Intro Physics Semester I **Name:**

Homework 4: Multi-Dimensional Kinematics: Homeworks are due as posted on the course web site. Multiple-choice questions will **NOT** be marked, but some of them will appear on exams. One or more full-answer questions may be marked as time allows for the grader. Hand-in the full-answer questions on other sheets of paper: i.e., not crammed onto the downloaded question sheets. Make the full-answer solutions sufficiently detailed that the grader can follow your reasoning, but you do **NOT** be verbose. Solutions will be posted eventually after the due date. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

1. "Let's play *Jeopardy*! For \$100, the answer is: The vectors

$$\vec{r}\,, \qquad \vec{v} = \frac{d\vec{r}}{dt}\,, \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}\,."$$

What are the most obvious quantities of ____________, Alex?

a) time   b) light   c) dynamics   d) multi-dimensional kinematics   e) rest

2. In Cartesian coordinates, kinematics is very simple if $x(t)$, $y(t)$, and $z(t)$ are specified because the unit vectors of Cartesian coordinates are:

a) variables.   b) dependent on time.   c) constants.   d) dependent on position.
e) fewer than two.

3. The unit vectors of polar and spherical polar coordinates are dependent on:

a) the angular coordinates of the displacement vector.
b) the magnitude of the displacement vector.   c) nothing.   d) time explicitly.   e) mass.

4. In Newtonian physics, the component motions of a particle in ____________ directions are independent of each other. This means that motion variables in one direction are **NOT** intrinsic functions of motion variables in ____________ directions. For example, a particular velocity in the x direction does **NOT** intrinsically set the velocity in the y direction. Now the initial or continuing conditions will set up relationships between motions in ____________ directions. But those conditions can be anything allowed by physics and relationships between the motions can be anything allowed by the conditions and physics. For example, the conditions could set the x direction velocity to be a constant $v_x = 1\,\mathrm{m/s}$ and the y direction velocity to a constant $v_y = 2\,\mathrm{m/s}$. Then one has, of course, that $v_y = 2v_x$, but the physically allowed conditions could have set the two velocity components to any constant values or anything else allowed by physics. There is **NO** intrinsic relationship between those velocity components. The independence of motions in ____________ directions is one of the amazing facts about the physical world and a vast simplification in dealing with it.

a) the same.   b) opposite.   c) both negative   d) orthogonal   e) dependent

5. "Let's play *Jeopardy*! For \$100, the answer is: Without qualifications, one usually means the non-powered flight of an object in the air or through space. The simplest in-air case is the one in which air drag is neglected. The science of such motions is ballistics.

What is ____________, Alex?

a) apparent motion   b) projectile motion   c) 1-dimensional motion
d) trigonometric motion   e) unstoppable motion

6. A ball is tossed into the air and falls to the ground some distance away. Consider its motion in the vertical direction only and neglect air drag.

a) The ball has a constant acceleration downward.
b) The ball first accelerates **UPWARD** on its rising path and then accelerates **DOWNWARD** on its falling path.
c) The ball first accelerates **DOWNWARD** on its rising path and then accelerates **UPWARD** on its falling path.
d) The ball does not accelerate at all.
e) The ball is always accelerating in the upward direction.

7. Which best describes the path of a ball thrown on level ground at an angle 30° above the horizontal as seen from a side view.

   a) Two straight lines that meet at an apex: one for the rising phase; one the declining phase. The rising phase line is **TWICE** the length of the declining phase line.
   b) A smooth curve that rises and falls with distance. As far as the eye can tell, the curve could be parabolic.
   c) Two straight lines that meet at an apex: one for the rising phase; one the declining phase. The rising phase line is **HALF** the length of the declining phase line.
   d) A smooth curve that rises and falls with distance, but suddenly breaks off and descends vertically.
   e) A smooth curve that rises and falls with distance and then rises and falls again with distance. A Bactrian camel curve.

8. The horizontal range formula for projectile motion near the Earth's surface and neglecting air drag is:

   a) $x_{\text{max}} = \dfrac{8v_0^2}{g}\sin(2\theta)$.   b) $x_{\text{max}} = \dfrac{v_0^2}{g}\cos(2\theta)$.   c) $x_{\text{max}} = \dfrac{v_0^2}{g}$.   d) $x_{\text{max}} = \dfrac{v_0^2}{g}\sin(2\theta)$.
   e) $x_{\text{max}} = \dfrac{4v_0^2}{g}\sin(2\theta)$.

9. The maximum horizontal range value for projectile motion near the Earth's surface for a given launch speed v_0 and neglecting air drag is obtained for launch angle:

   a) 30°.   b) 45°.   c) 60°.   d) 82.245°.   e) 90°.

10. The ratio of the maximum height formula (for projectile motion near the Earth's surface, neglecting air drag, and measuring from the launch height) to the horizontal range formula (for projectile motion near the Earth's surface and neglecting air drag) is ____________ and its value for the maximum range is _________.

    a) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \dfrac{1}{4}\tan\theta$; $\frac{1}{4}$ b) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \dfrac{1}{4}\tan\theta$; 1   c) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \tan\theta$; 1   d) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \tan\theta$; $\frac{1}{4}$
    e) $\dfrac{y_{\text{max}}}{x_{\text{range}}} = \dfrac{1}{4}\tan\theta$; ∞

11. "Let's play *Jeopardy*! For \$100, the answer is: Motion of something with respect to something else. To be a bit more explicit, say you have two objects. The relative displacement of object 2 from object 1 is defined to be

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \ .$$

    From this definition, the relative velocity and acceleration follow from differentiation:

$$\vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \text{and} \qquad \vec{a} = \vec{a}_2 - \vec{a}_1 \ .$$

    The description of the motion in these terms clarifies the initial answer statement."

    What is ____________, Alex?

    a) variable motion   b) relativistic motion   c) no motion   d) abosulte motion
    e) relative motion

12. You are in a featureless narrow room playing catch with a friend. How can you tell if the room is in a building or is a sealed compartment on super-smoothly running, non-accelerating train (or plane)? **HINT:** Review your whole life experience; try an experiment (but not while you are driving).

    a) When you throw the ball **ALONG** the long axis of the room, it would have different speeds (relative to the room) in the two possible directions if you were on a train.
    b) When you thow the ball **PERPENDICULAR** to the long axis of the room it would curve off a straight line (relative to the room) if you were on a train.
    c) On a train the thrown ball would zigzag wildly in flight.
    d) On a train the thrown ball would do loops in flight.
    e) There is no way to tell as long as the train motion is very smooth.

13. You are flying a plane. Air velocity (i.e., plane velocity relative to the air) is 40 mi/h due north. Wind velocity is 30 mi/h due west. What is the magnitude of ground velocity (i.e., the ground speed)?

a) 50 mi/h. b) -50 mi/h. c) 40 mi/h. d) 10 mi/h. e) 2500 mi/h.

14. A circle can be divided into:

    a) 360 divisions only. b) any number of divisions you like. c) 2π divisions only. d) π divisions only. e) 360 or 2π divisions only.

15. How many radians are there in a circle?

    a) π. b) 2π. c) 3π. d) 360°. e) 360.

16. The division of the circle into 360° was an arbitrary choice—and we don't know why. We just know the ancient Mesopotamian mathematicians and astronomers did it this way—you know Mesopotamia—ancient Iraq: "the cradle of civilization". Their choice was just adopted by the ancient Greeks and got passed on to us. In the French Revolutionary epoch, the decimal system was adopted for most measures, but the revolutionaries didn't get around (you might say) to the circle. We can guess that one reasons is that the ancient Mesopotamians had a preference for whole number arithmetic particularly in division and 360 has a lot of whole number factors. How many whole number (i.e., integer) factors does 360 have counting 1 and 360 itself?

    a) 24. b) 360. c) 6. d) 7. e) 12.

17. What is the approximate conversion factor from radians to degrees?

    a) 1/60 degrees/radian. b) π degrees/radian. c) 2π degrees/radian. d) 60 degrees/radian. e) 360 degrees/radian.

18. There are 2π radians in a circle. It's rather inconvenient that this means that there are $2\pi = 6.2831853\ldots$ radians in a circle which is an irrational number. For convenience, we could use the revolution (with sympbol Rev: vocalized rev) as a new unit: $1\,\text{Rev} = 2\pi$. One hundredth of an Rv would be a:

    a) exaRev. b) megaRev. c) kiloRev. d) deciRev. e) centiRev.

19. Approximately, at arm's length a finger subtends 1°, a fist 10°, and a spread hand 18°. These numbers, of course, vary a bit depending on person and exactly how the operation is done. What are these angles approximately in radians?

    a) 1/60, 1/6, and 1/3 radians. b) 60, 600, and 1800 radians. c) $\pi/12$, $\pi/3$, and $\pi/2$ radians. d) $\pi/12$, $\pi/3$, and π radians. e) $\pi/12$, $\pi/3$, and 2π radians.

20. Can you cover the Moon with your finger held at arm's length? **HINT:** You could try for yourself if you are not in a a test *mise en scène*.

    a) No. The Moon is much larger in angle than a finger. Just think how huge the Moon looks on the horizon sometimes.
    b) It depends critically on the size of one's finger and arm. People with huge hands can to it and those without can't.
    c) Yes. A finger at arm's length typically subtends about 10° and the Moon subtends 0.01°.
    d) No. The Moon's diameter is about 3470 km and a finger is about a centimeter or so in width.
    e) Usually yes. A finger at arm's length typically subtends about 1° and the Moon subtends 0.5°.

21. For small angles θ measured in radians and with increasing accuracy as θ goes to zero (where the formulas are in fact exact), one has the small angle approximations:

    a) $\sin\theta \approx \cos\theta \approx 1 - \frac{1}{2}\theta^2$. b) $\cos\theta \approx \tan\theta \approx 1 - \frac{1}{2}\theta^2$. c) $\sin\theta \approx \cos\theta \approx \theta$. d) $\cos\theta \approx \tan\theta \approx \theta$. e) $\sin\theta \approx \tan\theta \approx \theta$.

22. In 2-dimensional Cartesian coordinates, a displacement vector $\vec{r}$ is given by

    $$\vec{r} = (x, y) = x\hat{x} + y\hat{y} \ ,$$

    where the unit vectors $\hat{x}$ and $\hat{y}$ are constants. In polar coordinates,

    $$\vec{r} = (r, \theta) = r\hat{r} \ ,$$

    where the unit vector

    $$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \ .$$

The polar coordinates are obtained from the Cartesian ones by the formulae

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) .$$

In calculational work one must be aware that a negative argument of $\tan^{-1}$ is treated by calculators and computers as implying that $y > 0$ and $x < 0$. If the reverse is true, one must explicitly add or subtract $180°$ from the calculated result. The Cartesian components are obtained from the polar coordinates by the formulae

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta .$$

As well as $\hat{r}$, one needs another unit vector for polar coordinates that is perpendicular to the $\hat{r}$ and that is used for velocity and acceleration vectors and changes in the displacement vector. This is the unit vector ____________ given by

$$\underline{\qquad\qquad} = \vec{r}(\theta + 90°) = -\sin\theta\hat{x} + \cos\theta\hat{y} ,$$

a) $\hat{\alpha}$ b) $\hat{\omega}$ c) $\hat{n}$ d) $\hat{z}$ e) $\hat{\theta}$

23. "Let's play *Jeopardy*! For \$100, the answer is: The formulae

$$\begin{aligned}
\vec{r} &= r\hat{r} , \\
\vec{v} &= \frac{dr}{dt}\hat{r} + \frac{d\theta}{dt}\hat{\theta} , \\
\vec{a} &= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\hat{\theta} ,
\end{aligned}$$

where r is magnitude of displacement from the origin (or radial component of the displacement $\vec{r}$), $\hat{r}$ is the unit vector of the radial component, θ the angular component of $\vec{r}$, $\hat{\theta}$ is the unit vector of the angular coordinate, and one often writes $d\theta/dt$ is as ω which is called the angular velocity. The unit vectors are functions of the angular component:

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y} \qquad \text{and} \qquad \hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y} .$$

What are displacement, velocity, acceleration in ____________ coordinates, Alex?

a) polar b) Cartesian c) spherical polar d) elliptical e) hyperbolical

24. "Let's play *Jeopardy*! For \$100, the answer is: It is the acceleration in a case of circular motion."

What is ____________, Alex?

a) net acceleration b) centrifugal (center-fleeing) acceleration
c) centripetal (center-pointing) acceleration d) deceleration e) zero

25. The radial component of acceleration in polar coordinates

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2r}{dt^2} - r\omega^2$$

specializes to ____________ if the motion is circular and centered on the origin. In this case, the radial component of acceleration is called centripetal (meaning center pointing) since, in fact, it is always negative (i.e., the radial component of acceleration always points toward the origin). The radial component of velocity for circular motion is zero naturally and the angular component, often called the tangential velocity is given by

$$v_\theta = r\omega .$$

Usually, one drops the subscripts r and θ on a_r and v_θ if the quantities are identified by context.

a) $a_r = -r\omega = -\frac{v_\theta}{r}$ b) $a_r = r\omega^2 = \frac{v_\theta^2}{r}$ c) $a_r = -r\omega^2 = -\frac{v_\theta^2}{r}$ d) $a_r = \omega^2 = v_\theta^2$
e) $a_r = -\omega^2 = -v_\theta^2$

26. The formula for centripetal acceleration magnitude for circular motion is

$$a_{\rm cen} = \frac{v^2}{r} \ ,$$

where v is the tangential velocity of the motion and r is the radius of the circle. The centripetal acceleration ____________ with v and ____________ with r.

    a) increases linearly; decreases inverse linearly   b) increases quadratically; decreases inverse linearly   c) decreases inverse linearly; increases quadratically   d) increases quadratically; decreases inverse quadratically   e) constant; decreases inverse linearly

27. In uniform circular motion, position, velocity, and acceleration are continually changing. The velocity and acceleration magnitudes are:

    a) continually changing too.   b) continually changing and constant, respectively.
    c) constant and continually changing, respectively.   d) constant and undefined, respectively.
    e) both constant.

28. The formula for a general quadratic is

$$f(x) = ax^2 + bx + c \ .$$

The shape of a quadratic is actually a simple parabola with the vertical symmetry axis offset from the origin.

    a) Show this by completing the square in the function

$$f(x) = ax^2 + bx + c$$

    and find the x coordinate of the symmetry axis. The completed square form is called the vertex form.

    b) What is the formula for a transformed coordinate x' that explicitly turns the quadratic function into a simple parabola?

29. A projectile is launched from x-y origin which is on the ground level of large level plain. The x direction is the horizontal and the y direction is the vertical: upward is the positive y direction. The projectile is launched in the positive x direction. The initial launch speed is v_0 at angle θ above the horizontal. Air drag is neglected.

    a) Find the expressions for x and y position as functions of time t entirely in symbols and with any dependences on θ shown explicitly. Drop any symbols that stand for known zeros.

    b) Now find y as a function of x by eliminating t. Now find the horizontal range formula: i.e., the expression for x when the projectile returns to $y = 0$. Simplify the formula as much as is reasonably possible. **HINT:** The trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$ helps simplifying formula.

    c) Using the horizontal range formula, find by any means the angle θ a maximum range holding all the other variables constant. Briefly explain how you arrived at your answer. **HINT:** The sine function has only one maximum in the domain of its argument $[0^\circ, 180^\circ]$.

30. The women's volleyball court has a net height of 2.24 m and extends 9.0 m on either side of the net. On a jump serve, a player spikes (I think that's the word) the ball at 3.00 m above the court in a direction perpendicular to the net. The initial velocity is **HORIZONTAL** and the net is 8.0 m away from the server. Neglect air drag.

    This is a problem in which you want to find the conditions that lead to the desired result. It's really a pretty common kind of problem—in life as well as in physics.

    a) Sketch a cross-section diagram of the system: launch, court, net. Then sketch a general trajectory that lands before the net and one that lands on the other side of then (i.e., that clears the net).

    b) Now solve in **SYMBOLS** for initial velocity v_0 as a function of **ONLY** the variables x (horizontal position measured from the launch point) and y (height), and the constants y_0 (launch height) and g. The time variable t should be eliminated. For a general point (x, y) on a general trajectory (that had a horizontal launch velocity recall), the formula gives the initial velocity v_0 which will get the ball to that point.

Sketch a graph of v_0 as a function of y for constant x: indicate on sketch the v_0 for $y = 0$ in symbols and the location of the maximum of v_0. What is the meaning of this maximum? Then sketch v_0 as a function of x for constant y.

c) What is the minimum velocity v_0 needed for the ball to clear the net? Assume the ball is a point mass for this part.

d) What is the maximum velocity v_0 allowed if the ball is to stay in court? Assume no one touches the ball and assume the ball is a point mass for this part.

31. You are on the last run of the historic Orient Express train in 1939 traveling from Paris to Istanbul. Somewhere between Belgrade and Sofia, the train ominously starts accelerating in the reverse direction—which we will call the negative x direction. The **MAGNITUDE** of this acceleration is a_{tr}. The train is on a **STRAIGHT**, level line of track. Neglect **AIR DRAG**.

a) To make this problem clear, let the ground frame be the primed frame and the train frame be the unprimed frame. The coordinate transformation for any object (i.e, the general transformation) from the ground frame (the primed frame) to the train frame (the unprimed frame) in the x direction is

$$x = x' - x'_{\text{train}}$$

and in the y direction is

$$y = y' .$$

For the x direction, differentiate to obtain the general transformation for v_x (train frame velocity) and a_x (train frame acceleration). Note the given $a_{x,\text{train}} = -a_{\text{tr}}$.

b) Inside a train car, a projectile is launched in the positive x direction (i.e., in the forward direction). What is the horizontal acceleration in the train frame in terms of variables? What is the vertical acceleration in the train frame in terms of variables? Take the **UP** direction as the positive y direction. **HINT:** No elaborate calculations are needed. We are just looking for simple, short, symbol answers. Remember, there is no x acceleration for a projectile in the ground frame, but there is for the y acceleration in the ground frame.

c) You and your mysterious *compagnon de voyage* M. Achille find yourselves locked in your train car—and attempt some two-dimensional kinematics. What are the x and y positions relative to the car as functions of time t for the projectile launched from the origin (which is fixed to the car) at time zero with launch speed v_0 (relative to the car) and at an angle θ to the positive x direction: the angle is the range 0° to 90° relative to the train. Express these positions using the acceleration formulae found in part (b) and using time t, v_0 and θ. **HINT:** Remember that motion in orthogonal directions (i.e., perpendicular directions) is independent in the sense that only forces in those orthogonal directions affect the motion. Of course, the forces may have some dependencies among themselves, but that is at another level of description.

d) For reasons known to himself alone, M. Achille insists that you find the horizontal range formula for the projectile in the train frame: i.e., a formula giving the horizontal range (the x displacement from launch height to launch height) in terms of variables **NOT** including time t. Find that range formula and simplify it as much as reasonably possible.

e) What is the horizontal range formula for the train frame in the case that a_{tr} goes to zero?

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

---

1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67384(80) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2012, CODATA)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\mathrm{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

---

2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\mathrm{sphere}} = 4\pi \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

---

3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta+90^\circ)=\cos(\theta) \qquad \cos(\theta+90^\circ)=-\sin(\theta) \qquad \tan(\theta+90^\circ)=-\tan(\theta)$$

$$\sin(180^\circ-\theta)=\sin(\theta) \qquad \cos(180^\circ-\theta)=-\cos(\theta) \qquad \tan(180^\circ-\theta)=-\tan(\theta)$$

$$\sin(90^\circ-\theta)=\cos(\theta) \qquad \cos(90^\circ-\theta)=\sin(\theta) \qquad \tan(90^\circ-\theta)=\frac{1}{\tan(\theta)}=\cot(\theta)$$

$$\sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b) \qquad \cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b)$$

$$\sin(2a)=2\sin(a)\cos(a) \qquad \cos(2a)=\cos^2(a)-\sin^2(a)$$

$$\sin(a)\sin(b)=\frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right] \qquad \cos(a)\cos(b)=\frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]$$

$$\sin(a)\cos(b)=\frac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]$$

$$\sin^2\theta=\frac{1}{2}[1-\cos(2\theta)] \qquad \cos^2\theta=\frac{1}{2}[1+\cos(2\theta)] \qquad \sin(a)\cos(a)=\frac{1}{2}\sin(2a)$$

$$\cos(x)-\cos(y)=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x)+\cos(y)=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x)+\sin(y)=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

4 Approximation Formulae

$$\frac{\Delta f}{\Delta x}\approx\frac{df}{dx} \qquad \frac{1}{1-x}\approx 1+x\ :\ (x<<1)$$

$$\sin\theta\approx\theta \qquad \tan\theta\approx\theta \qquad \cos\theta\approx 1-\frac{1}{2}\theta^2 \qquad \text{all for } \theta<<1$$

5 Quadratic Formula

$$\text{If} \qquad 0=ax^2+bx+c\ , \qquad \text{then} \qquad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=-\frac{b}{2a}\pm\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}$$

Numerically robust solution (Press-178):

$$q = -\frac{1}{2}\left[b + \operatorname{sgn}(b)\sqrt{b^2 - 4ac}\right] \qquad x_1 = \frac{q}{a} \qquad x_2 = \frac{c}{q}$$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series
$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) \\ &= f(x_0) + (x - x_0) f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots \end{aligned}$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Five 1-dimensional equations of kinematics

Equation No.	Equation	Unwanted variable
1	$v = at + v_0$	Δx
2	$\Delta x = \frac{1}{2}at^2 + v_0 t$	v
3 (timeless eqn)	$v^2 - v_0^2 = 2a\Delta x$	t
4	$\Delta x = \frac{1}{2}(v + v_0)t$	a
5	$\Delta x = vt - \frac{1}{2}at^2$	v_1

Fiducial acceleration due to gravity (AKA little g) $\quad g = 9.8\,\mathrm{m/s^2}$

$$x_{\mathrm{rel}} = x_2 - x_1 \qquad v_{\mathrm{rel}} = v_2 - v_1 \qquad a_{\mathrm{rel}} = a_2 - a_1$$

$$x' = x - v_{\mathrm{frame}}t \qquad v' = v - v_{\mathrm{frame}} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\mathrm{avg}} = \frac{\Delta\vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\mathrm{avg}} = \frac{\Delta\vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0}t \qquad y = -\frac{1}{2}gt^2 + v_{y,0}t + y_0 \qquad v_{x,0} = v_0\cos\theta \qquad v_{y,0} = v_0\sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0\cos\theta} \qquad y = y_0 + x\tan\theta - \frac{x^2 g}{2v_0^2\cos^2\theta}$$

$$x_{\mathrm{for}\ y\ \mathrm{max}} = \frac{v_0^2\sin\theta\cos\theta}{g} \qquad y_{\mathrm{max}} = y_0 + \frac{v_0^2\sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin(2\theta)}{g} \qquad \theta_{\mathrm{for\ max}} = \frac{\pi}{4} \qquad x_{\mathrm{max}}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0\sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\tan} = r\alpha$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = v\omega$$

---

13 Very Basic Newtonian Physics

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i\vec{r}_i}{m_{\text{total}}} = \frac{\sum_{\text{sub}} m_{\text{sub}}\vec{r}_{\text{cm sub}}}{m_{\text{total}}} \qquad \vec{v}_{\text{cm}} = \frac{\sum_i m_i\vec{v}_i}{m_{\text{total}}} \qquad \vec{a}_{\text{cm}} = \frac{\sum_i m_i\vec{a}_i}{m_{\text{total}}}$$

$$\vec{r}_{\text{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\text{total}}}$$

$$\vec{F}_{\text{net}} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,\text{m/s}^2$$

$$\vec{F}_{\text{normal}} = -\vec{F}_{\text{applied}} \qquad F_{\text{linear}} = -kx$$

$$f_{\text{normal}} = \frac{T}{r} \qquad T = T_0 - F_{\text{parallel}}(s) \qquad T = T_0$$

$$F_{\text{f static}} = \min(F_{\text{applied}}, F_{\text{f static max}}) \qquad F_{\text{f static max}} = \mu_{\text{static}}F_{\text{N}} \qquad F_{\text{f kinetic}} = \mu_{\text{kinetic}}F_{\text{N}}$$

$$v_{\text{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\text{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\text{centripetal}} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\text{drag,lin}} = bv \qquad v_{\text{T}} = \frac{mg}{b} \qquad \tau = \frac{v_{\text{T}}}{g} = \frac{m}{b} \qquad v = v_{\text{T}}(1 - e^{-t/\tau})$$

$$F_{\text{drag,quad}} = bv^2 = \frac{1}{2}C\rho Av^2 \qquad v_{\text{T}} = \sqrt{\frac{mg}{b}}$$

---

14 Energy and Work

$$dW = \vec{F} \cdot d\vec{s} \qquad W = \int \vec{F} \cdot d\vec{s} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\text{mechanical}} = KE + PE$$

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F} \cdot \vec{v}$$

$$\Delta KE = W_{\text{net}} \quad \Delta PE_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \quad \Delta E = W_{\text{nonconservative}}$$

$$F = -\frac{dPE}{dx} \qquad \vec{F} = -\nabla PE \qquad PE = \frac{1}{2}kx^2 \qquad PE = mgy$$

15 **Momentum**

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta KE_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\text{cm}} = \vec{F}_{\text{net non-flux}} + (\vec{v}_{\text{flux}} - \vec{v}_{\text{cm}})\frac{dm}{dt} = \vec{F}_{\text{net non-flux}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_0 + v_{\text{ex}} \ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

16 **Collisions**

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\, dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\text{net}}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$KE_{\text{total } f} = KE_{\text{total } i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\text{rel}'} = -v_{\text{rel}} \qquad \text{1-d Elastic Collision Expressions}$$

17 **Rotational Kinematics**

$$2\pi = 6.2831853\ldots \qquad \frac{1}{2\pi} = 0.15915494\ldots$$

$$\frac{180^\circ}{\pi} = 57.295779\ldots \approx 60^\circ \qquad \frac{\pi}{180^\circ} = 0.017453292\ldots \approx \frac{1}{60^\circ}$$

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\omega = \alpha t + \omega_0 \qquad \Delta\theta = \frac{1}{2}\alpha t^2 + \omega_0 t \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t \qquad \Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$

---

18 **Rotational Dynamics**

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{\tau}_{\mathrm{net}} = \frac{d\vec{L}}{dt}$$

$$L_z = RP_{xy}\sin\gamma_L \qquad \tau_z = RF_{xy}\sin\gamma_\tau \qquad L_z = I\omega \qquad \tau_{z,\mathrm{net}} = I\alpha$$

$$I = \sum_i m_i R_i^2 \qquad I = \int R^2 \rho\, dV \qquad I_{\text{parallel axis}} = I_{\mathrm{cm}} + mR_{\mathrm{cm}}^2 \qquad I_z = I_x + I_y$$

$$I_{\text{cyl,shell,thin}} = MR^2 \qquad I_{\text{cyl}} = \frac{1}{2}MR^2 \qquad I_{\text{cyl,shell,thick}} = \frac{1}{2}M(R_1^2 + R_2^2)$$

$$I_{\text{rod,thin,cm}} = \frac{1}{12}ML^2 \qquad I_{\text{sph,solid}} = \frac{2}{5}MR^2 \qquad I_{\text{sph,shell,thin}} = \frac{2}{3}MR^2$$

$$a = \frac{g\sin\theta}{1 + I/(mr^2)}$$

$$KE_{\mathrm{rot}} = \frac{1}{2}I\omega^2 \qquad dW = \tau_z\, d\theta \qquad P = \frac{dW}{dt} = \tau_z\omega$$

$$\Delta KE_{\mathrm{rot}} = W_{\mathrm{net}} = \int \tau_{z,\mathrm{net}}\, d\theta \qquad \Delta PE_{\mathrm{rot}} = -W = -\int \tau_{z,\mathrm{con}}\, d\theta$$

$$\Delta E_{\mathrm{rot}} = KE_{\mathrm{rot}} + \Delta PE_{\mathrm{rot}} = W_{\mathrm{non,rot}} \qquad \Delta E = \Delta KE + KE_{\mathrm{rot}} + \Delta PE = W_{\mathrm{non}} + W_{\mathrm{rot}}$$

---

19 **Static Equilibrium**

$$\vec{F}_{\mathrm{ext,net}} = 0 \qquad \vec{\tau}_{\mathrm{ext,net}} = 0 \qquad \vec{\tau}_{\mathrm{ext,net}} = \tau'_{\mathrm{ext,net}} \quad \text{if } F_{\mathrm{ext,net}} = 0$$

$$0 = F_{\text{net }x} = \sum F_x \qquad 0 = F_{\text{net }y} = \sum F_y \qquad 0 = \tau_{\text{net}} = \sum \tau$$

20 Gravity

$$\vec{F}_{1\text{ on }2} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \qquad \vec{g} = -\frac{GM}{r^2}\hat{r} \qquad \oint \vec{g}\cdot d\vec{A} = -4\pi GM$$

$$PE = -\frac{Gm_1m_2}{r_{12}} \qquad V = -\frac{GM}{r} \qquad v_{\text{escape}} = \sqrt{\frac{2GM}{r}} \qquad v_{\text{orbit}} = \sqrt{\frac{GM}{r}}$$

$$P^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \qquad P = \left(\frac{2\pi}{\sqrt{GM}}\right) r^{3/2} \qquad \frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} = \text{Constant}$$

$$R_{\text{Earth,mean}} = 6371.0\,\text{km} \qquad R_{\text{Earth,equatorial}} = 6378.1\,\text{km} \qquad M_{\text{Earth}} = 5.9736\times 10^{24}\,\text{kg}$$

$$R_{\text{Earth mean orbital radius}} = 1.495978875\times 10^{11}\,\text{m} = 1.0000001124\,\text{AU} \approx 1.5\times 10^{11}\,\text{m} \approx 1\,\text{AU}$$

$$R_{\text{Sun,equatorial}} = 6.955\times 10^{8} \approx 109\times R_{\text{Earth,equatorial}} \qquad M_{\text{Sun}} = 1.9891\times 10^{30}\,\text{kg}$$

21 Fluids

$$\rho = \frac{\Delta m}{\Delta V} \qquad p = \frac{F}{A} \qquad p = p_0 + \rho g d_{\text{depth}}$$

Pascal's principle	$p = p_{\text{ext}} - \rho g(y - y_{\text{ext}}) \qquad \Delta p = \Delta p_{\text{ext}}$
Archimedes principle	$F_{\text{buoy}} = m_{\text{fluid dis}}g = V_{\text{fluid dis}}\rho_{\text{fluid}}g$
equation of continuity for ideal fluid	$R_V = Av = \text{Constant}$
Bernoulli's equation	$p + \frac{1}{2}\rho v^2 + \rho g y = \text{Constant}$

22 Oscillation

$$P = f^{-1} \qquad \omega = 2\pi f \qquad F = -kx \qquad PE = \frac{1}{2}kx^2 \qquad a(t) = -\frac{k}{m}x(t) = -\omega^2 x(t)$$

$$\omega = \sqrt{\frac{k}{m}} \qquad P = 2\pi\sqrt{\frac{m}{k}} \qquad x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$E_{\text{mec total}} = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}kx_{\text{max}}^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$P = 2\pi\sqrt{\frac{I}{mgr}} \qquad P = 2\pi\sqrt{\frac{r}{g}}$$

23 Waves

$$\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2} \qquad v = \sqrt{\frac{F_{\mathrm{T}}}{\mu}} \qquad y = f(x \mp vt)$$

$$y = y_{\mathrm{max}}\sin[k(x \mp vt)] = y_{\mathrm{max}}\sin(kx \mp \omega t)$$

$$\mathrm{Period} = \frac{1}{f} \qquad k = \frac{2\pi}{\lambda} \qquad v = f\lambda = \frac{\omega}{k} \qquad P \propto y_{\mathrm{max}}^2$$

$$y = 2y_{\mathrm{max}}\sin(kx)\cos(\omega t) \qquad n = \frac{L}{\lambda/2} \qquad L = n\frac{\lambda}{2} \qquad \lambda = \frac{2L}{n} \qquad f = n\frac{v}{2L}$$

$$v = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} \qquad n\lambda = d\sin(\theta) \qquad \left(n + \frac{1}{2}\right)\lambda = d\sin(\theta)$$

$$I = \frac{P}{4\pi r^2} \qquad \beta = (10\,\mathrm{dB}) \times \log\left(\frac{I}{I_0}\right)$$

$$f = n\frac{v}{4L} \ : \ n = 1, 3, 5, \ldots \qquad f_{\mathrm{medium}} = \frac{f_0}{1 - v_0/v_{\mathrm{medium}}}$$

$$f' = f\left(1 - \frac{v'}{v}\right) \qquad f = \frac{f'}{1 - v'/v}$$

24 Thermodynamics

$$dE = dQ - dW = T\,dS - p\,dV$$

$$T_{\mathrm{K}} = T_{\mathrm{C}} + 273.15\,\mathrm{K} \qquad T_{\mathrm{F}} = 1.8 \times T_{\mathrm{C}} + 32^\circ\mathrm{F}$$

$$Q = mC\Delta T \qquad Q = mL$$

$$PV = NkT \qquad P = \frac{2}{3}\frac{N}{V}KE_{\mathrm{avg}} = \frac{2}{3}\frac{N}{V}\left(\frac{1}{2}mv_{\mathrm{RMS}}^2\right)$$

$$v_{\mathrm{RMS}} = \sqrt{\frac{3kT}{m}} = 2735.51\ldots \times \sqrt{\frac{T/300}{A}}$$

$$PV^{\gamma} = \text{constant} \qquad 1 < \gamma \leq \frac{5}{3} \qquad v_{\text{sound}} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{-V(\partial P/\partial V)_S}{m(N/V)}} = \sqrt{\frac{\gamma kT}{m}}$$

$$\varepsilon = \frac{W}{Q_{\text{H}}} = \frac{Q_{\text{H}} - Q_{\text{C}}}{W} = 1 - \frac{Q_{\text{C}}}{Q_{\text{H}}} \qquad \eta_{\text{heating}} = \frac{Q_{\text{H}}}{W} = \frac{Q_{\text{H}}}{Q_{\text{H}} - Q_{\text{C}}} = \frac{1}{1 - Q_{\text{C}}/Q_{\text{H}}} = \frac{1}{\varepsilon}$$

$$\eta_{\text{cooling}} = \frac{Q_{\text{C}}}{W} = \frac{Q_{\text{H}} - W}{W} = \frac{1}{\varepsilon} - 1 = \eta_{\text{heating}} - 1$$

$$\varepsilon_{\text{Carnot}} = 1 - \frac{T_{\text{C}}}{T_{\text{H}}} \qquad \eta_{\text{heating,Carnot}} = \frac{1}{1 - T_{\text{C}}/T_{\text{H}}} \qquad \eta_{\text{cooling,Carnot}} = \frac{T_{\text{C}}/T_{\text{H}}}{1 - T_{\text{C}}/T_{\text{H}}}$$