^2 A\cos(\omega t)$.

SUGGESTED ANSWER: (d)

Wrong answers:

a) These are the solutions if it starts at $x = 0$ with positive velocity.
b) These are just the maximum value solutions.

Redaction: Jeffery, 2008jan01

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016 qmult 00240 1 1 1 easy memory: sinusoidal motion

10. In sinusoidal motion, an object's position as a function of time varies like a sine or:

    a) cosine curve. b) tangent curve. c) inverse tangent curve. d) inverse sine curve.
    e) straight line.

SUGGESTED ANSWER: (a)

Wrong answers:
b) Tangent curves have those infinities.

Redaction: Jeffery, 2008jan01

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016 qmult 00250 1 1 3 easy memory: angular frequency

11. A sinusoid repeats its behavior every time its argument increases by 2π or 360°. An oscillation in time described by a sinusoid repeats in a time period called the period of the oscillation. The argument of the sinusoid in this case is $\omega t + phi$, where ω is the angular frequency and ϕ is a phase constant set by initial conditions. For a period P, the angular frequency is given by:

    a) $\omega = 2\pi P$. b) $\omega = P/(2\pi)$. c) $\omega = 2\pi/P$. d) $\omega = 1/(2\pi P)$. e) $\omega = 2\pi P^2$.

SUGGESTED ANSWER: (c)

Wrong answers:
a) Not dimensionally correct.

Redaction: Jeffery, 2008jan01

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016 qmult 00260 1 1 3 easy memory: SHM time formulae related

12. For simple harmonic motion,

    a) $f = P = \omega$. b) $P = 1/f = 2\pi/\omega$, $f = 1/P^2 = \omega/(2\pi)^2$, $\omega = 2\pi f = 2\pi/P$.
    c) $P = 1/f = 2\pi/\omega$, $f = 1/P = \omega/(2\pi)$, $\omega = 2\pi f = 2\pi/P$.
    d) $P = 1/f = 2\pi/\omega$, $f = 1/P = 2\pi\omega$, $\omega = 2\pi f = 2\pi/P$.
    e) $P = 1/f = 1/\omega$, $f = 1/P = \omega/(2\pi)$, $\omega = 2\pi f = 1/P$.

SUGGESTED ANSWER: (c)

Wrong answers:
a) A nonsense answer.

Redaction: Jeffery, 2008jan01

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016 qmult 00270 3 4 1 hard deducto-memory: linearity of SHO equation

13. If $x_1(t)$ and $x_2(t)$ are solutions to a differential equation (DE), then the DE is linear if $c_1x_1(t) + c_2x_2(t)$ is also a solution where the c's are arbitrary constants. The differential equation for the simple harmonic oscillator

$$\frac{d^2x}{dt^2} = -\omega^2 x(t)$$

is:

    a) linear. b) non-linear. c) neither linear nor non-linear. d) both linear and non-linear.
    e) colinear.

SUGGESTED ANSWER: (a)

The SHO DE is linear by inspection.

Wrong answers:
c) Not possible I think.
d) Definitely not possible.
e) Mischievous.

Redaction: Jeffery, 2001jan01

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016 qmult 00280 1 1 2 easy memory: SHO energy

14. The mechanical energy of a 1-dimensional simple harmonic oscillator of mass m and force constant k is given by:

    a) $E = kx_{\rm max}^2 = mv_{\rm max}^2$.   b) $E = (1/2)kx_{\rm max}^2 = (1/2)mv_{\rm max}^2$.   c) $E = (1/2)kx^2$.
    d) $E = (1/2)mv^2$.   e) $E = mgy$.

    SUGGESTED ANSWER: (b)

    Wrong answers:
    c) This is just the potential energy which is time varying.
    d) This is just the kinetic energy which is time varying.

    Redaction: Jeffery, 2008jan01

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016 qmult 00290 1 1 3 easy memory: uniform circular motion and SHM

15. The projection of uniform circular motion on a line in the plane of rotation is:

    a) uniform circular motion.   b) a Tusi-couple motion   c) simple harmonic motion.
    d) oblique motion.   e) round motion.

    SUGGESTED ANSWER: (c)

    Wrong answers:
    b) Well no, but Tusi-couple motion is related.

    Fortran-95 Code

    Redaction: Jeffery, 2008jan01

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016 qmult 00300 1 1 4 easy memory: physical pendulum equation of motion

16. A phyiscal pendulum is any rigid object held by a free pivot axis: the axis supports the pendulum againnst gravity, but exerts no torque about itself. The equation of motion of the physical pendulum (i.e., Newton's 2nd law applied to the physical pendulum) if only gravity torques it is:

$$\frac{d^2\theta}{dt^2} = -\frac{rmg}{I}\sin\theta \ ,$$

    where θ is the angle a radius from the pivot axis to the center-of-mass axis makes with a downward vertical, r is the cylindrical radius from the pivot axis to the center-of-mass axis, m is the pendulum mass, and I is the pendulum rotational inertia about the pivot axis. This equation has no simple analytical solution because of:

    a) general considerations.   b) the I factor.   c) the r factor.   d) the $\sin\theta$ factor.
    e) darn good reasons.

    SUGGESTED ANSWER: (d)

    Wrong answers:
    e) Not the best answer in this context.

    Redaction: Jeffery, 2008jan01

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016 qmult 00310 1 4 5 easy deducto-memory: small angle approximation

17. "Let's play *Jeopardy*! For \$100, the answer is:

$$\sin\theta \approx \theta \ ,$$

    where θ is given in radians.

    What is a/an ____________, Alex?

    a) equality   b) approximate equality   c) inequality   d) trigonometric function
    e) small-angle approximation

    SUGGESTED ANSWER: (e)

Wrong answers:
b) Not a best answer in this context.

Redaction: Jeffery, 2008jan01

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016 qmult 00320 1 1 3 easy memory: physical pendulum in SHO approximation

18. The equation of motion of the physical pendulum (i.e., Newton's 2nd law applied to the physical pendulum) in the small-angle approximation is given by

$$\frac{d^2\theta}{dt^2} = -\frac{rmg}{I}\theta \ .$$

We recognize this equation as the differential equation of:

a) Tusi-couple motion. b) uniform circular motion. c) simple harmonic motion.
d) straight-line motion. e) uniform motion.

SUGGESTED ANSWER: (c)

Wrong answers:
b) Nope.

Redaction: Jeffery, 2008jan01

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016 qmult 00330 1 1 5 easy memory: simple pendulum period

19. The period of the physical pendulum (in the small-angle approximation) is

$$P = 2\pi\sqrt{\frac{I}{rmg}} \ .$$

If the pendulum is shrunk to a point-mass bob, we have the simple pendulum. In this case, ____________ and ____________.

a) $I = (1/2)mr^2$; $P = 2\pi\sqrt{r/(2g)}$. b) $I = mr^2$; $P = 2\pi r/g$. c) $I = mr$; $P = 2\pi r/g$.
d) $I = mr$; $P = 2\pi\sqrt{r/g}$. e) $I = mr^2$; $P = 2\pi\sqrt{r/g}$.

SUGGESTED ANSWER: (e)

Wrong answers:
b) Wrong dimensionally.

Redaction: Jeffery, 2008jan01

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016 qmult 00334 1 3 2 easy math: fiducial pendulum period

20. The period of a simple pendulum is given by

$$P = 2\pi\sqrt{\frac{r}{g}} \ ,$$

where r is the length of the pendulum and g is the gravitational fiedl. A fiducial pendulum period (i.e., period that can be used as a standard for reference or quick estimation) is obtained for a pendulum of length exactly **ONE METER** and assuming that $g = 9.8\,\mathrm{m/s^2}$ exactly. What is this fiducial period to 2 significant figures? To 4 significant figures?

a) 1.0 s and 1.010 s. b) 2.0 s and 2.007 s. c) 2.0 s and 2.01 s. d) 1.0 s and 1.01 s.
e) 1.0 s and 1.1 s.

SUGGESTED ANSWER: (b)

The rough calculation is

$$P = 2\pi\sqrt{\frac{r}{g}} \approx 6.3 \times \frac{1}{3.3} \approx 2\,\mathrm{s} \ .$$

This result limits the possible right answers to answers (b) and (c). Since answer (c) has only 3 significant figures. Thus the right answer must be (b).

The more precise calculation to 4 significant figures is as follows:

$$P = 2\pi\sqrt{\frac{r}{g}} = 2.00708992\ldots \approx 2.007\,\mathrm{s}\ .$$

Fortran Code

```
        p=pi2*sqrt(1.d0/9.8d0)
        print*,pi2,p
!         6.283185307179586 2.007089923154493
```

Wrong answers:

c) This answer is only to 3 significant figures.

Redaction: Jeffery, 2001jan01

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016 qmult 00336 2 3 5 moderate math: pendulum period sans calculator

21. Without using a calculator (you're on your honor here), what is approximately the period of a simple pendulum of length 9 m? **HINT:** Just look at what the simple pendulum period formula and reflect on the answer to the fiducial pendulum period question.

    a) 9.8 s. b) 162 s. c) 2.0 s. d) 1.0 s. e) 6.0 s.

SUGGESTED ANSWER: (e) For these modern, calculation-challenged students this is at least a moderate question.

Wrong answers:

a) This is g and it looks suspiciously wrong.
c) Whatever the answer is, it can't be the same as the fiducial pendulum period.
d) Whatever the answer is, it can't be less than the fiducial pendulum period.

Redaction: Jeffery, 2001jan01

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016 qmult 00340 3 5 4 tough thinking: pendulum period sans mass

22. Time for deep thought. If you are clever (and not like your physics professor) before doing any elaborate dynamical analysis, why should you know that the period of the simple pendulum is independent of mass?

    a) Mass is density times volume. Volume never comes into the simple pendulum problem. Ergo mass never comes into the simple pendulum problem.
    b) There no mass in the kinematic equations and thus mass can never affect the motion of anything. The simple pendulum is included in the set of anything.
    c) Physical intuition.
    d) For any particle acted on only by gravity and workless forces of constraint (e.g., normal forces such as those of a frictionless slope or of a frictionless wire for a sliding bead), the only force that can cause acceleration is the component of gravity in the direction allowed by the constraints. This component has mass as one of its factors and is the F in $F = ma$. Thus, the mass in 2nd Newton's law applied to the system cancels out from both sides of the equation. The acceleration is thus determined by g, the direction of gravity downward, and the directions allowed by the forces of constraint. In principle, the whole kinematics of the particle can be determined from knowing the acceleration for any location and the initial conditions. In practice, it might require elaborate calculations including computer calculations to find the whole kinematics. But mass does not come into the calculation. The simple pendulum fits the case just outlined. The ideal free massless pendulum arm exerts no force in the direction the pendulum bob can move. There is no air drag. The only force that can cause acceleration is a component of gravity. Consequently, the whole kinematics of the pendulum bob is independent of mass. The pendulum period is part of the kinematics. Therefore the pendulum period is independent of mass.
    e) Socrates has mass.
       All humans have mass.
       Ergo all humans are Socrates.

SUGGESTED ANSWER: (d)

Given that some answers are nonsense the students might be able to deduce the right answer. Also the question obeys the longest-answer-is-right rule.

Wrong answers:

a) There is something wrong with this syllogism. Well first off in idealized physics we do have point masses and and a point mass has no density or volume, and so the major premise is wrong for idealized physics. But even we grant that the major premise is true in reality and the minor premise is admittedly true, the conclusion still does not follow. We could still work the problem without specifying volume or density if only mass was needed to solve the dynamics.

b) There is no mass is the kinematic equations, but there is acceleration. If acceleration depends on mass through $F = ma$, then the kinematics depends on mass. The fact that mass doesn't explicitly appear in the kinematic equations is not conclusive.

c) Physical intuition is mostly just experience with a lot of problems. In this case, physical intuition will work *pour moi* because, I've done the simple pendulum problem many, many times. But it's not the best answer. Physical intuition is fallible.

e) Irrelevant even if valid. And it's probably not valid. I haven't noticed myself being Socrates ever—maybe Euripides now and then. If I'd ever taken a course in logic, I'd know the right terminology to point out the logical flaw. I suppose Socrates belongs to the set of mass-havers. So do all humans, but that doesn't imply all humans belong to the set Socrates. A Venn diagram would make it clear. My apologies to Woody Allen (1935–).

Redaction: Jeffery, 2001jan01

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016 qmult 00350 1 5 5 easy thinking: pendulum clock and Galileo

23. You are Galileo Galilei (1564–1642) professor (untenured) of mathematics at the University of Pisa circa 1590. You are red-bearded and feisty (Italian word meaning ...). Tired of people making fun of your redundant name (exactly so in Latin: Galileus Galileus) as if you were Humbert Humbert or some such and bored with dropping balls off the Leaning Tower in the piazza, you seek calm in the adjacent Cathedral of Pisa. There you notice that the Cathedral lanterns oscillate in the wind with a constant period no matter what the amplitude of the oscillation provided the amplitude isn't too large and that period only varies with lantern cord length and not lantern mass as far as you can tell. At once—you are an incomparable genius after all—you realize that pendulums would make great regulators for clocks because:

a) it has **NEVER BEEN** been thought of before.

b) it has **BEEN** thought of before.

c) the hypnotic pendulum swinging motion would induce even deeper slumber in your less-amusing, clockwatching students.

d) even a bad idea can make money. All that is needed is a great advertizing campaign.

e) all clocks using pendulums as regulators for the motions of the hands and the energy input to keep the motions going would keep the periods of the hands very constant since small variations in amplitude that arise from somewhat irregular resistive and driver forces would have little effect on the pendulum period. Also the pendulum clocks could be kept synchronized to high accuracy despite varying amplitudes for the pendulum motion and masses for pendulum. Of course, the effective length of the pendulum does affect the period and has to be carefully adjusted for synchronization, but that is easy to do. It's one of those Eureka moments.

SUGGESTED ANSWER: (e)

Eliminating the nonsense answers leaves only one plausible answer. Of course this Galileo is fictional and the reasoning can only be very roughly along the lines of the real Galileo at some point in his life. He did experiment with pendulums and probably had the idea for a pendulum clock, but Christiaan Huygens (1629–1695) built the first real ones (Wikipedia 2007nov06).

Wrong answers:

c) It probably would, but that's probably not what Galileo thought. I don't think that classrooms in the late 16th centuries had wall clocks. Possibly students relied on clock bells or chimes. We've mostly regressed from that level of sophistication. Where's our bell?

d) The pendulum clock idea is not a bad idea.

Redaction: Jeffery, 2001jan01

016 qmult 00400 1 4 2 easy deducto-memory: damped oscillations

24. "Let's play *Jeopardy*! For \$100, the answer is: The motion of a system slowing down to rest at a stable equilibrium through a series of decaying oscillations about the equilibrium point."

    What is ____________, Alex?

    a) critically motion b) underdamped motion c) overdamped motion
    d) uniform circular motion e) all-wet motion

    SUGGESTED ANSWER: (b)

    Wrong answers:
    a) No that's when the damping is just strong enough to stop the motion as fast as it can all else being equal.
    c) No that's when the damping is so strong that it slows the return to equilibrium relative to the critically damped case.
    e) If you chose this answer, then . . . .

    Redaction: Jeffery, 2001jan01

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016 qmult 00530 1 4 4 easy deducto-memory: oscillatory resonance

25. "Let's play *Jeopardy*! For \$100, the answer is: The behavior of an oscillatory system driven at its natural or resonance frequency."

    What is ____________, Alex?

    a) underdamped oscillation b) simple harmonic motion c) uniform circular motion
    d) resonance or in resonance e) loco

    SUGGESTED ANSWER: (d)

    Wrong answers:
    e) Figuratively speaking maybe.

    Redaction: Jeffery, 2001jan01

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016 qmult 00540 1 4 1 easy deducto-memory: swing resonance of child

26. An oscillatory system driven at its resonance frequency will exhibit large oscillations. Every child—before they knew the name of torque—understood this when playing:

    a) on a swing. b) on a ladder. c) soccer. d) hopscotch. e) with matches.

    SUGGESTED ANSWER: (a)

    Wrong answers:
    e) You never did this, did you?

    Redaction: Jeffery, 2001jan01

Full Answer Problems:

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016 qfull 00210 1 3 0 easy math: SHO period, frequency, ω

27. There is a simple harmonic oscillator (SHO) that takes a time $\Delta t = 0.75\,\mathrm{s}$ before it begins to repeat. What are its (a) period P, (b) frequency f (in hertz), and (c) angular frequency ω (in radians per second)?

    SUGGESTED ANSWER:

    Well (a) $P = 0.75\,\mathrm{s}$, (b) $f = 1/P = 4/3 = 1.33\,\mathrm{Hz}$, (c) $\omega = 2\pi f = 8.4\,\mathrm{rad/s}$.

    Redaction: Jeffery, 2001jan01

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016 qfull 00220 2 5 0 moderate thinking: maximum force of a SHO, etc.

28. A body of mass $m = 0.12\,\mathrm{kg}$ is a simple harmonic oscillator (SHO) with equilibrium position $x = 0$, amplitude $x_{\mathrm{max}} = 8.5\,\mathrm{cm}$, period $P = 0.20\,\mathrm{s}$, and a linear (spring-like) restoring force.

a) What is the force constant of the linear (or Hooke's law) force?

b) What is the maximum absolute value of the force acting on the body?

c) What is the maximum absolute value of the acceleration of the body?

SUGGESTED ANSWER:

a) Behold:

$$1)\quad \omega=\sqrt{\frac{k}{m}} \qquad 2)\quad k=m\omega^2 \qquad 3)\quad P=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{k}} \qquad 4)\quad k=m\omega^2=m\left(\frac{2\pi}{P}\right)^2\approx 120\,\mathrm{N/m}$$

to about 2-digit accuracy.

b) Behold:

$$|F|_{\mathrm{max}}=|kx_{\mathrm{max}}|=10.5\,\mathrm{N}$$

to about 2-digit accuracy. This is about 2 lb.

c) Behold:

$$|a|_{\mathrm{max}}=|\omega^2 x_{\mathrm{max}}|=\frac{|F|_{\mathrm{max}}}{m}\approx 85\,\mathrm{m/s^2}$$

to about 2-digit accuracy.

Redaction: Jeffery, 2001jan01

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016 qfull 00250 3 3 0 tough math: general SHO solution results

29. The general solution of the 1-dimension simple harmonic oscillator problem,

$$m\frac{d^2x}{dt^2}=-kx \qquad \text{is} \qquad x(t)=A\cos(\omega t)+B\sin(\omega t)\ ,$$

where A and B are determined by the initial conditions and can have any values in general. We will not prove that is the most general solution: that is something for a math course.

a) Verify that the general solution is a solution and determine the expression for angular frequency ω.

b) As initial conditions at $t=0$, $x=x_0$ and $v=v_0$. Determine A and B in terms of the initial conditions.

c) Determine the ωt values for the stationary points (i.e., $\phi_{\mathrm{st}}=(\omega t)_{\mathrm{st}}$) within the range $\phi_{\mathrm{st}}\in[-\pi/2,3\pi/2]$.

Recall,

$$1)\quad \frac{y}{x}=\tan(\phi) \qquad 2)\quad \phi=\begin{cases}\arctan(x,y)+n\pi & \text{where } n=0,\pm1,\pm2,\ldots \text{ here and below.}\\ \arctan\left(\frac{y}{x}\right)+n\pi & \text{for } x\geq 0 \text{ for the computational arctangent function.}\\ \arctan\left(\frac{y}{x}\right)+\pi+n\pi & \text{for } x<0 \text{ for the computational arctangent function.}\end{cases}$$

For this problem, you will not know the x and y value signs.

d) Give a test determining whether a stationary point ϕ_{st} is maximum or minium. **HINT:** Recall, $d^2/dt^2<0$ for a maximum, $d^2/dt^2=0$ for an inflection point, and $d^2/dt^2>0$ for a minimum.

e) Verify that

$$x(t)=C\cos(\omega t+\phi)$$

is an equivalent expression for the general solution of the SHO problem: the amplitude C and phase factor ϕ are set by initial conditions and can take on any value, except of course that $\phi\in[-\pi,\pi]$.

In the verification process, find A and B as functions of C and ϕ: this shows that arbitrary C and ϕ can be replaced by A and B. If the proposed equivalent solution is to be really equivalent, arbitrary A and B must be replaceable by C and ϕ. Show that they are by finding C and ϕ as functions A and B.

SUGGESTED ANSWER:

a) Behold:

$$1) \quad x(t) = A\cos(\omega t) + B\sin(\omega t) \qquad 2) \quad \frac{dx}{dt} = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$$

$$3) \quad \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t) \ .$$

Clearly,

$$1) \quad \frac{d^2x}{dt^2} = -\omega^2 x \qquad 2) \quad \omega = \sqrt{\frac{k}{m}} \ .$$

b) At $t = 0$,

$$x = x_0 = A \qquad \text{and} \qquad v = v_0 = \omega B \ .$$

Thus,

$$A = x_0 \qquad \text{and} \qquad B = \frac{v_0}{\omega} \ .$$

c) Behold:

$$1) \quad \left.\frac{dx}{dt}\right|_{\rm st} = \left[-\omega A\sin(\omega t) + \omega B\cos(\omega t)\right]\Big|_{\rm st} = 0 \qquad 2) \quad \frac{B}{A} = \tan(\phi_{\rm st})$$

$$3) \quad \phi_{\rm st} = \arctan(A, B) + n\pi$$

$$4) \quad \phi_{\rm st} = \begin{cases} \arctan\left(\dfrac{B}{A}\right) & \text{for one stationary point } \phi_{\rm st} \in [-\pi/2, 3\pi/2]. \\[2ex] \arctan\left(\dfrac{B}{A}\right) + \pi & \text{for the other stationary point } \phi_{\rm st} \in [-\pi/2, 3\pi/2]. \end{cases}$$

We cannot tell which is a maximum and which a minimum.

d) Behold:

$$\left.\frac{d^2x}{dt^2}\right|_{\rm st} = -\omega^2 x_{\rm st} = \begin{cases} -\omega^2 x_{\rm max} & \text{for } x_{\rm st} = x(\phi_{\rm st}) > 0. \\ -\omega^2 x_{\rm min} & \text{for } x_{\rm st} = x(\phi_{\rm st}) < 0. \end{cases}$$

e) Making use of a trigonometric identity,

$$\begin{aligned} x(t) &= C\cos(\omega t + \phi) \\ &= C\left[\cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)\right] \\ &= A\cos(\omega t) + B\sin(\omega t) \ , \end{aligned}$$

where we have set

$$A = C\cos(\phi) \qquad \text{and} \qquad B = -C\sin(\phi) \ .$$

Clearly, we have recovered the form of the general solution and verified that arbitrary C and ϕ can be replaced by A and B. Thus the proposed equivalent expression is equivalent to the original expression provided arbitrary A and B can be replaced by C and ϕ. Well, inverting the expressions for A and B gives

$$1) \quad C = \sqrt{A^2 + B^2} \qquad 2) \quad \phi = \begin{cases} -\arctan\left(\dfrac{B}{A}\right) & A > 0 \ ; \\[2ex] -\arctan\left(\dfrac{B}{A}\right) + \pi \ , & A < 0 \ ; \\[2ex] -\arcsin\left(\dfrac{B}{|B|}\right) = -\pi/2 & A = 0 \text{ and } B > 0; \\[2ex] -\arcsin\left(\dfrac{B}{|B|}\right) = \pi/2 & A = 0 \text{ and } B < 0. \end{cases}$$

Since there are inverse expressions, any A and B can be represented by C and ϕ. This completes the proof that the proposed alternative is indeed an equivalent general SHO solution.

Redaction: Jeffery, 2001jan01

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016 qfull 00260 3 5 0 tough thinking: springs in parallel, oscillations

30. You have a block of mass m sandwiched between a bunch of springs in parallel. The whole system is a 1-dimensional system. The springs are attached to opposing walls. Some springs are from the left and some are from the right. The block sits on a level frictionless floor. The springs are ideal. Each spring has a force constant k_i and equilibrium position x_i for the center of the block: i.e., x_i is where the block center would be in equilibrium if only spring i were attached to the block.

a) What is the expression for the net force on the mass?

b) Derive the appropriate single-spring equivalent k (i.e., force constant) and $x_{\rm eq}$ (i.e., equilibrium position) expressions such that the net force expression changes to

$$F = -k(x - x_{\rm eq}) \ .$$

Why is the $x_{\rm eq}$ the equilibrium position of the total system?

c) Derive the expression for the total system ω in terms of the individual spring angular frequencies ω_i and the total system period P in terms of the individual spring periods P_i.

SUGGESTED ANSWER:

a) Behold:

$$F = -\sum_i k_i(x - x_i) \ .$$

It doesn't matter if the spring is attached from the left or the right: either way $(x - x_i) > 0$ gives a negative direction force and $(x - x_i) < 0$ gives a positive direction force.

b) Behold:

$$\begin{aligned} F &= -\sum_i k_i(x - x_i) = -x\sum_i k_i + \sum_i k_i x_i = -kx + k\frac{\sum_i k_i x_i}{k} = -kx + kx_{\rm eq} \\ &= -k(x - x_{\rm eq}) \ , \end{aligned}$$

where we define the equivalent single-spring equivalents k and $x_{\rm eq}$ by

$$k = \sum_i k_i \qquad \text{and} \qquad x_{\rm eq} = \frac{\sum_i k_i x_i}{k} \ .$$

Note, the equivalent k is just the sum of k_i and the equivalent equilibrium position is just the k_i weighted mean of the individual spring equilibrium positions. The equilibrium position total system is just $x_{\rm eq}$ since when $x = x_{\rm eq}$ the net force is zero.

c) From 2nd law analysis, it follows immediately for

$$F = -k(x - x_{\rm eq})$$

that the total system ω is $\omega = \sqrt{k/m}$, and thus

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{\sum_i k_i}{m}} = \sqrt{\sum_i \omega_i^2} \ .$$

Now the total system period $P = 2\pi/\omega$ and $P_i = 2\pi/\omega_i$. Thus,

$$\frac{1}{P} = \sqrt{\sum_i \frac{1}{P_i^2}} \qquad \text{or} \qquad P = \frac{1}{\sqrt{\sum_i 1/P_i^2}} \ .$$

We have now pretty much solved the general N springs-in-parallel problem. Ah, the curse of generality as my old friend Francesco would say.

Redaction: Jeffery, 2001jan01

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016 qfull 00270 3 5 0 tough thinking: two-phase motion, collision and SHM
Extra keywords: conservation of momentum, conservation of energy.

31. There is a hunk of mass 5 kg on a frictionless horizontal surface attached to a horizontal ideal spring. The hunk is initially in the equilibrium position at rest with zero energy. A bullet on the opposite side from the spring, but traveling on the spring's axis, hits the hunk and embeds. The bullet mass is 0.050 kg and its speed is 500 m/s (which is supersonic not that it matters).

   a) What is the initial speed of the hunk just after the bullet hits?

   b) The spring is compressed for 2 meters before rebounding. What is its spring constant?

   c) What is the **ANGULAR FREQUENCY** and the **MAGNITUDE OF THE MAXIMUM ACCELERATION** of the simple harmonic motion that results from the bullet impact?

SUGGESTED ANSWER:

This is a two-phase problem. First is the collision phase which in the collision approximation is assumed to be instantaneous and conserve linear momentum. Second is the simple harmonic motion phase which starts from the equilibrium position with maximum speed. Given the initial conditions of this system $x = -x_{\rm max}\sin(\omega t)$. The minus sign just means the motion is to the negative at time zero. The sine function has the same oscillatory behavior as the cosine function, but with a shift along the x-axis to account for the different initial conditions at time zero.

a) The collision is perfectly inelastic: the colliders stick together. Thus the momentum conservation equation becomes

$$P_{\rm ini} = m_{\rm bu}v_{\rm bu} + m_{\rm hu}v_{\rm hu} = (m_{\rm bu} + m_{\rm hu})v_{\rm fin} = P_{\rm fin} \ ,$$

where "ini," "fin," "bu,", and "hu" stand for initial, final, bullet, and hunk, respectively. In this case, $v_{\rm hu} = 0$, and so

$$v_{\rm fin} = \frac{m_{\rm bu}v_b}{m_{\rm bu} + m_{\rm hu}} = 4.95\,{\rm m/s} \ .$$

The final velocity of the collision, $v_{\rm fin}$, is the initial and maximum velocity of the subsequent SHM.

b) In the SHM phase, mechanical energy is assumed conserved. Thus in the work-energy theorem

$$\Delta E = W_{\rm nc} \ ,$$

we have the work done by nonconservative forces $W_{\rm nc} = 0$ and $\Delta E = 0$. Let

$$m = m_{\rm bu} + m_{\rm hu}$$

and $v_{\rm max} = v_{\rm fin}$ from part (a). At the initiation of the motion all the mechanical energy is kinetic energy since displacement is zero. At the maximum compression all the mechanical energy is potential energy since the velocity is zero. Thus

$$E = \frac{1}{2}mv_{\rm max}^2 = \frac{1}{2}kx_{\rm max}^2 \ ,$$

and so

$$k = \frac{mv_{\rm max}^2}{x_{\rm max}^2} = 30.9\,{\rm N/m}$$

is the spring constant.

c) The angular frequency is $\omega = \sqrt{k/m} = 2.48\,{\rm rad/s}$ and the magnitude of the maximum acceleration is $a_{\rm max} = \omega^2 x_{\rm max} = 12.3\,{\rm m/s}^2$.

Redaction: Jeffery, 2001jan01

016 qfull 00350 3 5 0 tough thinking: physical pendulum and simple pendulum

32. Consider a physical pendulum in the small amplitude limit with a frictionless pivot axis. In this case, there is a linear restoring torque and the system exhibits simple harmonic motion. The period for the physical pendulum is

$$P = 2\pi\sqrt{\frac{I}{mgr}} \ ,$$

where I is the rotational inertia about the pivot point, m is the mass of the body that makes up the pendulum, and r is the distance from the pivot axis to the center-of-mass axis of the body.

a) Rewrite the formula substituting the parallel axis theorem formula for the rotational inertia:

$$I = mr^2 + I_{\rm cm} \ ,$$

where m is the body mass, r is the distance from pivot axis to the center-of-mass axis, and $I_{\rm cm}$ is the rotational inertia about the body center-of-mass axis. Simplify the new formula so there a term in the numerator under the square root symbol that is $I_{\rm cm}/(mr)$.

b) What is the limiting behavior of the period formula when $I_{\rm cm}/m$ goes to zero with r fixed? What is the pendulum called in this limit?

c) What is the limiting behavior of the period formula as $I_{\rm cm}/m$ goes to infinity with r fixed? What is the explication for the limiting behavior?

d) Now consider $I_{\rm cm}/m$ fixed. Give small r and large r limiting behaviors for the period formula. What is the large r formula? What is the limiting behavior as r goes to zero and as r goes to infinity?

e) For fixed $I_{\rm cm}/m$, solve for the stationary points $r_{\rm min}$ and $r_{\rm max}$ that give the minimum and (stationary point) maximum period values and give those values. **HINT:** The $r_{\rm max}$ value is not finite.

f) Sketch the plot of the physical and simple pendulum periods as a function of r for $r \in [0, \infty]$. **HINT:** What must the period curve look like as it rises to the stationary maximum period.
Is the simple pendulum period ever greater or equal to the physical pendulum period?

g) The physical pendulum is more realistic than the simple pendulum and its period formula gives the right behavior as r becomes small. Why, in fact, does the simple pendulum period formula give the wrong behavior as r becomes small. **HINT:** The simple pendulum assumes a massless pendulum arm and a point mass bob for all r.

SUGGESTED ANSWER:

a) Behold:

$$P = 2\pi\sqrt{\frac{mr^2 + I_{\rm cm}}{mgr}} = 2\pi\sqrt{\frac{r + I_{\rm cm}/(mr)}{g}} \ .$$

b) Behold:

$$P = 2\pi\sqrt{\frac{r}{g}} \ .$$

The pendulum in this limit is called the simple pendulum.

c) Behold:

$$\lim_{I/m\to\infty} P = \infty \ .$$

In this case the rotational inertia of the pendulum is so great that the pendulum never moves.

d) The small and large r formulae are, respectively,

$$P = 2\pi\sqrt{\frac{I_{\rm cm}/m}{gr}} \qquad \text{and} \qquad P = 2\pi\sqrt{\frac{r}{g}} \ .$$

The large r formula is just the simple pendulum formula again. As r gets large the center of mass component of the inertia $I_{\rm cm}$ becomes negligigle compared to the mr^2 component and the physical pendulum asymptotically becomes the simple pendulum.

e) To find the stationary points $r_{\rm min}$ and $r_{\rm max}$ take the derivative of P with respect to r and set it to zero. Behold:

$$\frac{dP}{dr} = \frac{2\pi}{\sqrt{g}}\frac{1}{2}\frac{1}{\sqrt{r + I_{\rm cm}/(mr)}}\left(1 - \frac{I_{\rm cm}}{mr^2}\right) = 0$$

which leads to

$$r_{\rm min} = \sqrt{\frac{I_{\rm cm}}{m}} \qquad \text{and} \qquad r_{\rm max} = \infty \ .$$

The minimum period occurs exactly where the non-center-of-mass rotational inertia equals the center of mass rotational inertia: i.e.,

$$I = mr_{\rm min}^2 + I_{\rm cm} = I_{\rm cm} + I_{\rm cm} = 2I_{\rm cm} \ .$$

The minimum period itself is

$$P = 2\pi\sqrt{\frac{2}{g}}\left(\frac{I_{\rm cm}}{m}\right)^{1/4} \ .$$

The maximum period

$$P = \infty \ .$$

Note, the other infinity at $r = 0$ is a maximum, but not a stationary point. It is a singularity.

f) You will have to imagine the plot. The physical pendulum period has infinities at $r = 0$ (a singulariy) and $r = \infty$ (which is stationary point at infinity which means the slope is asymptotically approaching zero at infinity). At $r = \sqrt{I_{\rm cm}/m}$, it reaches a minimum of

$$P = 2\pi\sqrt{\frac{2}{g}}\left(\frac{I_{\rm cm}}{m}\right)^{1/4} \ .$$

The simple pendulum period grows from zero at $r = 0$ to infinity at $r = \infty$. It asymptotically approaches the physical period as r goes to infinity.

For finite r, the simple pendulum period is always less than the physical pendulum period:

$$P_{\rm sim} = 2\pi\sqrt{\frac{r}{g}} \le 2\pi\sqrt{\frac{r + I_{\rm cm}/(mr)}{g}} = P_{\rm phy} \ ,$$

where the equality holds asymptotically as r goes to infinity.

g) The assumption of point mass pendulum for all r must fail as r becomes comparable and smaller than the size of any real bob. In fact, when r becomes that small, clearly the pendulum has turned into a physical pendulum.

To go beyond the required answer, since the rotational inertia of physical pendulum never goes to zero, its period goes to infinity as r goes to zero. When $r = 0$, the physical pendulum is in neutral equilibrium. If put at rest in any orientation, it will stay at rest.

Redaction: Jeffery, 2001jan01

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

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1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67430(15) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(circa 2025)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\text{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

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2 Geometrical Formulae

$$C_{\text{cir}} = 2\pi r \qquad A_{\text{cir}} = \pi r^2 \qquad A_{\text{sph}} = 4\pi r^2 \qquad V_{\text{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\text{sphere}} = 4\pi \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

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3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1-x} \approx 1 + x \;:\; (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

5 Quadratic Formula

$$\text{If} \qquad 0 = ax^2 + bx + c\,, \qquad \text{then} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

Numerically robust solution (Press-178):

$$q = -\frac{1}{2}\left[b + \mathrm{sgn}(b)\sqrt{b^2 - 4ac}\right] \qquad x_1 = \frac{q}{a} \qquad x_2 = \frac{c}{q}$$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$\theta = \arctan(a_x, a_y) + n\pi = \arctan_{\mathrm{computational}}\left(\frac{a_y}{a_x}\right) + \pi? + n\pi$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series

$$f(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0)$$
$$= f(x_0) + (x - x_0)f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\mathrm{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\mathrm{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Five 1-dimensional equations of kinematics

Equation No.	Equation	Unwanted variable
1	$v = at + v_0$	Δx
2	$\Delta x = \frac{1}{2}at^2 + v_0 t$	v
3 (timeless eqn)	$v^2 - v_0^2 = 2a\Delta x$	t
4	$\Delta x = \frac{1}{2}(v + v_0)t$	a
5	$\Delta x = vt - \frac{1}{2}at^2$	v_1

Fiducial acceleration due to gravity (AKA little g) $g = 9.8\,\mathrm{m/s^2}$

$$x_{\mathrm{rel}} = x_2 - x_1 \qquad v_{\mathrm{rel}} = v_2 - v_1 \qquad a_{\mathrm{rel}} = a_2 - a_1$$

$$x' = x - v_{\mathrm{frame}} t \qquad v' = v - v_{\mathrm{frame}} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\mathrm{avg}} = \frac{\Delta \vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\mathrm{avg}} = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0} t \qquad y = -\frac{1}{2}gt^2 + v_{y,0} t + y_0 \qquad v_{x,0} = v_0 \cos\theta \qquad v_{y,0} = v_0 \sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0 \cos\theta} \qquad y = y_0 + x\tan\theta - \frac{x^2 g}{2v_0^2 \cos^2\theta}$$

$$x_{\mathrm{for}\ y\ \mathrm{max}} = \frac{v_0^2 \sin\theta\cos\theta}{g} \qquad y_{\mathrm{max}} = y_0 + \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2v_0^2 \sin\theta\cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g} \qquad \theta_{\mathrm{for\ max}} = \frac{\pi}{4} \qquad x_{\mathrm{max}}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0 \sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\text{tan}} = r\alpha$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = v\omega$$

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13 Very Basic Newtonian Physics

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i\vec{r}_i}{m_{\text{total}}} = \frac{\sum_{\text{sub}} m_{\text{sub}}\vec{r}_{\text{cm sub}}}{m_{\text{total}}} \qquad \vec{v}_{\text{cm}} = \frac{\sum_i m_i\vec{v}_i}{m_{\text{total}}} \qquad \vec{a}_{\text{cm}} = \frac{\sum_i m_i\vec{a}_i}{m_{\text{total}}}$$

$$\vec{r}_{\text{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\text{total}}}$$

$$\vec{F}_{\text{net}} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,\text{m/s}^2$$

$$\vec{F}_{\text{normal}} = -\vec{F}_{\text{applied}} \qquad F_{\text{linear}} = -kx$$

$$f_{\text{normal}} = \frac{T}{r} \qquad T = T_0 - F_{\text{parallel}}(s) \qquad T = T_0$$

$$F_{\text{f static}} = \min(F_{\text{applied}}, F_{\text{f static max}}) \qquad F_{\text{f static max}} = \mu_{\text{static}}F_{\text{N}} \qquad F_{\text{f kinetic}} = \mu_{\text{kinetic}}F_{\text{N}}$$

$$v_{\text{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\text{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\text{centripetal}} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\text{drag,lin}} = bv \qquad v_{\text{T}} = \frac{mg}{b} \qquad \tau = \frac{v_{\text{T}}}{g} = \frac{m}{b} \qquad v = v_{\text{T}}(1 - e^{-t/\tau})$$

$$F_{\text{drag,quad}} = bv^2 = \frac{1}{2}C\rho Av^2 \qquad v_{\text{T}} = \sqrt{\frac{mg}{b}}$$

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14 Energy and Work

$$dW = \vec{F}\cdot d\vec{s} \qquad W = \int \vec{F}\cdot d\vec{s} \qquad K = \frac{1}{2}mv^2 \qquad E_{\text{mechanical}} = K + U$$

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F}\cdot\vec{v}$$

$$\Delta K = W_{\text{net}} \quad \Delta U_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \quad \Delta E = W_{\text{nonconservative}}$$

$$F = -\frac{dU}{dx} \qquad \vec{F} = -\nabla U \qquad U = \frac{1}{2}kx^2 \qquad U = mgy$$

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15 **Momentum**

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta K_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\text{cm}} = \vec{F}_{\text{net non-flux}} + (\vec{v}_{\text{flux}} - \vec{v}_{\text{cm}})\frac{dm}{dt} = \vec{F}_{\text{net non-flux}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_0 + v_{\text{ex}}\ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

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16 **Collisions**

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\,dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\text{net}}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$K_{\text{total }f} = K_{\text{total }i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\text{rel}'} = -v_{\text{rel}} \qquad \text{1-d Elastic Collision Expressions}$$

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17 **Rotational Kinematics**

$$2\pi = 6.2831853\ldots \qquad \frac{1}{2\pi} = 0.15915494\ldots$$

$$\frac{180^\circ}{\pi} = 57.295779\ldots \approx 60^\circ \qquad \frac{\pi}{180^\circ} = 0.017453292\ldots \approx \frac{1}{60^\circ}$$

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\omega = \alpha t + \omega_0 \qquad \Delta\theta = \frac{1}{2}\alpha t^2 + \omega_0 t \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t \qquad \Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$

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18 Rotational Dynamics

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{\tau}_{\rm net} = \frac{d\vec{L}}{dt}$$

$$L_z = RP_{xy}\sin\gamma_L \qquad \tau_z = RF_{xy}\sin\gamma_\tau \qquad L_z = I\omega \qquad \tau_{z,\rm net} = I\alpha$$

$$I = \sum_i m_i R_i^2 \qquad I = \int R^2 \rho\, dV \qquad I_{\text{parallel axis}} = I_{\rm cm} + mR_{\rm cm}^2 \qquad I_z = I_x + I_y$$

$$I_{\text{cyl,shell,thin}} = MR^2 \qquad I_{\rm cyl} = \frac{1}{2}MR^2 \qquad I_{\text{cyl,shell,thick}} = \frac{1}{2}M(R_1^2 + R_2^2)$$

$$I_{\text{rod,thin,cm}} = \frac{1}{12}ML^2 \qquad I_{\text{sph,solid}} = \frac{2}{5}MR^2 \qquad I_{\text{sph,shell,thin}} = \frac{2}{3}MR^2$$

$$a = \frac{g\sin\theta}{1 + I/(mr^2)}$$

$$K_{\rm rot} = \frac{1}{2}I\omega^2 \qquad dW = \tau_z\, d\theta \qquad P = \frac{dW}{dt} = \tau_z\omega$$

$$\Delta K_{\rm rot} = W_{\rm net} = \int \tau_{z,\rm net}\, d\theta \qquad \Delta U_{\rm rot} = -W = -\int \tau_{z,\rm con}\, d\theta$$

$$\Delta E_{\rm rot} = K_{\rm rot} + \Delta U_{\rm rot} = W_{\text{non,rot}} \qquad \Delta E = \Delta K + K_{\rm rot} + \Delta U = W_{\rm non} + W_{\rm rot}$$

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19 Static Equilibrium

$$\vec{F}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = \tau'_{\text{ext,net}} \quad \text{if } F_{\text{ext,net}} = 0$$

$$0 = F_{\text{net } x} = \sum F_x \qquad 0 = F_{\text{net } y} = \sum F_y \qquad 0 = \tau_{\text{net}} = \sum \tau$$

20 Gravity

$$\vec{F}_{1 \text{ on } 2} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \qquad \vec{g} = -\frac{GM}{r^2}\hat{r} \qquad \oint \vec{g}\cdot d\vec{A} = -4\pi GM$$

$$U = -\frac{Gm_1m_2}{r_{12}} \qquad V = -\frac{GM}{r} \qquad v_{\text{escape}} = \sqrt{\frac{2GM}{r}} \qquad v_{\text{orbit}} = \sqrt{\frac{GM}{r}}$$

$$P^2 = \left(\frac{4\pi^2}{GM}\right)r^3 \qquad P = \left(\frac{2\pi}{\sqrt{GM}}\right)r^{3/2} \qquad \frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} = \text{Constant}$$

$$R_{\text{Earth,mean}} = 6371.0\,\text{km} \qquad R_{\text{Earth,equatorial}} = 6378.1\,\text{km} \qquad M_{\text{Earth}} = 5.9736\times 10^{24}\,\text{kg}$$

$$R_{\text{Earth mean orbital radius}} = 1.495978875\times 10^{11}\,\text{m} = 1.0000001124\,\text{AU} \approx 1.5\times 10^{11}\,\text{m} \approx 1\,\text{AU}$$

$$R_{\text{Sun,equatorial}} = 6.955\times 10^{8} \approx 109\times R_{\text{Earth,equatorial}} \qquad M_{\text{Sun}} = 1.9891\times 10^{30}\,\text{kg}$$

21 Fluids

$$\rho = \frac{\Delta m}{\Delta V} \qquad p = \frac{F}{A} \qquad p = p_0 + \rho g d_{\text{depth}}$$

Pascal's principle	$p = p_{\text{ext}} - \rho g(y - y_{\text{ext}}) \qquad \Delta p = \Delta p_{\text{ext}}$
Archimedes principle	$F_{\text{buoy}} = m_{\text{fluid dis}}g = V_{\text{fluid dis}}\rho_{\text{fluid}}g$
equation of continuity for ideal fluid	$R_V = Av = \text{Constant}$
Bernoulli's equation	$p + \frac{1}{2}\rho v^2 + \rho g y = \text{Constant}$

22 Oscillation

$$P = f^{-1} \qquad \omega = 2\pi f \qquad F = -kx \qquad U = \frac{1}{2}kx^2 \qquad a(t) = -\frac{k}{m}x(t) = -\omega^2 x(t)$$

$$\omega = \sqrt{\frac{k}{m}} \qquad P = 2\pi\sqrt{\frac{m}{k}} \qquad x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$E_{\text{mec total}} = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}kx_{\text{max}}^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$P = 2\pi\sqrt{\frac{I}{mgr}} \qquad P = 2\pi\sqrt{\frac{r}{g}}$$

23 Waves

$$\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2} \qquad v = \sqrt{\frac{F_{\mathrm{T}}}{\mu}} \qquad y = f(x - vt)$$

$$y = y_{\max}\sin[k(x - vt)] = y_{\max}\sin(kx - \omega t) \qquad v = \frac{\omega}{k}$$

$$\text{Period} = \frac{1}{f} \qquad k = \frac{2\pi}{\lambda} \qquad v = f\lambda = \frac{\omega}{k} \qquad P \propto y_{\max}^2$$

$$y = 2y_{\max}\sin(kx)\cos(\omega t) \qquad n = \frac{L}{\lambda/2} \qquad L = n\frac{\lambda}{2} \qquad \lambda = \frac{2L}{n} \qquad f = n\frac{v}{2L}$$

$$v = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} \qquad n\lambda = d\sin(\theta) \qquad \left(n + \frac{1}{2}\right)\lambda = d\sin(\theta)$$

$$I = \frac{P}{4\pi r^2} \qquad \beta = (10\,\mathrm{dB}) \times \log\left(\frac{I}{I_0}\right)$$

$$f = n\frac{v}{4L} \;:\; n = 1, 3, 5, \ldots \qquad f_{\text{medium}} = \frac{f_0}{1 - v_0/v_{\text{medium}}}$$

$$f' = f\left(1 - \frac{v'}{v}\right) \qquad f = \frac{f'}{1 - v'/v}$$

24 Thermodynamics

$$dE = dQ - dW = T\,dS - p\,dV$$

$$T_{\mathrm{K}} = T_{\mathrm{C}} + 273.15\,\mathrm{K} \qquad T_{\mathrm{F}} = 1.8 \times T_{\mathrm{C}} + 32^\circ\mathrm{F}$$

$$Q = mC\Delta T \qquad Q = mL$$

$$PV = NkT \qquad P = \frac{2}{3}\frac{N}{V}K_{\text{avg}} = \frac{2}{3}\frac{N}{V}\left(\frac{1}{2}mv_{\mathrm{RMS}}^2\right)$$

$$v_{\mathrm{RMS}} = \sqrt{\frac{3kT}{m}} = 2735.51\ldots \times \sqrt{\frac{T/300}{A}}$$

$$PV^\gamma = \text{constant} \qquad 1 < \gamma \leq \frac{5}{3} \qquad v_{\text{sound}} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{-V(\partial P/\partial V)_S}{m(N/V)}} = \sqrt{\frac{\gamma kT}{m}}$$

$$\varepsilon = \frac{W}{Q_{\text{H}}} = \frac{Q_{\text{H}} - Q_{\text{C}}}{W} = 1 - \frac{Q_{\text{C}}}{Q_{\text{H}}} \qquad \eta_{\text{heating}} = \frac{Q_{\text{H}}}{W} = \frac{Q_{\text{H}}}{Q_{\text{H}} - Q_{\text{C}}} = \frac{1}{1 - Q_{\text{C}}/Q_{\text{H}}} = \frac{1}{\varepsilon}$$

$$\eta_{\text{cooling}} = \frac{Q_{\text{C}}}{W} = \frac{Q_{\text{H}} - W}{W} = \frac{1}{\varepsilon} - 1 = \eta_{\text{heating}} - 1$$

$$\varepsilon_{\text{Carnot}} = 1 - \frac{T_{\text{C}}}{T_{\text{H}}} \qquad \eta_{\text{heating,Carnot}} = \frac{1}{1 - T_{\text{C}}/T_{\text{H}}} \qquad \eta_{\text{cooling,Carnot}} = \frac{T_{\text{C}}/T_{\text{H}}}{1 - T_{\text{C}}/T_{\text{H}}}$$