Intro Physics Semester I Name:

Homework 4: Multi-Dimensional Kinematics: Homeworks are due as posted on the course web site. Multiple-choice questions will **NOT** be marked, but some of them will appear on exams. One or more full-answer questions may be marked as time allows for the grader. Hand-in the full-answer questions on other sheets of paper: i.e., not crammed onto the downloaded question sheets. Make the full-answer solutions sufficiently detailed that the grader can follow your reasoning, but you do **NOT** be verbose. Solutions will be posted eventually after the due date. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

---

004 qmult 00100 1 4 4 easy deducto-memory: kinematic quantities in multi-dimensions

1. "Let's play *Jeopardy*! For \$100, the answer is: The vectors

$$\vec{r}\ , \qquad \vec{v} = \frac{d\vec{r}}{dt}\ , \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}\ ."$$

What are the most obvious quantities of ____________, Alex?

a) time b) light c) dynamics d) multi-dimensional kinematics e) rest

SUGGESTED ANSWER: (d)

Wrong answers:
a) As Lurch would say AAAARGH.

Redaction: Jeffery, 2008jan01

---

004 qmult 00110 1 1 3 easy memory: unit vectors in Cartesian coordinates

2. In Cartesian coordinates, kinematics is very simple if $x(t)$, $y(t)$, and $z(t)$ are specified because the unit vectors of Cartesian coordinates are:

a) variables. b) dependent on time. c) constants. d) dependent on position.
e) fewer than two.

SUGGESTED ANSWER: (c)

Wrong answers:
e) A nonsense answer.

Redaction: Jeffery, 2008jan01

---

004 qmult 00120 1 1 1 easy memory: unit vectors in polar/spherical coordinates

3. The unit vectors of polar and spherical polar coordinates are dependent on:

a) the angular coordinates of the displacement vector.
b) the magnitude of the displacement vector. c) nothing. d) time explicitly. e) mass.

SUGGESTED ANSWER: (a)

Wrong answers:
d) Well no. They depend on the angular coordinates explicitly. If those coordinates depend on time, then there is an implicit time dependence. But there is no general time dependence.

Redaction: Jeffery, 2008jan01

---

004 qmult 00130 1 5 4 easy think: independent orthogonal motion

4. In Newtonian physics, the component motions of a particle in ____________ directions are independent of each other. This means that motion variables in one direction are **NOT** intrinsic functions of motion variables in ____________ directions. For example, a particular velocity in the x direction does **NOT** intrinsically set the velocity in the y direction. Now the initial or continuing conditions will set up relationships between motions in ____________ directions. But those conditions can be anything allowed by physics and relationships between the motions can be anything allowed by the conditions and physics. For example, the conditions could set the x direction velocity to be a constant $v_x = 1\,\mathrm{m/s}$ and the y direction velocity to a constant $v_y = 2\,\mathrm{m/s}$. Then one has, of course, that $v_y = 2v_x$, but the physically allowed conditions could have set the two velocity components to any constant values or

anything else allowed by physics. There is **NO** intrinsic relationship between those velocity components. The independence of motions in ____________ directions is one of the amazing facts about the physical world and a vast simplification in dealing with it.

a) the same. b) opposite. c) both negative d) orthogonal e) dependent

SUGGESTED ANSWER: (d)

This is easy because it's a leading question. Deduction tells you some answers are nonsensical. Space itself (on our current level anyway) imposes no intrinsic relationships on motions in orthogonal directions.

Wrong answers:
a) A nonsense answer.

Redaction: Jeffery, 2001jan01

---

004 qmult 00200 1 4 2 easy deducto-memory: projectile motion

5. "Let's play *Jeopardy*! For $100, the answer is: Without qualifications, one usually means the non-powered flight of an object in the air or through space. The simplest in-air case is the one in which air drag is neglected. The science of such motions is ballistics.

What is ____________, Alex?

a) apparent motion b) projectile motion c) 1-dimensional motion
d) trigonometric motion e) unstoppable motion

SUGGESTED ANSWER: (b)

Wrong answers:
d) A nonsense answer.

Redaction: Jeffery, 2008jan01

---

004 qmult 00210 2 1 1 mod. deducto-mem.: gravity is always downward on Earth
Extra keywords: physci

6. A ball is tossed into the air and falls to the ground some distance away. Consider its motion in the vertical direction only and neglect air drag.

a) The ball has a constant acceleration downward.
b) The ball first accelerates **UPWARD** on its rising path and then accelerates **DOWNWARD** on its falling path.
c) The ball first accelerates **DOWNWARD** on its rising path and then accelerates **UPWARD** on its falling path.
d) The ball does not accelerate at all.
e) The ball is always accelerating in the upward direction.

SUGGESTED ANSWER: (a)

An easy memory question. Acceleration due to gravity alone is always downward and is a nearly constant near the Earth's surface. The magnitude of acceleration due to gravity alone is the magnitude of the gravitational field. Near the Earth's surface the gravitational field magnitude has fiducial value $g = 9.8\,\mathrm{m/s^2}$.

Wrong answers:

Redaction: Jeffery, 2001jan01

---

004 qmult 00220 1 5 2 moderate thinking: projectile parabolic arc

7. Which best describes the path of a ball thrown on level ground at an angle 30° above the horizontal as seen from a side view.

a) Two straight lines that meet at an apex: one for the rising phase; one the declining phase. The rising phase line is **TWICE** the length of the declining phase line.
b) A smooth curve that rises and falls with distance. As far as the eye can tell, the curve could be parabolic.

c) Two straight lines that meet at an apex: one for the rising phase; one the declining phase. The rising phase line is **HALF** the length of the declining phase line.
d) A smooth curve that rises and falls with distance, but suddenly breaks off and descends vertically.
e) A smooth curve that rises and falls with distance and then rises and falls again with distance. A Bactrian camel curve.

SUGGESTED ANSWER: (b)

An easy thinking question. People should be able to identify the only answer that corresponds to common observation. And some may have heard that projectile motion is parabolic aside from air drag effects. Neglecting air drag, the path is, in fact, a parabolic arc as function of the horizontal coordinate x. To understand this note that the vertical height is parabolic with time. The x distance is linear in time. Therefore the vertical height is parabolic with time.

Wrong answers:
e) A two-hump camel curve?

Redaction: Jeffery, 2001jan01

---

004 qmult 00230 1 1 4 easy memory: horizontal range formula

8. The horizontal range formula for projectile motion near the Earth's surface and neglecting air drag is:

a) $x_{\rm max} = \dfrac{8v_0^2}{g}\sin(2\theta)$. b) $x_{\rm max} = \dfrac{v_0^2}{g}\cos(2\theta)$. c) $x_{\rm max} = \dfrac{v_0^2}{g}$. d) $x_{\rm max} = \dfrac{v_0^2}{g}\sin(2\theta)$.
e) $x_{\rm max} = \dfrac{4v_0^2}{g}\sin(2\theta)$.

SUGGESTED ANSWER: (d)

Here's the quick derivation. In the two independent directions one has

$$x = (v_0\cos\theta)t \qquad \text{and} \qquad y = -\frac{1}{2}gt^2 + (v_0\sin\theta)t \ ,$$

where we've taken the inital launch position to be the origin to avoid fruitless generality and the meaning of the symbols is obvious. As a function of x, we find that

$$y = -\frac{1}{2}\frac{gx^2}{v_0^2\cos^2\theta} + x\tan\theta \ .$$

The equation is a quadratic for x. There two solutions for $y = 0$. One is the launch solution $x = 0$. The other follows from

$$0 = -\frac{1}{2}\frac{gx}{v_0^2\cos^2\theta} + \tan\theta \ .$$

We find

$$x = \frac{2v_0^2}{g}\sin\theta\cos\theta = \frac{v_0^2}{g}\sin(2\theta) \ ,$$

where we have used a trigonometric identity. That completes the derivation.

Wrong answers:
c) This is the maximum horizontal range formula.

Redaction: Jeffery, 2008jan01

---

004 qmult 00240 1 1 2 easy memory: horizontal range maximum angle

9. The maximum horizontal range value for projectile motion near the Earth's surface for a given launch speed v_0 and neglecting air drag is obtained for launch angle:

a) 30°. b) 45°. c) 60°. d) 82.245°. e) 90°.

SUGGESTED ANSWER: (b)

Wrong answers:
e) You are shooting the object straight up.

Redaction: Jeffery, 2008jan01

---

004 qmult 00260 1 1 1 easy memory: ratio of height to range formulae

10. The ratio of the maximum height formula (for projectile motion near the Earth's surface, neglecting air drag, and measuring from the launch height) to the horizontal range formula (for projectile motion near the Earth's surface and neglecting air drag) is ____________ and its value for the maximum range is ________.

a) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \dfrac{1}{4}\tan\theta$; $\frac{1}{4}$ b) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \dfrac{1}{4}\tan\theta$; 1 c) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \tan\theta$; 1 d) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \tan\theta$; $\frac{1}{4}$
e) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \dfrac{1}{4}\tan\theta$; ∞

SUGGESTED ANSWER: (a)

The horizontal range and maximum height formulae are, respectively,

$$x_{\text{range}} = \frac{2v_0^2}{g}\sin\theta\cos\theta = \frac{v_0^2}{g}\sin(2\theta) \qquad \text{and} \qquad y_{\text{max}} = \frac{v_0^2}{2g}\sin^2\theta \ .$$

Taking the ratio gives

$$\frac{y_{\text{max}}}{x_{\text{range}}} = \frac{1}{4}\tan\theta \ .$$

For the maximum range for which $\theta = 45^\circ$, one finds

$$\frac{y_{\text{max}}}{x_{\text{range}}} = \frac{1}{4} \ .$$

Wrong answers:

e) The value is what you'd get for $\theta = 90^\circ$. In this case, the height is finite, but the range has gone to zero.

Redaction: Jeffery, 2008jan01

---

004 qmult 00300 1 4 5 easy deducto-memory: relative motion explained

11. "Let's play *Jeopardy*! For \$100, the answer is: Motion of something with respect to something else. To be a bit more explicit, say you have two objects. The relative displacement of object 2 from object 1 is defined to be

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \ .$$

From this definition, the relative velocity and acceleration follow from differentiation:

$$\vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \text{and} \qquad \vec{a} = \vec{a}_2 - \vec{a}_1 \ .$$

The description of the motion in these terms clarifies the initial answer statement."

What is ____________, Alex?

a) variable motion b) relativistic motion c) no motion d) abosulte motion
e) relative motion

SUGGESTED ANSWER: (e)

Wrong answers:

d) Exactly wrong.

Redaction: Jeffery, 2008jan01

---

004 qmult 00350 2 5 5 moderate thinking: inertial frame motion is relative

12. You are in a featureless narrow room playing catch with a friend. How can you tell if the room is in a building or is a sealed compartment on super-smoothly running, non-accelerating train (or plane)? **HINT:** Review your whole life experience; try an experiment (but not while you are driving).

a) When you throw the ball **ALONG** the long axis of the room, it would have different speeds (relative to the room) in the two possible directions if you were on a train.
b) When you thow the ball **PERPENDICULAR** to the long axis of the room it would curve off a straight line (relative to the room) if you were on a train.
c) On a train the thrown ball would zigzag wildly in flight.
d) On a train the thrown ball would do loops in flight.
e) There is no way to tell as long as the train motion is very smooth.

SUGGESTED ANSWER: (e)

A moderate thinking question. People have been on trains and planes and cars and they know nothing weird happens as long as there is no acceleration. Fundamentally, non-accelerating frames are inertial frames and the laws of physics are the same relative to all of them. But in the context of a 5-minute quiz, it could be tough.

Wrong answers:

Redaction: Jeffery, 2001jan01

---

004 qmult 00360 2 3 1 moderate math: ground speed from Pyth. theorem
Extra keywords: physci
13. You are flying a plane. Air velocity (i.e., plane velocity relative to the air) is 40 mi/h due north. Wind velocity is 30 mi/h due west. What is the magnitude of ground velocity (i.e., the ground speed)?

a) 50 mi/h. b) −50 mi/h. c) 40 mi/h. d) 10 mi/h. e) 2500 mi/h.

SUGGESTED ANSWER: (a)

This question was a natural for my pilot students at Middle Tennessee State University (MTSU). The magnitude of the ground velocity can be found by the Pythagorean theorem in this case because the two velocity vectors to be added are at right angles to each other. Deduction should also give the answer. But more formally:

$$\vec{v}_{\rm air} = (0, 40) \qquad \text{and} \qquad \vec{v}_{\rm wind} = (30, 0) \ .$$

Thus,

$$\vec{v}_{\rm ground} = (30, 40) \qquad \text{and} \qquad v_{\rm ground} = \sqrt{30^2 + 40^2} = 50\,\text{mi/h} \ .$$

Wrong answers:
b) Magnitude or speed is never negative.
d) No. This speed is only possible if the plane direction were opposite the wind direction.
e) You forgot to take the square root. But no subsonic aircraft velocity and wind velocity can give you this speed.

Redaction: Jeffery, 2001jan01

---

004 qmult 00400 1 1 2 easy memory: dividing a circle
14. A circle can be divided into:

a) 360 divisions only. b) any number of divisions you like. c) 2π divisions only. d) π divisions only. e) 360 or 2π divisions only.

SUGGESTED ANSWER: (b)

Wrong answers:
a) A nonsense answer. **Redaction:** Jeffery, 2008jan01

---

004 qmult 00410 1 1 2 easy memory: radians in a circle 1
15. How many radians are there in a circle?

a) π. b) 2π. c) 3π. d) 360°. e) 360.

SUGGESTED ANSWER: (b)

Wrong answers:

e) The trick answer.

Redaction: Jeffery, 2001jan01

---

004 qmult 00420 1 5 1 easy thinking: 24 factors in 360

16. The division of the circle into 360° was an arbitrary choice—and we don't know why. We just know the ancient Mesopotamian mathematicians and astronomers did it this way—you know Mesopotamia—ancient Iraq: "the cradle of civilization". Their choice was just adopted by the ancient Greeks and got passed on to us. In the French Revolutionary epoch, the decimal system was adopted for most measures, but the revolutionaries didn't get around (you might say) to the circle. We can guess that one reasons is that the ancient Mesopotamians had a preference for whole number arithmetic particularly in division and 360 has a lot of whole number factors. How many whole number (i.e., integer) factors does 360 have counting 1 and 360 itself?

a) 24. b) 360. c) 6. d) 7. e) 12.

SUGGESTED ANSWER: (a)

Below are the whole number factors of 360 table format:

count	factor	complement factor
2	1	360
4	2	180
6	3	120
8	4	90
10	5	72
12	6	60
14	8	45
16	9	40
18	10	36
20	12	30
22	15	24
24	18	20

Wrong answers:
b) A specious guess.

Redaction: Jeffery, 2008jan01

---

004 qmult 00430 1 3 4 easy math: radian to degree conversion

17. What is the approximate conversion factor from radians to degrees?

a) 1/60 degrees/radian. b) π degrees/radian. c) 2π degrees/radian. d) 60 degrees/radian. e) 360 degrees/radian.

SUGGESTED ANSWER: (d)

Behold

$$180^\circ = \pi \ , \qquad \text{and so} \qquad \frac{180^\circ}{\pi} \approx 57.2958 \approx 60\,\text{degrees/radian} \ .$$

Wrong answers:
a) Wrong conversion factor: this is for degrees to radians.

Redaction: Jeffery, 2008jan01

---

004 qmult 00440 1 5 5 easy thinking: the 2 pi unit ti

18. There are 2π radians in a circle. It's rather inconvenient that this means that there are $2\pi = 6.2831853\ldots$ radians in a circle which is an irrational number. For convenience, we could use the revolution (with sympbol Rev: vocalized rev) as a new unit: $1\,\text{Rev} = 2\pi$. One hundredth of an Rv would be a:

a) exaRev. b) megaRev. c) kiloRev. d) deciRev. e) centiRev.

SUGGESTED ANSWER: (e)

I think the idea of revolutions makes sense to me. We could then drop this non-metric degree unit and use centiRevs (3.6°) and milliRevs (0.36°) for most purposes. But no one ever listens to me.

Wrong answers:

a) Eek, 10^{18} ti.

Redaction: Jeffery, 2008jan01

---

004 qmult 00450 1 3 1 easy math: hand angular measure

19. Approximately, at arm's length a finger subtends 1°, a fist 10°, and a spread hand 18°. These numbers, of course, vary a bit depending on person and exactly how the operation is done. What are these angles approximately in radians?

a) 1/60, 1/6, and 1/3 radians. b) 60, 600, and 1800 radians.
c) $\pi/12$, $\pi/3$, and $\pi/2$ radians. d) $\pi/12$, $\pi/3$, and π radians. e) $\pi/12$, $\pi/3$, and 2π radians.

SUGGESTED ANSWER: (a)

I've used the conversion factor (π radians)/(180° approximated to (1 radian/60°.

Wrong answers:

b) This looks like a conversion from radians to degrees where one uses the approximate conversion factor 60 degrees/radian.

Redaction: Jeffery, 2008jan01

---

004 qmult 00460 1 5 5 easy thinking: covering the Moon

20. Can you cover the Moon with your finger held at arm's length? **HINT:** You could try for yourself if you are not in a a test *mise en scène*.

a) No. The Moon is much larger in angle than a finger. Just think how huge the Moon looks on the horizon sometimes.
b) It depends critically on the size of one's finger and arm. People with huge hands can to it and those without can't.
c) Yes. A finger at arm's length typically subtends about 10° and the Moon subtends 0.01°.
d) No. The Moon's diameter is about 3470 km and a finger is about a centimeter or so in width.
e) Usually yes. A finger at arm's length typically subtends about 1° and the Moon subtends 0.5°.

SUGGESTED ANSWER: (e)

Wrong answers:

d) Yes, this makes sense.

Redaction: Jeffery, 2008jan01

---

004 qmult 00470 1 1 5 easy memory: small angle approximations

21. For small angles θ measured in radians and with increasing accuracy as θ goes to zero (where the formulas are in fact exact), one has the small angle approximations:

a) $\sin\theta \approx \cos\theta \approx 1 - \frac{1}{2}\theta^2$. b) $\cos\theta \approx \tan\theta \approx 1 - \frac{1}{2}\theta^2$. c) $\sin\theta \approx \cos\theta \approx \theta$.
d) $\cos\theta \approx \tan\theta \approx \theta$. e) $\sin\theta \approx \tan\theta \approx \theta$.

SUGGESTED ANSWER: (e)

The proof of these approximations follows from the Taylor expansions of sine and tangent about $\theta = 0$: i.e.,

$$\sin\theta = \sum_{n=0} \frac{1}{(2n+1)!}\theta^{2n+1} = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 - \frac{1}{5040}\theta^7 + \ldots\ ,$$
$$\tan\theta = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + \frac{17}{315}\theta^7 + \ldots\ ,$$

where one can get these from

`http://en.wikipedia.org/wiki/Tangent_function#Series_definitions` .

Wrong answers:
a) One has $\cos\theta \approx 1-(1/2)\theta^2$ for small angles, in fact.
b) One has $\cos\theta \approx 1-(1/2)\theta^2$ for small angles, in fact.

Redaction: Jeffery, 2008jan01

---

004 qmult 00500 1 1 5 easy memory: polar coordinates

22. In 2-dimensional Cartesian coordinates, a displacement vector $\vec{r}$ is given by

$$\vec{r} = (x, y) = x\hat{x} + y\hat{y} \ ,$$

where the unit vectors $\hat{x}$ and $\hat{y}$ are constants. In polar coordinates,

$$\vec{r} = (r, \theta) = r\hat{r} \ ,$$

where the unit vector

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \ .$$

The polar coordinates are obtained from the Cartesian ones by the formulae

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \ .$$

In calculational work one must be aware that a negative argument of $\tan^{-1}$ is treated by calculators and computers as implying that $y > 0$ and $x < 0$. If the reverse is true, one must explicitly add or subtract 180° from the calculated result. The Cartesian components are obtained from the polar coordinates by the formulae

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \ .$$

As well as $\hat{r}$, one needs another unit vector for polar coordinates that is perpendicular to the $\hat{r}$ and that is used for velocity and acceleration vectors and changes in the displacement vector. This is the unit vector ____________ given by

$$\underline{\qquad\qquad} = \vec{r}(\theta + 90^\circ) = -\sin\theta\hat{x} + \cos\theta\hat{y} \ ,$$

a) $\hat{\alpha}$ b) $\hat{\omega}$ c) $\hat{n}$ d) $\hat{z}$ e) $\hat{\theta}$

SUGGESTED ANSWER: (e)

Wrong answers:
c) This is the usual symbol for the normal unit vector.

Redaction: Jeffery, 2008jan01

---

004 qmult 00530 1 4 1 easy deducto-memory: acceleration in polar coordinates

23. "Let's play *Jeopardy*! For \$100, the answer is: The formulae

$$\vec{r} = r\hat{r} \ ,$$
$$\vec{v} = \frac{dr}{dt}\hat{r} + \frac{d\theta}{dt}\hat{\theta} \ ,$$
$$\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\theta} \ ,$$

where r is magnitude of displacement from the origin (or radial component of the displacement $\vec{r}$), $\hat{r}$ is the unit vector of the radial component, θ the angular component of $\vec{r}$, $\hat{\theta}$ is the unit vector of the angular coordinate, and one often writes $d\theta/dt$ is as ω which is called the angular velocity. The unit vectors are functions of the angular component:

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \qquad \text{and} \qquad \hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y} \ .$$

What are displacement, velocity, acceleration in ____________ coordinates, Alex?

a) polar b) Cartesian c) spherical polar d) elliptical e) hyperbolical

SUGGESTED ANSWER: (a) See Fr-557 and

`http://en.wikipedia.org/wiki/Kinematics#Cylindrical_coordinates` .

Wrong answers:
c) You really don't want to know.
d) I really don't want to know.

Redaction: Jeffery, 2008jan01

---

004 qmult 00550 1 4 3 easy deducto-memory: centripetal acceleration defined sort of
Extra keywords: physci

24. "Let's play *Jeopardy*! For $100, the answer is: It is the acceleration in a case of circular motion."

What is ____________, Alex?

a) net acceleration b) centrifugal (center-fleeing) acceleration
c) centripetal (center-pointing) acceleration d) deceleration e) zero

SUGGESTED ANSWER: (c)

Wrong answers:
e) The magnitude of the velocity is unchanging, but the direction changes continually. Hence there is an acceleration.

Redaction: Jeffery, 2001jan01

---

004 qmult 00552 1 1 3 easy memory: centripetal acceleration formula 1

25. The radial component of acceleration in polar coordinates

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2r}{dt^2} - r\omega^2$$

specializes to ____________ if the motion is circular and centered on the origin. In this case, the radial component of acceleration is called centripetal (meaning center pointing) since, in fact, it is always negative (i.e., the radial component of acceleration always points toward the origin). The radial component of velocity for circular motion is zero naturally and the angular component, often called the tangential velocity is given by

$$v_\theta = r\omega \ .$$

Usually, one drops the subscripts r and θ on a_r and v_θ if the quantities are identified by context.

a) $a_r = -r\omega = -\dfrac{v_\theta}{r}$ b) $a_r = r\omega^2 = \dfrac{v_\theta^2}{r}$ c) $a_r = -r\omega^2 = -\dfrac{v_\theta^2}{r}$ d) $a_r = \omega^2 = v_\theta^2$
e) $a_r = -\omega^2 = -v_\theta^2$

SUGGESTED ANSWER: (c)

Wrong answers:
a) Not dimensionally correct.

Redaction: Jeffery, 2008jan01

---

004 qmult 00556 2 5 2 moderate thinking memory: cent. accel. behavior

26. The formula for centripetal acceleration magnitude for circular motion is

$$a_{\rm cen} = \frac{v^2}{r} \ ,$$

where v is the tangential velocity of the motion and r is the radius of the circle. The centripetal acceleration ____________ with v and ____________ with r.

a) increases linearly; decreases inverse linearly b) increases quadratically; decreases inverse linearly c) decreases inverse linearly; increases quadratically d) increases quadratically; decreases inverse quadratically e) constant; decreases inverse linearly

SUGGESTED ANSWER: (b)

Wrong answers:
e) As Lurch would say: "Aaaarh."

Redaction: Jeffery, 2001jan01

---

004 qmult 00560 1 1 5 easy memory: uniform circular motion

27. In uniform circular motion, position, velocity, and acceleration are continually changing. The velocity and acceleration magnitudes are:

a) continually changing too. b) continually changing and constant, respectively. c) constant and continually changing, respectively. d) constant and undefined, respectively. e) both constant.

SUGGESTED ANSWER: (e)

Wrong answers:
d) A nonsense answer.

Redaction: Jeffery, 2008jan01

---

004 qfull 00200 2 3 0 moderate math: quadratic parabolic I

28. The formula for a general quadratic is

$$f(x) = ax^2 + bx + c \ .$$

The shape of a quadratic is actually a simple parabola with the vertical symmetry axis offset from the origin.

a) Show this by completing the square in the function

$$f(x) = ax^2 + bx + c$$

and find the x coordinate of the symmetry axis. The completed square form is called the vertex form.

b) What is the formula for a transformed coordinate x' that explicitly turns the quadratic function into a simple parabola?

SUGGESTED ANSWER:

a) Behold:

$$\begin{aligned} f(x) = ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right] \\ &= a\left\{\left[x - \left(-\frac{b}{2a}\right)\right]^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right\} \ . \end{aligned}$$

Now it is clear that the function gives a simple parabola with symmetry axis at $x = -b/(2a)$.

b) The required formula is

$$x' = x + \frac{b}{2a} \ .$$

Redaction: Jeffery, 2008jan01

---

004 qfull 00210 3 5 0 tough thinking: horizontal range formula algebra
Extra keywords: Suitable for the algebra-based course.

29. A projectile is launched from x-y origin which is on the ground level of large level plain. The x direction is the horizontal and the y direction is the vertical: upward is the positive y direction. The projectile is launched in the positive x direction. The initial launch speed is v_0 at angle θ above the horizontal. Air drag is neglected.

a) Find the expressions for x and y position as functions of time t entirely in symbols and with any dependences on θ shown explicitly. Drop any symbols that stand for known zeros.

b) Now find y as a function of x by eliminating t. Now find the horizontal range formula: i.e., the expression for x when the projectile returns to $y = 0$. Simplify the formula as much as is reasonably possible. **HINT:** The trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$ helps simplifying formula.

c) Using the horizontal range formula, find by any means the angle θ a maximum range holding all the other variables constant. Briefly explain how you arrived at your answer. **HINT:** The sine function has only one maximum in the domain of its argument $[0°, 180°]$.

SUGGESTED ANSWER:

a) From the kinematic equations for constant acceleration, one immediately finds:

$$x = (v_0 \cos\theta)\, t$$

and

$$y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)\, t \ .$$

b) Well

$$t = \frac{x}{v_0 \cos\theta} \ .$$

Substituting this into the expression for y gives

$$y = -\frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2 + x\tan\theta \ .$$

Well by setting the last expression for y equal to zero we obtain

$$0 = -\frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2 + x\tan\theta \ .$$

One solution for x is $x = 0$ which is just the launch solution. The other solution is the horizontal range solution which is nonzero. Since it is non-zero, we can divide through by this solution x and obtain a linear equation for x:

$$0 = -\frac{1}{2}g\left(\frac{x}{v_0^2\cos^2\theta}\right) + \tan\theta \ .$$

Solving for x gives

$$x = \frac{2v_0^2}{g}\sin\theta\cos\theta = \frac{v_0^2}{g}\sin(2\theta)$$

(See also HRW-57). This solution is the horizontal range formula.

c) After an exhausting search that took hours and hours, I found that $\theta = 45°$ gave $\sin(2\theta)$ a maximum value of about 1 and therefore a maximum range for the projectile.

All right I lied, I didn't do that. I know that the sine function in the domain for its argument $(0°, 180°)$ has a maximum of 1 for $90°$. Thus, the range is maximum for $2\theta = 90°$ or $\theta = 45°$. Of course, if I didn't know where the maximum was I could have used calculus to find it, but calculus is *hors de combat* as *les Français* might say in algebra-based introductory physics.

Redaction: Jeffery, 2008jan01

004 qfull 00230 3 3 0 tough math: projectile motion in volleyball

30. The women's volleyball court has a net height of 2.24 m and extends 9.0 m on either side of the net. On a jump serve, a player spikes (I think that's the word) the ball at 3.00 m above the court in a direction perpendicular to the net. The initial velocity is **HORIZONTAL** and the net is 8.0 m away from the server. Neglect air drag.

    This is a problem in which you want to find the conditions that lead to the desired result. It's really a pretty common kind of problem—in life as well as in physics.

    a) Sketch a cross-section diagram of the system: launch, court, net. Then sketch a general trajectory that lands before the net and one that lands on the other side of then (i.e., that clears the net).

    b) Now solve in **SYMBOLS** for initial velocity v_0 as a function of **ONLY** the variables x (horizontal position measured from the launch point) and y (height), and the constants y_0 (launch height) and g. The time variable t should be eliminated. For a general point (x, y) on a general trajectory (that had a horizontal launch velocity recall), the formula gives the initial velocity v_0 which will get the ball to that point.

       Sketch a graph of v_0 as a function of y for constant x: indicate on sketch the v_0 for $y = 0$ in symbols and the location of the maximum of v_0. What is the meaning of this maximum? Then sketch v_0 as a function of x for constant y.

    c) What is the minimum velocity v_0 needed for the ball to clear the net? Assume the ball is a point mass for this part.

    d) What is the maximum velocity v_0 allowed if the ball is to stay in court? Assume no one touches the ball and assume the ball is a point mass for this part.

SUGGESTED ANSWER:

a) You will have to imagine the sketch.

b) The kinematic equations for position as a function of time are for the x and y directions, respectively,

$$x = v_0 t \qquad \text{and} \qquad y = -\frac{1}{2}gt^2 + y_0 \ .$$

Now we want to find v_0 as a function of x, y_0, y, and g. So clearly we must combine the above formulae and eliminate t. There are more than one way to do this. The one that comes to mind first is to find t as a function of y. Behold:

$$t = \sqrt{\frac{y - y_0}{-(1/2)g}} = \sqrt{\frac{2(y_0 - y)}{g}} \ .$$

The negative solution for time was for the mythical extrapolation of the kinematic equations to negative times before the spiking event. Using our expression for time and the x position expression, we find

$$v_0 = \frac{x}{t} = x\sqrt{\frac{g}{2(y_0 - y)}} \ .$$

You will have imagine the plot of v_0 versus y. For x fixed, one one can see that v_0 rises from a minimum $x\sqrt{g/(2y_0)}$ for $y = 0$ (which is when the ball lands on the ground at x) and goes to maximum of infinity for $y = y_0$. The latter case means that you need infinite initial horizontal velocity to keep the ball from falling downward when $x > 0$.

You will have imagine the plot of v_0 versus x. But you can see that it is a simple linear plot. The slope is $\sqrt{g/[2(y_0 - y)]}$. The behavior of v_0 as function of x shows that for fixed y, one requires requires a higher initial velocity as x increases. Another way of looking at it is that to put the arrival at height y farther away (i.e., at increased x) requires increasing v_0.

This problem illustrates how initial conditions can be determined so that the outcome is what one desires.

c) The final x position in this case is fixed and is $x = 8$ m measured relative to the launch position. Thus the v_0 can be viewed as a function of final y. Since v_0 increases with y, the minimum launch speed to clear the net will be for y equal to the net height. Thus,

$$v_0 \approx 8 \times \sqrt{\frac{10}{2 \times (3.00 - 2.24)}} \approx 8 \times \sqrt{\frac{10}{1.5}} \approx 20\,\mathrm{m/s}$$

to about 1-digit accuracy. To better accuracy

$$v_0 = 20.3\,\mathrm{m/s}\ .$$

d) In this case, the final height is fixed at $y = 0$ which is the minimum landing height of the ball. Now v_0 increases with x and the maximum in-court x is the distance from the spiker to the far end of the court (i.e., $x = 17\,\mathrm{m}$), and so the maximum velocity for the ball staying in-court is given by

$$v_0 \approx 17.0 \times \sqrt{\frac{10}{2 \times (3.00 - 0.00)}} \approx 17.0 \times \sqrt{\frac{10}{6.00}} \approx 22\,\mathrm{m/s}$$

to about 1-digit accuracy. To better accuracy

$$v_0 = 21.7\,\mathrm{m/s}\ .$$

The speed range for a good serve is very narrow under the given conditions. I suspect that in actual play there is probably a broader range. Mediocre servers probably just shoot somewhat upward at less-than-killer speed to make sure the ball gets over and stays in play. Expert servers may be able hit the horizontal launch speed rather closely, but through practice not calculation. By the way the spiking height of 3.00 m (9.84 ft) is really high it seems to me: maybe the spiker is an Olympian.

Fortran Code

```
        print*
        conv=3.28  ! conversion factor for meters to feet
        xx1=8.
        g=9.8
        y1=2.24
        y0=3.
        y0feet=y0*conv
        vv1=xx1*sqrt( g/ (2.*(y0-y1)) )
        xx2=17.
        y2=0.
        vv2=xx2*sqrt( g/ (2.*(y0-y2)) )
        print*,'vv1,vv2,y0feet'
        print*,vv1,vv2,y0feet
*             20.3133354  21.7263279  9.84
```

Redaction: Jeffery, 2001jan01

---

004 qfull 00240 3 3 0 tough math: orient express

31. You are on the last run of the historic Orient Express train in 1939 traveling from Paris to Istanbul. Somewhere between Belgrade and Sofia, the train ominously starts accelerating in the reverse direction—which we will call the negative x direction. The **MAGNITUDE** of this acceleration is a_{tr}. The train is on a **STRAIGHT**, level line of track. Neglect **AIR DRAG**.

a) To make this problem clear, let the ground frame be the primed frame and the train frame be the unprimed frame. The coordinate transformation for any object (i.e, the general transformation) from the ground frame (the primed frame) to the train frame (the unprimed frame) in the x direction is

$$x = x' - x'_{\mathrm{train}}$$

and in the y direction is

$$y = y'\ .$$

For the x direction, differentiate to obtain the general transformation for v_x (train frame velocity) and a_x (train frame acceleration). Note the given $a_{x,\text{train}} = -a_{\text{tr}}$.

b) Inside a train car, a projectile is launched in the positive x direction (i.e., in the forward direction). What is the horizontal acceleration in the train frame in terms of variables? What is the vertical acceleration in the train frame in terms of variables? Take the **UP** direction as the positive y direction. **HINT:** No elaborate calculations are needed. We are just looking for simple, short, symbol answers. Remember, there is no x acceleration for a projectile in the ground frame, but there is for the y acceleration in the ground frame.

c) You and your mysterious *compagnon de voyage* M. Achille find yourselves locked in your train car—and attempt some two-dimensional kinematics. What are the x and y positions relative to the car as functions of time t for the projectile launched from the origin (which is fixed to the car) at time zero with launch speed v_0 (relative to the car) and at an angle θ to the positive x direction: the angle is the range $0°$ to $90°$ relative to the train. Express these positions using the acceleration formulae found in part (b) and using time t, v_0 and θ. **HINT:** Remember that motion in orthogonal directions (i.e., perpendicular directions) is independent in the sense that only forces in those orthogonal directions affect the motion. Of course, the forces may have some dependencies among themselves, but that is at another level of description.

d) For reasons known to himself alone, M. Achille insists that you find the horizontal range formula for the projectile in the train frame: i.e., a formula giving the horizontal range (the x displacement from launch height to launch height) in terms of variables **NOT** including time t. Find that range formula and simplify it as much as reasonably possible.

e) What is the horizontal range formula for the train frame in the case that a_{tr} goes to zero?

SUGGESTED ANSWER:

a) Behold:

$$\begin{aligned} x &= x' - x'_{\text{train}} \\ v_x &= v'_x - v'_{x,\text{train}} \\ a_x &= a'_x - a'_{x,\text{train}} = a'_x - (-a_{\text{tr}}) = a'_x + a_{\text{tr}} \ . \end{aligned}$$

b) Behold:

$$\begin{aligned} a_x &= 0 - (-a_{\text{tr}}) = a_{\text{tr}} \\ a_y &= -g \ . \end{aligned}$$

To make it concrete why the x direction acceleration is positive, imagine you are in suddenly decelerating car (i.e., a negatively accelerating car). In order not to be atom-by-atom thrown forward, tension forces in your seat belt or a pressure force in your airbag has to act on you. And inside your body, tension, pressure, etc. forces act throughout to decelerate you with the car. To put it in more physicsy terms, the car acceleration relative to the ground acts as an inertial force per unit mass inside the car that acts on you atom-by-atom like gravity (and in general relativity, gravity is considered an inertial force) and to resist the inertial forces on your body and inside it, contact forces (i.e., pressure, tension, friction, shear resistance, etc.) must act. If the car deceleration is very large, the contact forces can be unpleasant.

c) Applying the constant-acceleration kinematic equations to the two independent directions, we find

$$x = \frac{1}{2}a_{\text{tr}}t^2 + (v_0\cos\theta)t$$

and

$$y = -\frac{1}{2}gt^2 + (v_0\sin\theta)t \ .$$

d) Finding the horizontal range formula begins by setting $y = 0$ in the equation for the y position. When that is done, there are two solutions for time t. One is just the launch time $t = 0$. The other solution gives the time when the projectile has returned to the height $y = 0$. Carrying out the aforeaid procedure, we set y equal to zero in the y position equation and divide through by t (which is nonzero, but not yet known) and obtain

$$0 = -\frac{1}{2}gt + v_0 \sin\theta \ .$$

Solving for time gives

$$t = \frac{2v_0 \sin\theta}{g} \ .$$

Substituting this time into the x position equation gives

$$x = \frac{2a_{\rm tr} v_0^2}{g^2} \sin^2\theta + \frac{2v_0^2}{g} \cos\theta \sin\theta$$

or

$$x = \frac{2a_{\rm tr} v_0^2}{g^2} \sin^2\theta + \frac{v_0^2}{g} \sin(2\theta) \ .$$

The last formula is the horizontal range formula for the accelerated-relative-to-the-ground train frame.

e) If $a_{\rm tr} \to 0$, then the horizontal range formula reduces to

$$x = \frac{v_0^2}{g} \sin(2\theta)$$

which is just the ordinary horizontal range formula for projectile motion relative to the ground or frames unaccelerated with respect to the ground.

NOTE: M. Achille may be the brother of Hercule Poirot, but then again Hercule Poirot may never have had a brother—except temporarily. But if M. Achille is not Poirot's brother, then he could be ...

Redaction: Jeffery, 2008jan01

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

---

1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67384(80) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2012, CODATA)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\mathrm{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

---

2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\mathrm{sphere}} = 4\pi \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

---

3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1-x} \approx 1 + x \;:\; (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

5 Quadratic Formula

$$\text{If} \qquad 0 = ax^2 + bx + c\,, \qquad \text{then} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

Numerically robust solution (Press-178):

$$q = -\frac{1}{2}\left[b + \mathrm{sgn}(b)\sqrt{b^2 - 4ac}\right] \qquad x_1 = \frac{q}{a} \qquad x_2 = \frac{c}{q}$$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) \\ &= f(x_0) + (x - x_0) f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots \end{aligned}$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\mathrm{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\mathrm{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$$

Five 1-dimensional equations of kinematics

Equation No.	Equation	Unwanted variable
1	$v = at + v_0$	Δx
2	$\Delta x = \frac{1}{2}at^2 + v_0 t$	v
3 (timeless eqn)	$v^2 - v_0^2 = 2a\Delta x$	t
4	$\Delta x = \frac{1}{2}(v + v_0)t$	a
5	$\Delta x = vt - \frac{1}{2}at^2$	v_1

Fiducial acceleration due to gravity (AKA little g) $\quad g = 9.8\,\mathrm{m/s^2}$

$$x_{\mathrm{rel}} = x_2 - x_1 \qquad v_{\mathrm{rel}} = v_2 - v_1 \qquad a_{\mathrm{rel}} = a_2 - a_1$$

$$x' = x - v_{\mathrm{frame}}t \qquad v' = v - v_{\mathrm{frame}} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\mathrm{avg}} = \frac{\Delta\vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\mathrm{avg}} = \frac{\Delta\vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0}t \qquad y = -\frac{1}{2}gt^2 + v_{y,0}t + y_0 \qquad v_{x,0} = v_0\cos\theta \qquad v_{y,0} = v_0\sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0\cos\theta} \qquad y = y_0 + x\tan\theta - \frac{x^2 g}{2v_0^2\cos^2\theta}$$

$$x_{\mathrm{for}\ y\ \mathrm{max}} = \frac{v_0^2\sin\theta\cos\theta}{g} \qquad y_{\mathrm{max}} = y_0 + \frac{v_0^2\sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin(2\theta)}{g} \qquad \theta_{\mathrm{for\ max}} = \frac{\pi}{4} \qquad x_{\mathrm{max}}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0\sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\text{tan}} = r\alpha$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = v\omega$$

---

13 Very Basic Newtonian Physics

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i\vec{r}_i}{m_{\text{total}}} = \frac{\sum_{\text{sub}} m_{\text{sub}}\vec{r}_{\text{cm sub}}}{m_{\text{total}}} \qquad \vec{v}_{\text{cm}} = \frac{\sum_i m_i\vec{v}_i}{m_{\text{total}}} \qquad \vec{a}_{\text{cm}} = \frac{\sum_i m_i\vec{a}_i}{m_{\text{total}}}$$

$$\vec{r}_{\text{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\text{total}}}$$

$$\vec{F}_{\text{net}} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,\text{m/s}^2$$

$$\vec{F}_{\text{normal}} = -\vec{F}_{\text{applied}} \qquad F_{\text{linear}} = -kx$$

$$f_{\text{normal}} = \frac{T}{r} \qquad T = T_0 - F_{\text{parallel}}(s) \qquad T = T_0$$

$$F_{\text{f static}} = \min(F_{\text{applied}}, F_{\text{f static max}}) \qquad F_{\text{f static max}} = \mu_{\text{static}}F_{\text{N}} \qquad F_{\text{f kinetic}} = \mu_{\text{kinetic}}F_{\text{N}}$$

$$v_{\text{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\text{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\text{centripetal}} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\text{drag,lin}} = bv \qquad v_{\text{T}} = \frac{mg}{b} \qquad \tau = \frac{v_{\text{T}}}{g} = \frac{m}{b} \qquad v = v_{\text{T}}(1 - e^{-t/\tau})$$

$$F_{\text{drag,quad}} = bv^2 = \frac{1}{2}C\rho Av^2 \qquad v_{\text{T}} = \sqrt{\frac{mg}{b}}$$

---

14 Energy and Work

$$dW = \vec{F} \cdot d\vec{s} \qquad W = \int \vec{F} \cdot d\vec{s} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\text{mechanical}} = KE + PE$$

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F} \cdot \vec{v}$$

$$\Delta KE = W_{\text{net}} \quad \Delta PE_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \quad \Delta E = W_{\text{nonconservative}}$$

$$F = -\frac{dPE}{dx} \qquad \vec{F} = -\nabla PE \qquad PE = \frac{1}{2}kx^2 \qquad PE = mgy$$

---

15 **Momentum**

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta KE_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\text{cm}} = \vec{F}_{\text{net non-flux}} + (\vec{v}_{\text{flux}} - \vec{v}_{\text{cm}})\frac{dm}{dt} = \vec{F}_{\text{net non-flux}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_0 + v_{\text{ex}} \ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

---

16 **Collisions**

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\, dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\text{net}}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$KE_{\text{total } f} = KE_{\text{total } i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\text{rel}'} = -v_{\text{rel}} \qquad \text{1-d Elastic Collision Expressions}$$

---

17 **Rotational Kinematics**

$$2\pi = 6.2831853\ldots \qquad \frac{1}{2\pi} = 0.15915494\ldots$$

$$\frac{180^\circ}{\pi} = 57.295779\ldots \approx 60^\circ \qquad \frac{\pi}{180^\circ} = 0.017453292\ldots \approx \frac{1}{60^\circ}$$

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\omega = \alpha t + \omega_0 \qquad \Delta\theta = \frac{1}{2}\alpha t^2 + \omega_0 t \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t \qquad \Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$

18 Rotational Dynamics

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

$$L_z = RP_{xy}\sin\gamma_L \qquad \tau_z = RF_{xy}\sin\gamma_\tau \qquad L_z = I\omega \qquad \tau_{z,\text{net}} = I\alpha$$

$$I = \sum_i m_i R_i^2 \qquad I = \int R^2 \rho\, dV \qquad I_{\text{parallel axis}} = I_{\text{cm}} + mR_{\text{cm}}^2 \qquad I_z = I_x + I_y$$

$$I_{\text{cyl,shell,thin}} = MR^2 \qquad I_{\text{cyl}} = \frac{1}{2}MR^2 \qquad I_{\text{cyl,shell,thick}} = \frac{1}{2}M(R_1^2 + R_2^2)$$

$$I_{\text{rod,thin,cm}} = \frac{1}{12}ML^2 \qquad I_{\text{sph,solid}} = \frac{2}{5}MR^2 \qquad I_{\text{sph,shell,thin}} = \frac{2}{3}MR^2$$

$$a = \frac{g\sin\theta}{1 + I/(mr^2)}$$

$$KE_{\text{rot}} = \frac{1}{2}I\omega^2 \qquad dW = \tau_z\, d\theta \qquad P = \frac{dW}{dt} = \tau_z\omega$$

$$\Delta KE_{\text{rot}} = W_{\text{net}} = \int \tau_{z,\text{net}}\, d\theta \qquad \Delta PE_{\text{rot}} = -W = -\int \tau_{z,\text{con}}\, d\theta$$

$$\Delta E_{\text{rot}} = KE_{\text{rot}} + \Delta PE_{\text{rot}} = W_{\text{non,rot}} \qquad \Delta E = \Delta KE + KE_{\text{rot}} + \Delta PE = W_{\text{non}} + W_{\text{rot}}$$

19 Static Equilibrium

$$\vec{F}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = \vec{\tau}'_{\text{ext,net}} \quad \text{if } F_{\text{ext,net}} = 0$$

$$0 = F_{\text{net } x} = \sum F_x \qquad 0 = F_{\text{net } y} = \sum F_y \qquad 0 = \tau_{\text{net}} = \sum \tau$$

20 Gravity

$$\vec{F}_{1 \text{ on } 2} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \qquad \vec{g} = -\frac{GM}{r^2}\hat{r} \qquad \oint \vec{g}\cdot d\vec{A} = -4\pi GM$$

$$PE = -\frac{Gm_1m_2}{r_{12}} \qquad V = -\frac{GM}{r} \qquad v_{\text{escape}} = \sqrt{\frac{2GM}{r}} \qquad v_{\text{orbit}} = \sqrt{\frac{GM}{r}}$$

$$P^2 = \left(\frac{4\pi^2}{GM}\right)r^3 \qquad P = \left(\frac{2\pi}{\sqrt{GM}}\right)r^{3/2} \qquad \frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} = \text{Constant}$$

$$R_{\text{Earth,mean}} = 6371.0\,\text{km} \qquad R_{\text{Earth,equatorial}} = 6378.1\,\text{km} \qquad M_{\text{Earth}} = 5.9736\times10^{24}\,\text{kg}$$

$$R_{\text{Earth mean orbital radius}} = 1.495978875\times10^{11}\,\text{m} = 1.0000001124\,\text{AU} \approx 1.5\times10^{11}\,\text{m} \approx 1\,\text{AU}$$

$$R_{\text{Sun,equatorial}} = 6.955\times10^{8} \approx 109\times R_{\text{Earth,equatorial}} \qquad M_{\text{Sun}} = 1.9891\times10^{30}\,\text{kg}$$

21 Fluids

$$\rho = \frac{\Delta m}{\Delta V} \qquad p = \frac{F}{A} \qquad p = p_0 + \rho g d_{\text{depth}}$$

$$\begin{aligned}
\text{Pascal's principle} \quad & p = p_{\text{ext}} - \rho g(y - y_{\text{ext}}) \qquad \Delta p = \Delta p_{\text{ext}} \\
\text{Archimedes principle} \quad & F_{\text{buoy}} = m_{\text{fluid dis}}g = V_{\text{fluid dis}}\rho_{\text{fluid}}g \\
\text{equation of continuity for ideal fluid} \quad & R_V = Av = \text{Constant} \\
\text{Bernoulli's equation} \quad & p + \frac{1}{2}\rho v^2 + \rho g y = \text{Constant}
\end{aligned}$$

22 Oscillation

$$P = f^{-1} \qquad \omega = 2\pi f \qquad F = -kx \qquad PE = \frac{1}{2}kx^2 \qquad a(t) = -\frac{k}{m}x(t) = -\omega^2 x(t)$$

$$\omega = \sqrt{\frac{k}{m}} \qquad P = 2\pi\sqrt{\frac{m}{k}} \qquad x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$E_{\text{mec total}} = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}kx_{\max}^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$P = 2\pi\sqrt{\frac{I}{mgr}} \qquad P = 2\pi\sqrt{\frac{r}{g}}$$

23 Waves

$$\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2} \qquad v = \sqrt{\frac{F_{\mathrm{T}}}{\mu}} \qquad y = f(x \mp vt)$$

$$y = y_{\max}\sin[k(x \mp vt)] = y_{\max}\sin(kx \mp \omega t)$$

$$\text{Period} = \frac{1}{f} \qquad k = \frac{2\pi}{\lambda} \qquad v = f\lambda = \frac{\omega}{k} \qquad P \propto y_{\max}^2$$

$$y = 2y_{\max}\sin(kx)\cos(\omega t) \qquad n = \frac{L}{\lambda/2} \qquad L = n\frac{\lambda}{2} \qquad \lambda = \frac{2L}{n} \qquad f = n\frac{v}{2L}$$

$$v = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} \qquad n\lambda = d\sin(\theta) \qquad \left(n + \frac{1}{2}\right)\lambda = d\sin(\theta)$$

$$I = \frac{P}{4\pi r^2} \qquad \beta = (10\,\mathrm{dB}) \times \log\left(\frac{I}{I_0}\right)$$

$$f = n\frac{v}{4L} \; : \; n = 1, 3, 5, \ldots \qquad f_{\text{medium}} = \frac{f_0}{1 - v_0/v_{\text{medium}}}$$

$$f' = f\left(1 - \frac{v'}{v}\right) \qquad f = \frac{f'}{1 - v'/v}$$

24 Thermodynamics

$$dE = dQ - dW = T\,dS - p\,dV$$

$$T_{\mathrm{K}} = T_{\mathrm{C}} + 273.15\,\mathrm{K} \qquad T_{\mathrm{F}} = 1.8 \times T_{\mathrm{C}} + 32^\circ\mathrm{F}$$

$$Q = mC\Delta T \qquad Q = mL$$

$$PV = NkT \qquad P = \frac{2}{3}\frac{N}{V}KE_{\text{avg}} = \frac{2}{3}\frac{N}{V}\left(\frac{1}{2}mv_{\text{RMS}}^2\right)$$

$$v_{\text{RMS}} = \sqrt{\frac{3kT}{m}} = 2735.51\ldots \times \sqrt{\frac{T/300}{A}}$$

$$PV^{\gamma} = \text{constant} \qquad 1 < \gamma \leq \frac{5}{3} \qquad v_{\text{sound}} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{-V(\partial P/\partial V)_S}{m(N/V)}} = \sqrt{\frac{\gamma kT}{m}}$$

$$\varepsilon = \frac{W}{Q_{\text{H}}} = \frac{Q_{\text{H}} - Q_{\text{C}}}{W} = 1 - \frac{Q_{\text{C}}}{Q_{\text{H}}} \qquad \eta_{\text{heating}} = \frac{Q_{\text{H}}}{W} = \frac{Q_{\text{H}}}{Q_{\text{H}} - Q_{\text{C}}} = \frac{1}{1 - Q_{\text{C}}/Q_{\text{H}}} = \frac{1}{\varepsilon}$$

$$\eta_{\text{cooling}} = \frac{Q_{\text{C}}}{W} = \frac{Q_{\text{H}} - W}{W} = \frac{1}{\varepsilon} - 1 = \eta_{\text{heating}} - 1$$

$$\varepsilon_{\text{Carnot}} = 1 - \frac{T_{\text{C}}}{T_{\text{H}}} \qquad \eta_{\text{heating,Carnot}} = \frac{1}{1 - T_{\text{C}}/T_{\text{H}}} \qquad \eta_{\text{cooling,Carnot}} = \frac{T_{\text{C}}/T_{\text{H}}}{1 - T_{\text{C}}/T_{\text{H}}}$$