Introductory Physics II: Calculus-Based: Exam 4 Name:

Instructions: There are 20 multiple-choice problems worth 1 mark each for a total of 20 marks altogether. Choose the **BEST** answer, completion, etc. Read all responses carefully. **NOTE** long detailed responses won't depend on hidden keywords: keywords in such responses are bold-faced capitalized.

There are **THREE** full-answer problems worth 10 marks each for a total of 30 marks altogether. Answer them all. It is important that you **SHOW** (**SHOW, SHOW, SHOW**) how you got the answer for the full-answer problem. Don't give up on problems where you can't do the first part: sometimes later parts can be done independently. Some full-answer problems may extend over multiple-pages: make sure you have answered everything. And **BOX-IN** your final answers. There are 2 scratch pages for additional calculations or to rewrite answers if needed. In the latter case, make clear which answer is to be marked. You can ask for more paper if needed.

This is a **CLOSED-BOOK: CLOSED-NOTES: NO-ONLINE-ACCESS: NO-AI** exam. **NO** cheat sheets allowed. An **EQUATION SHEET** is provided. Calculators are permitted—but **ONLY** for calculations. There are **SCRATCH PAGES** for auxiliary calculations. Remember your name (and write it down on the exam too).

The exam is out of 50 marks altogether and is a 60-minute exam.

Multiple Choice Questions:

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031 qmult 00120 1 1 2 easy memory: Faraday's law of induction 3

1. Faraday's law of induction (in its differential or Maxwell-Faraday equation form) implies that time-varying ____________ induce ____________.

   a) electric fields; magnetic fields b) magnetic fields; electric fields
   c) magnetic fields; capacitor fields d) magnetic fields; inductor fields
   e) electric fields; capacitor fields

   SUGGESTED ANSWER: (b)

   Wrong answers:

   Redaction: Jeffery, 2001jan01

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031 qmult 00150 1 1 4 easy memory: magnetic flux defined
Extra keywords: physci

2. Magnetic flux is defined by the differential formula:

   a) $d\vec{\Phi}_B = \vec{E} \times d\vec{A}$. b) $\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int B \cdot d\vec{A}$. c) $d\Phi_B = \vec{E} \cdot d\vec{A}$.
   d) $d\Phi_B = \vec{B} \cdot d\vec{A}$. e) $d\vec{\Phi}_B = \vec{B} \times d\vec{A}$.

   SUGGESTED ANSWER: (d)

   The MKS unit of magnetic flux is a hoary old EM unit, the weber or $\mathrm{Wb} = \mathrm{T{\cdot}m^2}$ It is actually a pretty big unit of flux. It is named for Wilhelm Eduard Weber (1804–1891), a German and so pronounced veber by the cognoscenti. Frankly, I've little use for the weber. I just use the MKS unit of magnetic flux.

But maybe there are fields where it's webers this and webers that and "Chuck him out, the brute!"

Wrong answers:
b) This is Faraday's law of induction itself.
c) This is electric flux.

Redaction: Jeffery, 2001jan01

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031 qmult 00330 2 3 4 moderate math: flux rule for motional emf example 2
Extra keywords: Can be used for an algebra-based course.

3. Say you have two infinite parallel wires a distance ℓ apart and two conducting bars. You lay the bars across the wires: they are perpendicular to the wires. There is a uniform magnetic field perpendicular to the bars and wires. You move the **LEFTMOST** bar to the **RIGHT** at constant **SPEED** v_1 and the **RIGHTMOST** bar to the **LEFT** at constant **SPEED** v_2. Assume the bars do **NOT** touch each other in the time frame of interest. What is the **MAGNITUDE** of the induced emf around the loop formed by the wires and bars and what does this emf do?

a) $\mathcal{E} = \ell(-v_1 + v_2)B$ and opposes the increase in magnetic flux.
b) $\mathcal{E} = -\ell(v_1 + v_2)B$ and reinforces the increase in magnetic flux.
c) $\mathcal{E} = \ell v_1 B$ and has no effect on the magnetic flux.
d) $\mathcal{E} = \ell(v_1 + v_2)B$ and opposes the increase in magnetic flux.
e) $\mathcal{E} = \ell(v_1 + v_2)$ and has no effect on the magnetic flux.

SUGGESTED ANSWER: (d)

In this case, the second form of the flux rule for motional emf gives the answer simply

$$\mathcal{E} = \oint \vec{v} \times \vec{B} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int_{\text{linked}} \vec{B} \cdot d\vec{A} = \pm B(v_1 + v_2)\ell$$

and the emf whatever its sign causes a backreaction B-field that opposing the increasing magnetic flux as we know from Lenz's law.

Wrong answers:
c) It has an effect.

Redaction: Jeffery, 2008jan01

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031 qmult 00610 1 4 1 easy deducto-memory: electric motor 2
Extra keywords: mathematical physics

4. "Let's play *Jeopardy*! For \$100, the answer is: It is a device that consists of a rotating coil with an alternating current that is located in a magnetic field. It outputs kinetic energy."

What is a/an ____________, Alex?

a) electric motor b) solenoid c) commutator d) half-ring e) brush

SUGGESTED ANSWER: (a)

Wrong answers:

b) As Lurch would say AAAARGH.

Redaction: Jeffery, 2008jan01

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032 qmult 00202 1 4 1 easy deducto-memory: inductor defined 2

5. An inductor is a device designed:

a) to have a **SELF-INDUCED EMF** when a time-varying current (implying a time-varying magnetic field) **PASSES** through it.
b) to have a **SELF-INDUCED EMF** when a time-varying current (implying a time-varying magnetic field) **DOES NOT PASS** through it.
c) to have a **CAPACITANCE** when a time-varying current (implying a time-varying magnetic field) **PASSES** through it.
d) to have a **CAPACITANCE** when a time-varying current (implying a time-varying magnetic field) **DOES NOT PASS** through it.
e) for use by the draft board.

SUGGESTED ANSWER: (a)

Wrong answers:

e) Being born 20 miles west of the New York state line, that's something I didn't have to worry about when growing up in the 1960s.

Redaction: Jeffery, 2001jan01

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032 qmult 00210 1 1 4 easy memory: mutual inductance transformer

6. An example of mutual inductance device is a/an:

a) generator. b) distributor. c) contributor. d) transformer.
e) informer

SUGGESTED ANSWER: (d)

Wrong answers:

e) In ancient Rome, they had a certain cachet.

Redaction: Jeffery, 2008jan01

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032 qmult 00250 1 1 5 easy deducto-memory: inductor and current

Extra keywords: inductor and current

7. An inductor resists changes in:

a) its immanence. b) its resistance. c) its capacitance.
d) its inductance. e) the current through it.

SUGGESTED ANSWER: (e)

Wrong answers:

a) It's immance of what I wonder.

Redaction: Jeffery, 2001jan01

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032 qmult 00300 1 1 2 easy memory: ideal solenoid inductance

8. The ideal solenoid has no magnetic field outside and magnetic field inside

$$\vec{B} = \mu_0 n I \hat{z} \ ,$$

where n is the number of turns per unit length, I is the current through the solenoid, and z is the symmetry axis with its sense determined by a right-hand rule from the current flow in the turns. From this formula, the solenoid length ℓ, area A, and volume $V_{\rm vol}$, and Faraday's law of induction

$$\mathcal{E} = -\frac{d\Phi}{dt} \ ,$$

one can easily calculate the ideal solenoid inductance to be:

a) $L = \mu_0 n^2 V_{\rm vol} \ell/A$. b) $L = \mu_0 n^2 V_{\rm vol}$. c) $L = \mu_0 n^2 V_{\rm vol} A/\ell$.
d) $L = \mu_0 n^2$. e) $L = \mu_0 n V_{\rm vol} A/\ell$.

SUGGESTED ANSWER: (b)

The magnetic flux of single turn of the solenoid is

$$\Phi_{\rm turn} = \int \vec{B} \cdot d\vec{A} = BA = \mu_0 n I A \ ,$$

where we have aligned the area vector with the z axis. The total magnetic flux is

$$\Phi = N\Phi_{\rm turn} = \mu_0 N n I A = \mu_0 n^2 I A \ell = \mu_0 n^2 I V_{\rm vol}$$

Now applying Faraday's law, we get

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\mu_0 n^2 V_{\rm vol} \frac{dI}{dt} \ .$$

Thus, we recognize the inductance to be

$$L = \mu_0 n^2 V_{\rm vol} \ .$$

See SJ-899 for confirmation.

Wrong answers:

d) An inductance per unit volume. Not a useful quantity I'd guess.

Fortran-95 Code

Redaction: Jeffery, 2008jan01

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032 qmult 00410 1 1 1 easy memory: RL circuit behavior

9. In an RC circuit the potential across a capacitor is proportional to the integral of current and in an RL circuit the potential across an inductor is proportional to:

a) the derivative of current. b) the 2nd derivative of current.
c) the current. d) the current squared. e) the square current.

SUGGESTED ANSWER: (a)

Wrong answers:

e) The square current is an unhip current.

Redaction: Jeffery, 2001jan01

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033 qmult 00100 1 1 3 easy memory: AC definition 1

10. Alternating current or AC means that current and potential vary about a mean value (usually zero) and usually (but not always)

 a) trapezoidally. b) parabolically. c) sinusoidally. d) geometrically. e) wretchedly.

SUGGESTED ANSWER: (c)

Wrong answers:

e) A nonsense answer.

Redaction: Jeffery, 2008jan01

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033 qmult 00130 1 1 2 easy memory: frequency, period, and all that

11. An AC potential can usually be described by a sinusoid: i.e., by

$$V = V_0 \cos(\omega t + \theta_0) ,$$

where V_0 is the amplitude, ω is the angular frequency, and θ_0 is a constant phase angle. If $\theta_0 = -90°$, then

$$V = V_0 \sin(\omega t) .$$

In actual fact, the value of θ_0 can usually be chosen for convenience (and the choice is often 0 or, so that one has a sine function, $-90°$) since the phase angle of the sinusoid at time zero is unimportant for the time average behavior or the behavior over a cycle which is usually all that is of interest in describing an AC circuit in purely periodic operation. The angular frequency ω, ____________ (cycles per unit time) f, and period p (not to be confused with power P) are related by

$$\omega = 2\pi f = \frac{2\pi}{p} , \qquad f = \frac{\omega}{2\pi} = \frac{1}{p} , \qquad p = \frac{1}{f} = \frac{2\pi}{\omega} .$$

 a) RPM b) frequency c) fraction d) hertz e) function

SUGGESTED ANSWER: (b)

Wrong answers:

a) Some folks use RPM as a synonym for frequency, but only when frequency is in RPM (revolutions per minute) does it make any sense.

Redaction: Jeffery, 2008jan01

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033 qmult 00140 1 1 4 easy memory: unit of frequency

12. The MKS unprefixed unit of frequency is the:

 a) kilovolt (kV). b) volt (V) c) revolution per minute (RPM). d) hertz (Hz). e) kilohertz (kHz).

SUGGESTED ANSWER: (d)

Wrong answers:
a) Now does this look unprefixed?

Redaction: Jeffery, 2008jan01

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033 qmult 00160 1 1 1 easy memory: integral averages

13. In AC work, one often needs to know the time averages over one period p of $\cos^2(\omega t)$, $\sin^2(\omega t)$ and $\cos(\omega t)\sin(\omega t)$. The values, respectively, are:

a) 1/2, 1/2, 0. b) 1/2, 1/2, 1/2. c) 1/2, 0, 1/2. d) 0, 1/2, 0.
e) 1/2, 0, 0.

SUGGESTED ANSWER: (a)

First, note that

$$\frac{1}{P}\int_0^P f(\omega t), dt = \frac{1}{2\pi}\int_0^{2\pi} f(x), dx \ ,$$

where $x = \omega t$ and $\omega P = 2\pi$. With this transformation

$$\frac{1}{P}\int_0^P \cos^2(\omega t)\,dt \ , \qquad \frac{1}{P}\int_0^P \sin^2(\omega t)\,dt \ , \qquad \frac{1}{P}\int_0^P \cos(\omega t)\sin(\omega t)\,dt$$

become, respectively,

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^2 x\,dx \ , \qquad \frac{1}{2\pi}\int_0^{2\pi} \sin^2 x\,dx \ , \qquad \frac{1}{2\pi}\int_0^{2\pi} \cos(x)\sin(x)\,dx \ .$$

Using trig identities, we find

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}[1+\cos(2x)]\,dx \ , \qquad \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}[1-\cos(2x)]\,dx \ , \qquad \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\sin(2x)\,dx$$

which become

$$\frac{1}{2\pi}\left(\frac{1}{2}\right)\left[x+\frac{\sin(2x)}{2}\right]\Bigg|_0^{2\pi} \ , \qquad \frac{1}{2\pi}\left(\frac{1}{2}\right)\left[x-\frac{\sin(2x)}{2}\right]\Bigg|_0^{2\pi} \ , \qquad \frac{1}{2\pi}\left(\frac{1}{2}\right)\left[-\frac{\cos(2x)}{2}\right]\Bigg|_0^{2\pi}$$

which become

$$\frac{1}{2} \ , \qquad \frac{1}{2} \ , \qquad 0 \ .$$

Wrong answers:
e) Oh, c'mon.

Redaction: Jeffery, 2008jan01

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033 qmult 00170 1 1 5 easy memory: sinusoidal RMS quantities

14. If you have a sinusoidally varying quantity (like AC potential or current) with amplitude A_0, the root-mean-square (RMS) average for the quantity is:

    a) $2A_0$. b) $1/(2A_0)$. c) $1/(A_0\sqrt{2})$. d) $A_0/2$. e) $A_0/\sqrt{2}$.

    SUGGESTED ANSWER: (e)

    Wrong answers:
    a) This is the full range of the variation.

    Redaction: Jeffery, 2008jan01

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033 qmult 00210 1 1 3 easy memory: impedance magnitude

15. For a given circuit branch (which could be a device or circuit system), the ratio of the amplitude of the potential drop across the branch to the amplitude of the current through it is the non-complex number form of or modulus of the:

    a) reflectance Z. b) frequency Z. c) impedance Z. d) reluctance Z. e) redolence Z.

    SUGGESTED ANSWER: (c) The term impedance can be used either for the complex number impedance or its modulus. Wikipedia (Wikipedia: Electrical impedance) gives the first usage as correct, but intro physics textbooks can give the second (e.g., OpenStax Volume 2, p. 633). Context must decide

    Wrong answers:
    a) A nonsense answer.

    Redaction: Jeffery, 2008jan01

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033 qmult 00230 1 1 2 easy memory: phase angle

16. For a circuit branch (which could be a device or circuit system), you have potential drop across and current through given by, respectively,

    $$V = V_0 \cos(\omega t) \qquad \text{and} \qquad I = I_0 \cos(\omega t - \phi)$$

    The non-complex number impedance (or modulus of the complex number impedance) is ____________ and the phase angle is ____________. Formally, the current lags the potential by ϕ, but if $\phi = 0$, the two are in phase angle and if ϕ is negative, then the current leads the potential.

    a) $\dfrac{I_0}{V_0}$; ϕ b) $\dfrac{V_0}{I_0}$; ϕ c) $\dfrac{V_0}{I_0}$; ωt d) $\dfrac{V_0}{I_0}$; ω e) $\dfrac{I_0}{V_0}$; ω

    SUGGESTED ANSWER: (b)

    Wrong answers:
    d) ω is the angular frequency

    Redaction: Jeffery, 2008jan01

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033 qmult 00240 1 1 2 easy memory: AC power, phase angle

17. For a circuit branch (which could be a device or circuit system), you have potential drop/rise across and current through given by, respectively,

    $$V = V_0 \cos(\omega t) \qquad \text{and} \qquad I = I_0 \cos(\omega t - \phi) \ .$$

The time-average power output/input of the branch is given by:

a) $P = I_0V_0$.
b) $P = I_{\text{rms}}V_{\text{rms}}\cos(\phi)$, where $\cos(\phi)$ is the power factor.
c) $P = I_{\text{rms}}V_{\text{rms}}$.
d) $P = I_0V_0\cos(\phi)$, where $\cos(\phi)$ is the power factor.
e) $P = I_{\text{rms}}/V_{\text{rms}}$.

SUGGESTED ANSWER: (b)

The general formula

$$P = IV$$

works for time-dependent systems as long as current is constant in space—or so I believe. Thus, the instantaneous power for our AC system is

$$P = IV = I_0V_0\cos(\omega t)\cos(\omega t-\phi) = I_0V_0\left[\cos^2(\omega t)\cos(\phi)+\cos(\omega t)\sin(\omega t)\sin(\phi)\right] \ ,$$

where we have used the trig identity $\cos(x\pm y) = \cos(x)\cos(y)\mp\sin(x)\sin(y)$ (Wikipedia: List of trigonometric identities: Angle sum and difference identities). The time-average power is then

$$P_{\text{ave}} = \frac{1}{p}\int_0^p IV\,dt = I_0V_0\frac{1}{p}\int_0^p \left[\cos^2(\omega t)\cos(\phi)+\cos(\omega t)\sin(\omega t)\sin(\phi)\right]\,dt = I_0V_0\left[\frac{1}{2}\cos(\phi)+0\right]$$
$$= \frac{1}{2}I_0V_0\cos(\phi) = I_{\text{rms}}V_{\text{rms}}\cos(\phi) \ , \qquad \text{where} \qquad I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \qquad \text{and} \qquad V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \ .$$

Compactly:

$$P_{\text{ave}} = I_{\text{rms}}V_{\text{rms}}\cos(\phi) \ ,$$

where $\cos(\phi)$ is the power factor.

Wrong answers:

c) True only in the special case that $\phi = 0$.

Redaction: Jeffery, 2001jan01

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033 qmult 00242 1 1 4 easy memory: phase angle of an AC circuit

18. The time-average power supplied by a AC emf is 100 W. The RMS potential rise and currents for the device are, respectively 20 V and 10 A. What are the power factor and phase?

a) 1/2 and 0°. b) 1/3 and 30°. c) 1/2 and 45°. d) 1/2 and 60°.
e) 2 and 90°.

SUGGESTED ANSWER: (d)

Behold:

$$\text{power factor} = \cos(\phi) = \frac{P_{\text{ave}}}{I_{\text{rms}}V_{\text{rms}}} = \frac{1}{2} \qquad \text{and} \qquad \phi = \cos^{-1}\left(\frac{P_{\text{ave}}}{I_{\text{rms}}V_{\text{rms}}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} = 60^\circ \ .$$

Wrong answers:

a) A nonsense answer.

Fortran-95 Code

Redaction: Jeffery, 2008jan01

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033 qmult 00250 1 1 1 easy memory: resistor impedance

19. Ohm's law is the relationship between voltage drop across a resistor and current through it. The law is

$$v = iR$$

where v is voltage drop, i is current, and R is resistance. Given simple one-loop circuit with only an resistor of resistance R and AC power source with

$$v = V_0 \cos(\omega t) \ ,$$

what is the non-complex number impedance and the phase angle for current?

a) R and 0°. b) $1/R$ and 0°. c) $1/R$ and 90°. d) $1/R$ and $-90°$.
e) R and 90°.

SUGGESTED ANSWER: (a)

Behold:

$$i = \frac{v}{R} = \frac{V_0}{R} \cos(\omega t) = I_0 \cos(\omega t) \ ,$$

where $I_0 = V_0/R$. So the amplitude ratio is just $R = V_0/I_0$ and the phase angle is obviously 0°.

Wrong answers:

b) Half right, but half wrong

Redaction: Jeffery, 2008jan01

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033 qmult 00260 1 1 3 easy memory: capacitor impedance

20. The capacitor formula is

$$C = \frac{q}{v} \ ,$$

where v is instantaneous voltage drop across the capacitor (i.e., $v(t)$), q is the instantaneous capacitor charge (i.e., $q(t)$), and C is the capacitance. Given a simple one-loop circuit with only a capacitor of capacitance C and AC power source with

$$v = V_0 \cos(\omega t) \ ,$$

what is the non-complex number impedance (which is called the capacitive reactance) and the phase angle for current? Assume the current is flowing into positive plate of the capacitor.

a) $1/(\omega C)$ and 90°. b) ωC and 90°. c) $1/(\omega C)$ and $-90°$.
d) ωC and $-90°$. e) $1/\omega$ and $-90°$.

SUGGESTED ANSWER: (c)

Behold:

$$i = \frac{dq}{dt} = C\frac{dv}{dt} = -\omega CV_0 \sin(\omega t) = \omega CV_0 \cos[\omega t - (-90^\circ)] \ ,$$

where

1) $\cos(\omega t - \phi) = \cos(\omega t)\cos(\phi) + \sin(\omega t)\sin(\phi) = -\sin(\omega t)$ 2) $\sin(\phi) = -1$ 3) $\phi = -90^\circ$,

where we have used the trig identity $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$ (Wikipedia: List of trigonometric identities: Angle sum and difference identities). So the impedance (i.e., the ratio of potential amplitude to current amplitude) is $1/(\omega C)$ and the phase angle is -90°.

The phase angle means there is an angular lag ϕ. So when the potential is at ωt (e.g., $\omega t = 90^\circ$ giving $\cos(\omega t) = 0$), then the current is at $\omega t - \phi$ (e.g., $\omega t - \phi = 90^\circ - 90^\circ = 0^\circ$ giving $\cos(\omega t) = 1$ for $\omega t = 90^\circ$ and $\phi = 90^\circ$). In the current case, the phase angle is -90° meaning the lag is negative and current actually leads the potential. It's tricky to get the phase angle shift direction right.

Wrong answers:

a) All wrong.

Redaction: Jeffery, 2008jan01

Full Answer Problems:

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031 qfull 00510 2 3 0 moderate math: bike electric generator

21. Consider a bike electric generator that powers a headlight. The (motional) emf of the generator is given by

$$\mathcal{E} = NAB\omega_{\text{gen}} \sin(\omega_{\text{gen}} t) \ ,$$

where N is the number of turns in the generator coil, A is the coil area, B is the magnitude of the magnetic field in the generator, ω_{gen} is the angular velocity of the the generator coil, and t is time. We assume $\omega_{\text{gen}} > 0$ to avoid useless generality.

a) What is $\mathcal{E}_{\text{max}}$ (i.e., the maximum value of $\mathcal{E}$)?

b) What is ω_{gen} in terms of $\mathcal{E}_{\text{max}}$, N, A, and B? Note, ω_{gen} depends on the bike speed, and so $\mathcal{E}_{\text{max}}$ is not a fixed parameter.

c) The generator is cranked by a small wheel of radius r_{gen} that is driven by the bike wheel. The outermost radii of the two wheels directly contact each other and there is a no-slip condition between the two wheels. What is the formula for the speed of the bike wheel (i.e., v_{bike}) in terms of the emf $\mathcal{E}_{\text{max}}$?

d) What is the bike translational speed v_{trans} in terms of the emf $\mathcal{E}_{\text{max}}$ given the no-slip condition between ground and bike wheel?

e) What is the bike translational speed v_{trans} given $r_{\text{gen}} = 0.010\,\text{m}$, $\mathcal{E}_{\text{max}} = 5.0\,\text{V}$, $N = 100$, $B = 0.10\,\text{T}$, $A = 1.0 \times 10^{-3}\,\text{m}^2$? Note, the emf has been measured somehow.

SUGGESTED ANSWER:

a) Clearly,

$$\mathcal{E}_{\rm max} = NAB\omega_{\rm gen}$$

since $\sin(\omega_{\rm gen}t)$ reaches a maximum of 1 once each period. The absolute value $|\mathcal{E}|$ reaches a maximum of $\mathcal{E}_{\rm max}$ twice per cycle since $|\sin(\omega_{\rm gen}t)|$ reaches a maximum of 1 twice per period.

b) Behold:

$$\omega_{\rm gen} = \frac{\mathcal{E}_{\rm max}}{NAB} \ .$$

c) The no-slip condition implies

$$v_{\rm bike} = v_{\rm gen} = r_{\rm gen}\omega_{\rm gen} \ , \qquad \text{and thus} \qquad v_{\rm bike} = \frac{r_{\rm gen}\mathcal{E}_{\rm max}}{NAB} \ .$$

d) The no-slip condition implies that the bike wheel and ground at point of contact are at rest relative to each other. The wheel velocity at contact is at $-v_{\rm bike}$ relative to the wheel center, and thus the ground is at $-v_{\rm bike}$ relative to the wheel center. Thus, the wheel center relative to the ground is moving at velocity $v_{\rm bike}$. But the wheel center velocity is just the translational velocity of the whole bike. Thus

$$v_{\rm trans} = v_{\rm bike} = \frac{r_{\rm gen}\mathcal{E}_{\rm max}}{NAB} \ .$$

e) Behold:

$$v_{\rm trans} = \frac{r_{\rm gen}\mathcal{E}_{\rm max}}{NAB} = 5.0\,{\rm m/s} \ .$$

Fortran-95 Code

```
print*
r=.01d0
emf=5.d0
xn=100.d0
a=.001d0
b=.1d0
v=(r*emf)/(xn*a*b)
print*,'v=',v  !  4.999999999999999
```

Redaction: Jeffery, 2008jan01

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031 qfull 00590 2 3 0 mod math: energy density in a big solenoid

22. Say you had a big, ideal solenoid with an inductance of $L = 10^6\,$H and volume of $V_{\rm vol} = 125\,{\rm m}^3$. The current in the solenoid is $I = 100\,$A.

a) Calculate the magnetic field energy density without calculating the B-field? Note, the energy stored in an inductor is

$$dU_B = P\,dt = ILI'\,dt = IL\,dI \ , \qquad \text{and so} \qquad U_B = \int_0^I iL\,di = \frac{1}{2}LI^2$$

which is not a potential energy, but it is analogous to a potential energy for circuits. Note, we are using the fact that the energy stored in the inductor is magnetic field energy. **HINT:** This is easy.

b) Calculate the number of turns per unit length in symbols representing knowns (i.e., L, $V_{\rm vol}$, and I) and numerically.

c) Calculate the magnetic field in terms of symbols representing knowns and numerically.

d) Now calculate the energy density from the magnetic field itself in symbols and numerically. Does the answer agree with the part (a) answer?

SUGGESTED ANSWER:

a) Behold

$$u_{\rm B} = \frac{U_{\rm B}}{V_{\rm vol}} = \frac{(1/2)LI^2}{V_{\rm vol}} = 4.0 \times 10^7\,{\rm J/m^3}\ .$$

b) Given that the solenoid is ideal,

$$L = \mu_0 n^2 V_{\rm vol}\ ,$$

where $V_{\rm vol}$ is volume and n is the number of turns per unit length. Solving for the number of turns per unit length gives

$$n = \sqrt{\frac{L}{\mu_0 V_{\rm vol}}} = 79800 \qquad {\rm turns/m}\ .$$

c) For an ideal solenoid,

$$B = \mu_0 n I = \sqrt{\frac{\mu_0 L}{V_{\rm vol}}} I = 10.0\,{\rm T}\ .$$

This is a fairly large magnetic field for the terrestrial environment.

d) The energy density in terms of knowns is

$$u_{\rm B} = \frac{1}{2\mu_0}B^2 = \frac{1}{2\mu_0}\frac{\mu_0 L}{V_{\rm vol}} = \frac{(1/2)LI^2}{V_{\rm vol}} = 4.0 \times 10^7\,{\rm J/m^3}\ .$$

The answers agree both in symbols and numerically.

Fortran Code

```
print*
pi=acos(-1.)
perm=4.*pi*1.e-7
xL=1.e+6
vol=125.
xi=100
en=.5*xl*xi**2/vol
```

```
      turnden=sqrt(xL/(perm*vol))
      bfield=perm*turnden*xi
      print*,'en,turnden,bfield'
      print*,en,turnden,bfield
*  40000000.  79788.4531  10.0265131
```

Redaction: Jeffery, 2001jan01

---

032 qfull 00424 3 5 0 tough thinking: RL circuit discharge

Extra keywords: a sub-useful question maybe good for tests

23. You have a one-loop circuit with **ONLY** a resistor R and inductor L: there is no fixed emf source. At time zero, the current in the circuit is I_0.

a) Draw the circuit diagram and choose a fiducial current direction.

b) Write down Kirchhoff's voltage law for the circuit. The voltage drops across the resistor and inductor are both taken in the direction of current flow. If there is a actually a voltage rise, then the drop turns out to be implicitly negative.

c) The equation in the part (b) answer is actually a differential equation for current. Solve this equation for I as function of time. **HINTS:** Remember the initial current is I_0: i.e., $I(t=0)=I_0$. Re-order the equation so that both sides of the equal sign can be integrated. Let $\tau = L/R$. If your current goes to infinity, you may have made a sign error.

d) Draw a plot of I/I_0 as a function of t/τ.

e) What is the formula for the power dissipated in the resistor at any time t in terms of variables including t explicitly?

f) Integrate the expression for power dissipated in the resistor from time zero to infinity. Given that the initial energy stored in the inductor is $U_L = \frac{1}{2}LI_0^2$, has energy all the energy been dissipated.

SUGGESTED ANSWER:

a) You will have to imagine the diagram.

b) Behold:

$$LI' + IR = 0 \ .$$

The above Kirchhoff's voltage law equation was evaluated going in the designated direction of current flow which makes both the LI' and IR terms formally potential drops. One of these terms must be implicitly negative. Because of our choice of initial current going in the positive current direction, we know that LI' must be initially negative: i.e., $I' < 0$ at time zero.

c) Behold:

$$1)\quad I' = -\frac{1}{\tau}I \qquad 2)\quad \frac{di}{i} = -\frac{dt}{\tau} \qquad 3)\quad \ln(i)\Big|_0^I = -\frac{t}{\tau} \qquad 4)\quad I = I_0 e^{-t/\tau} \ ,$$

where τ is the LR time constant (or the e-folding time).

To go beyond the required answer: Note, the current falls off exponentially with time and reaches zero at time infinity. Because of finite accuracy in measuring current and current fluctuations due to noise of some kind, the the current is effectively zero in most circuits after some number of e-folding times: i.e., it is zero within uncertainty and/or fluctuation size. We note that the direction of current flow never changes sign. Thus, if $I_0 > 0$ as we have chosen, then $I > 0$ always.

d) You will have to imagine the diagram.

e) Behold:

$$P = IV_{\text{drop in resistor}} = I^2 R = I_0^2 R e^{-2t/\tau} \ .$$

f) Behold:

$$U_{\text{dis}} = \int_0^\infty P\,dt = I_0^2 R \int_0^\infty e^{-2t/\tau}\,dt = I_0^2 R \left(-\frac{\tau}{2}\right)(0-1) = I_0^2 R \left(-\frac{L/R}{2}\right)(-1) = \frac{1}{2} L I_0^2 \ .$$

The energy dissipated in the resistor is exactly equal to the initial energy stored in the inductor, and so all the initial energy has been dissipated.

Redaction: Jeffery, 2001jan01

SCRATCH PAGE

SCRATCH PAGE

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

---

1 Constants

$$
\begin{aligned}
c &= 2.997\ 924\ 58 \times 10^{8}\,\mathrm{m/s} \approx 2.998 \times 10^{8}\,\mathrm{m/s} \approx 3 \times 10^{8}\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \\
&\quad \text{exact by definition} \\
e &= 1.602\ 176\ 634 \times 10^{-19}\,\mathrm{C} \qquad \text{exact by definition} \\
G &= 6.674\ 30(15) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(circa 2025)} \\
g &= 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value} \\
k &= \frac{1}{4\pi\varepsilon_0} = 8.987\ 551\ 785\ 972(14) \times 10^{9} \approx 8.99 \times 10^{9} \approx 10^{10}\,\mathrm{N\,m^2/C^2} \\
k_{\text{Boltzmann}} &= 1.380\ 649 \times 10^{-23}\,\mathrm{J/K} = 0.8617\ 333\ 262 \times 10^{-4}\,\mathrm{eV/K} \\
&\approx 10^{-4}\,\mathrm{eV/K} \qquad \text{exact by definition in both forms} \\
m_e &= 9.109\ 383\ 7139(28) \times 10^{-31}\,\mathrm{kg} = 0.510\ 998\ 950\ 69(16)\,\mathrm{MeV} \\
m_p &= 1.672\ 621\ 923\ 95(52) \times 10^{-27}\,\mathrm{kg} = 938.272\ 089\ 32(29)\,\mathrm{MeV} \\
\varepsilon_0 &= 8.854\ 187\ 8188(14) \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11}\,\mathrm{C^2/(N\,m^2)} \qquad \text{vacuum permittivity} \\
\mu_0 &= [1.256\ 637\ 061\ 27(20)] \times 10^{-6}\,\mathrm{N/A^2} \approx 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{vacuum permeability}
\end{aligned}
$$

---

2 Geometrical Formulae

$$
C_{\text{cir}} = 2\pi r \qquad A_{\text{cir}} = \pi r^2 \qquad A_{\text{sph}} = 4\pi r^2 \qquad V_{\text{sph}} = \frac{4}{3}\pi r^3
$$

$$
\Omega_{\text{sphere}} = 4\pi \qquad d\Omega = \sin\theta\, d\theta\, d\phi
$$

---

3 Trigonometry Formulae

$$
\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1
$$

$$
\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}
$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a - b) - \cos(a + b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a - b) + \sin(a + b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

---

4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1-x} \approx 1 + x \ : \ (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

---

5 Quadratic Formula

If $\quad 0 = ax^2 + bx + c\ ,\quad$ then $\quad x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = -\dfrac{b}{2a} \pm \sqrt{\left(\dfrac{b}{2a}\right)^2 - \dfrac{c}{a}}$

Numerically robust solution (Press-178):

$$q = -\frac{1}{2}\left[b + \mathrm{sgn}(b)\sqrt{b^2-4ac}\right] \qquad x_1 = \frac{q}{a} \qquad x_2 = \frac{c}{q}$$

---

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$\theta = \begin{cases} \arctan\left(a_x, a_y\right) + n\pi = \arctan_{\text{computational}}\left(\dfrac{a_y}{a_x}\right) + \pi \mathrm{H}_{\text{Heaviside}}(-a_x)\mathrm{sgn}(a_y) + n\pi & \text{for } a_x \neq 0. \\ \dfrac{\pi}{2} + n\pi & \text{for } a_x = 0 \text{ and } a_y > 0. \\ -\dfrac{\pi}{2} + n\pi & \text{for } a_x = 0 \text{ and } a_y < 0. \end{cases}$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

---

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series
$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) \\ &= f(x_0) + (x - x_0)f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots \end{aligned}$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

---

8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$$

Five 1-dimensional equations of kinematics

Equation No.	Equation	Unwanted variable
1	$v = at + v_0$	Δx
2	$\Delta x = \frac{1}{2}at^2 + v_0 t$	v
3 (timeless eqn)	$v^2 - v_0^2 = 2a\Delta x$	t
4	$\Delta x = \frac{1}{2}(v + v_0)t$	a
5	$\Delta x = vt - \frac{1}{2}at^2$	v_1

Fiducial acceleration due to gravity (AKA little g) $\quad g = 9.8\,\mathrm{m/s^2}$

$$x_{\mathrm{rel}} = x_2 - x_1 \qquad v_{\mathrm{rel}} = v_2 - v_1 \qquad a_{\mathrm{rel}} = a_2 - a_1$$

$$x' = x - v_{\mathrm{frame}} t \qquad v' = v - v_{\mathrm{frame}} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\mathrm{avg}} = \frac{\Delta \vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\mathrm{avg}} = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0} t \qquad y = -\frac{1}{2}gt^2 + v_{y,0}t + y_0 \qquad v_{x,0} = v_0 \cos\theta \qquad v_{y,0} = v_0 \sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0 \cos\theta} \qquad y = y_0 + x\tan\theta - \frac{x^2 g}{2v_0^2 \cos^2\theta}$$

$$x_{\text{for } y \text{ max}} = \frac{v_0^2 \sin\theta\cos\theta}{g} \qquad y_{\mathrm{max}} = y_0 + \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2v_0^2 \sin\theta\cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g} \qquad \theta_{\text{for max}} = \frac{\pi}{4} \qquad x_{\mathrm{max}}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0 \sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

---

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

---

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\tan} = r\alpha$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = v\omega$$

---

13 Very Basic Newtonian Physics

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i\vec{r}_i}{m_{\text{total}}} = \frac{\sum_{\text{sub}} m_{\text{sub}}\vec{r}_{\text{cm sub}}}{m_{\text{total}}} \qquad \vec{v}_{\text{cm}} = \frac{\sum_i m_i\vec{v}_i}{m_{\text{total}}} \qquad \vec{a}_{\text{cm}} = \frac{\sum_i m_i\vec{a}_i}{m_{\text{total}}}$$

$$\vec{r}_{\text{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\text{total}}}$$

$$\vec{F}_{\text{net}} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,\text{m/s}^2$$

$$\vec{F}_{\text{normal}} = -\vec{F}_{\text{applied}} \qquad F_{\text{linear}} = -kx$$

$$F_{\text{f static}} = \min(F_{\text{applied}}, F_{\text{f static max}}) \qquad F_{\text{f static max}} = \mu_{\text{static}} F_{\text{N}} \qquad F_{\text{f kinetic}} = \mu_{\text{kinetic}} F_{\text{N}}$$

$$v_{\text{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\text{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\text{centripetal}} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\text{drag,lin}} = bv \qquad v_{\text{T}} = \frac{mg}{b} \qquad \tau = \frac{v_{\text{T}}}{g} = \frac{m}{b} \qquad v = v_{\text{T}}(1 - e^{-t/\tau})$$

$$F_{\text{drag,quad}} = bv^2 = \frac{1}{2}C\rho A v^2 \qquad v_{\text{T}} = \sqrt{\frac{mg}{b}}$$

14 Energy and Work

$$dW = \vec{F} \cdot d\vec{s} \qquad W = \int \vec{F} \cdot d\vec{s} \qquad K = \frac{1}{2}mv^2 \qquad E_{\text{mechanical}} = K + U$$

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F} \cdot \vec{v}$$

$$\Delta K = W_{\text{net}} \quad \Delta U_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \quad \Delta E = W_{\text{nonconservative}}$$

$$F = -\frac{dU}{dx} \qquad \vec{F} = -\nabla U \qquad U = \frac{1}{2}kx^2 \qquad U = mgy$$

15 Momentum

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta K_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\rm cm} = \vec{F}_{\rm net\ non\text{-}flux} + (\vec{v}_{\rm flux} - \vec{v}_{\rm cm})\frac{dm}{dt} = \vec{F}_{\rm net\ non\text{-}flux} + \vec{v}_{\rm rel}\frac{dm}{dt}$$

$$v = v_0 + v_{\rm ex}\ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

---

16 Collisions

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\,dt \qquad \vec{F}_{\rm avg} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\rm net}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\rm cm} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\rm total}}$$

$$K_{{\rm total}\ f} = K_{{\rm total}\ i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\rm rel'} = -v_{\rm rel} \qquad \text{1-d Elastic Collision Expressions}$$

---

17 Rotational Kinematics

$$2\pi = 6.2831853\ldots \qquad \frac{1}{2\pi} = 0.15915494\ldots$$

$$\frac{180^\circ}{\pi} = 57.295779\ldots\ \text{deg/rad} \approx 60\,\text{rad/deg} \qquad \frac{\pi}{180^\circ} = 0.017453292\ldots\ \text{rad/deg} \approx \frac{1}{60}\,\text{rad/deg}$$

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\omega = \alpha t + \omega_0 \qquad \Delta\theta = \frac{1}{2}\alpha t^2 + \omega_0 t \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t \qquad \Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$

18 Rotational Dynamics

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

$$L_z = RP_{xy}\sin\gamma_L \qquad \tau_z = RF_{xy}\sin\gamma_\tau \qquad L_z = I\omega \qquad \tau_{z,\text{net}} = I\alpha$$

$$I = \sum_i m_i R_i^2 \qquad I = \int R^2 \rho\, dV \qquad I_{\text{parallel axis}} = I_{\text{cm}} + mR_{\text{cm}}^2 \qquad I_z = I_x + I_y$$

$$I_{\text{cyl,shell,thin}} = MR^2 \qquad I_{\text{cyl}} = \frac{1}{2}MR^2 \qquad I_{\text{cyl,shell,thick}} = \frac{1}{2}M(R_1^2 + R_2^2)$$

$$I_{\text{rod,thin,cm}} = \frac{1}{12}ML^2 \qquad I_{\text{sph,solid}} = \frac{2}{5}MR^2 \qquad I_{\text{sph,shell,thin}} = \frac{2}{3}MR^2$$

$$a = \frac{g\sin\theta}{1 + I/(mr^2)}$$

$$K_{\text{rot}} = \frac{1}{2}I\omega^2 \qquad dW = \tau_z\, d\theta \qquad P = \frac{dW}{dt} = \tau_z\omega$$

$$\Delta K_{\text{rot}} = W_{\text{net}} = \int \tau_{z,\text{net}}\, d\theta \qquad \Delta U_{\text{rot}} = -W = -\int \tau_{z,\text{con}}\, d\theta$$

$$\Delta E_{\text{rot}} = K_{\text{rot}} + \Delta U_{\text{rot}} = W_{\text{non,rot}} \qquad \Delta E = \Delta K + K_{\text{rot}} + \Delta U = W_{\text{non}} + W_{\text{rot}}$$

19 Static Equilibrium

$$\vec{F}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = \tau'_{\text{ext,net}} \quad \text{if } F_{\text{ext,net}} = 0$$

$$0 = F_{\text{net }x} = \sum F_x \qquad 0 = F_{\text{net }y} = \sum F_y \qquad 0 = \tau_{\text{net}} = \sum \tau$$

20 **Gravity**

$$\vec{F}_{1\text{ on }2} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \qquad \vec{g} = -\frac{GM}{r^2}\hat{r} \qquad \oint \vec{g}\cdot d\vec{A} = -4\pi GM$$

$$U = -\frac{Gm_1m_2}{r_{12}} \qquad V = -\frac{GM}{r} \qquad v_{\text{escape}} = \sqrt{\frac{2GM}{r}} \qquad v_{\text{orbit}} = \sqrt{\frac{GM}{r}}$$

$$P^2 = \left(\frac{4\pi^2}{GM}\right)r^3 \qquad P = \left(\frac{2\pi}{\sqrt{GM}}\right)r^{3/2} \qquad \frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} = \text{Constant}$$

$$R_{\text{Earth,mean}} = 6371.0\,\text{km} \qquad R_{\text{Earth,equatorial}} = 6378.1370\,\text{km} \qquad M_{\text{Earth}} = 5.9736\times10^{24}\,\text{kg}$$

$$1\,\text{AU} = 1.495978707\times10^{11}\,\text{m} \qquad \text{exact by definition}$$

$$R_{\text{Earth mean orbital radius}} = 1.495978875\times10^{11}\,\text{m} = 1.0000001124\,\text{AU} \approx 1.5\times10^{11}\,\text{m} \approx 1\,\text{AU}$$

$$R_{\text{Sun,equatorial}} = 6.955\times10^{8}\ \approx 109\times R_{\text{Earth,equatorial}} \qquad M_{\text{Sun}} = 1.9891\times10^{30}\,\text{kg}$$

21 **Fluids**

$$\rho = \frac{\Delta m}{\Delta V} \qquad p = \frac{F}{A} \qquad p = p_0 + \rho g d_{\text{depth}}$$

Pascal's principle $\quad p = p_{\text{ext}} - \rho g(y - y_{\text{ext}}) \qquad \Delta p = \Delta p_{\text{ext}}$

Archimedes principle $\quad F_{\text{buoy}} = m_{\text{fluid dis}}g = V_{\text{fluid dis}}\rho_{\text{fluid}}g$

equation of continuity for ideal fluid $\quad R_V = Av = \text{Constant}$

Bernoulli's equation $\quad p + \frac{1}{2}\rho v^2 + \rho g y = \text{Constant}$

22 Oscillation

$$P = f^{-1} \qquad \omega = 2\pi f \qquad F = -kx \qquad U = \frac{1}{2}kx^2 \qquad a(t) = -\frac{k}{m}x(t) = -\omega^2 x(t)$$

$$\omega = \sqrt{\frac{k}{m}} \qquad P = 2\pi\sqrt{\frac{m}{k}} \qquad x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$E_{\text{mec total}} = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}kx_{\text{max}}^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$P = 2\pi\sqrt{\frac{I}{mgr}} \qquad P = 2\pi\sqrt{\frac{r}{g}}$$

23 Waves

$$\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2} \qquad v = \sqrt{\frac{F_{\text{T}}}{\mu}} \qquad y = f(x - vt)$$

$$y = y_{\text{max}}\sin[k(x - vt)] = y_{\text{max}}\sin(kx - \omega t) \qquad v = \frac{\omega}{k}$$

$$\text{Period} = \frac{1}{f} \qquad k = \frac{2\pi}{\lambda} \qquad v = f\lambda = \frac{\omega}{k} \qquad P \propto y_{\text{max}}^2$$

$$y = 2y_{\text{max}}\sin(kx)\cos(\omega t) \qquad n = \frac{L}{\lambda/2} \qquad L = n\frac{\lambda}{2} \qquad \lambda = \frac{2L}{n} \qquad f = n\frac{v}{2L}$$

$$v = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} \qquad n\lambda = d\sin(\theta) \qquad \left(n + \frac{1}{2}\right)\lambda = d\sin(\theta)$$

$$I = \frac{P}{4\pi r^2} \qquad \beta = (10\,\text{dB}) \times \log\left(\frac{I}{I_0}\right)$$

$$f = n\frac{v}{4L} \;:\; n = 1, 3, 5, \ldots \qquad f_{\text{medium}} = \frac{f_0}{1 - v_0/v_{\text{medium}}}$$

$$f' = f\left(1 - \frac{v'}{v}\right) \qquad f = \frac{f'}{1 - v'/v}$$

24 Thermodynamics

$$dE = dQ - dW = T\,dS - p\,dV$$

$$T_{\text{K}} = T_{\text{C}} + 273.15\,\text{K} \qquad T_{\text{F}} = 1.8 \times T_{\text{C}} + 32°\text{F}$$

$$Q = mC\Delta T \qquad Q = mL$$

$$PV = NkT \qquad P = \frac{2}{3}\frac{N}{V}K_{\text{avg}} = \frac{2}{3}\frac{N}{V}\left(\frac{1}{2}mv_{\text{RMS}}^2\right)$$

$$v_{\text{RMS}} = \sqrt{\frac{3kT}{m}} = 2735.51\ldots \times \sqrt{\frac{T/300}{A}}$$

$$PV^{\gamma} = \text{constant} \qquad 1 < \gamma \leq \frac{5}{3} \qquad v_{\text{sound}} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{-V(\partial P/\partial V)_S}{m(N/V)}} = \sqrt{\frac{\gamma kT}{m}}$$

$$\varepsilon = \frac{W}{Q_{\text{H}}} = \frac{Q_{\text{H}} - Q_{\text{C}}}{W} = 1 - \frac{Q_{\text{C}}}{Q_{\text{H}}} \qquad \eta_{\text{heating}} = \frac{Q_{\text{H}}}{W} = \frac{Q_{\text{H}}}{Q_{\text{H}} - Q_{\text{C}}} = \frac{1}{1 - Q_{\text{C}}/Q_{\text{H}}} = \frac{1}{\varepsilon}$$

$$\eta_{\text{cooling}} = \frac{Q_{\text{C}}}{W} = \frac{Q_{\text{H}} - W}{W} = \frac{1}{\varepsilon} - 1 = \eta_{\text{heating}} - 1$$

$$\varepsilon_{\text{Carnot}} = 1 - \frac{T_{\text{C}}}{T_{\text{H}}} \qquad \eta_{\text{heating,Carnot}} = \frac{1}{1 - T_{\text{C}}/T_{\text{H}}} \qquad \eta_{\text{cooling,Carnot}} = \frac{T_{\text{C}}/T_{\text{H}}}{1 - T_{\text{C}}/T_{\text{H}}}$$

25 Electrostatics

$$\vec{F}_{1 \text{ on } 2} = \frac{kq_1q_2}{r_{1,2}^2}\hat{r}_{1,2} \qquad \vec{F} = q\vec{E} \qquad \vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \qquad \vec{E}(\vec{r}\,) = \frac{kq}{r^2}\hat{r} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad k = \frac{1}{4\pi\varepsilon_0}$$

$$\vec{E}(\vec{r}\,) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|^3}(\vec{r} - \vec{r}_i) \qquad \vec{E}(\vec{r}\,) = \int_{\text{all space}} \frac{k\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}(\vec{r} - \vec{r}\,')\, d\vec{r}\,' \qquad \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

$$\vec{E}(\vec{r}\,) = \sum_i \frac{kq_i}{r^2}\left(\hat{r} - \frac{\vec{r}_i}{r} + 3\frac{r_i}{r}\cos\theta_i + \ldots\right)$$

$$q = \sum_i q_i \qquad \vec{p} = \sum_i \vec{r}_i q_i \qquad \vec{p} = 2q\vec{a} \qquad \vec{E}(\vec{r}\,) = \frac{kq}{r^2}\hat{r} + k\left[\frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3}\right]$$

$$\vec{E}(\vec{r}\,) = k\left[\frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3}\right] = \frac{kp}{r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n} \qquad \vec{\tau} = \vec{p} \times \vec{E} \qquad U = -\vec{p} \cdot \vec{E}$$

$$\Delta U = q\Delta V \qquad \Delta V = -\int_{\text{i}}^{\text{f}} \vec{E} \cdot d\vec{s} \qquad dV = -\vec{E} \cdot d\vec{s}$$

$$\vec{E} = -\nabla V \qquad \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$V = \frac{kq}{r} \qquad V(\vec{r}\,) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|} \qquad V(\vec{r}\,) = \int_{\text{all space}} \frac{k\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d\vec{r}\,' \qquad V(\vec{r}\,) = \frac{k\vec{p} \cdot \hat{r}}{r^2}$$

$$U = \frac{1}{2}\sum_{i,j,i \neq j} \frac{kq_iq_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2}\sum_i q_i V_i$$

$$U = \frac{1}{2}\int\int \frac{k\rho(\vec{r}\,)\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d\mathcal{V}\, d\mathcal{V}' = \frac{1}{2}\int \rho(\vec{r}\,)V(\vec{r}\,)\, d\mathcal{V}$$

$$C = \frac{Q}{V} \qquad C = \frac{\varepsilon_0 A}{d} \qquad C_{\text{parallel}} = \sum_i C_i \qquad \frac{1}{C_{\text{series}}} = \sum_i \frac{1}{C_i}$$

$$U_C = \frac{Q^2}{2C} = \frac{1}{2}CV^2 \qquad C_{\text{dielectric}} = \kappa C_{\text{vacuum}}$$

$$\vec{E}_{\text{macroscopic charge}} = \kappa \vec{E}_{\text{net}} \qquad \oint \kappa \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

26 Current and Circuits

$$I = \frac{dq}{dt} \qquad I = \int \vec{J} \cdot d\vec{A} \qquad J = \frac{I}{A} \qquad \vec{J} = nq\vec{v}_{\text{drift}} \qquad \vec{J} = \sigma \vec{E} \qquad V = IR$$

$$\sigma = \frac{nq^2\tau}{m} \qquad \rho = \frac{1}{\sigma} \qquad V = IR \qquad R = \rho\frac{L}{A} \qquad P = VI \qquad P = VI = I^2R = \frac{V^2}{R}$$

$$0 = \sum_i^{\text{node}} I_i \qquad 0 = \sum_i^{\text{loop}} \Delta V_i \qquad R_{\text{series}} = \sum_i R_i \qquad \frac{1}{R_{\text{parallel}}} = \sum_i \frac{1}{R_i}$$

$$\tau = RC \qquad I = \frac{\Delta V_{\text{initial}}}{R} e^{-t/\tau} \qquad \Delta V = \Delta V_{\text{final}} \left(1 - e^{-t/\tau}\right)$$

27 Magnetic Fields and Forces

$$\mu_0 = [1.256\ 637\ 061\ 27(20)] \times 10^{-6}\,\text{N/A}^2 \approx 4\pi \times 10^{-7}\,\text{N/A}^2 \qquad u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2$$

$$1\,\text{N/A}^2 = 1\,\text{T} \cdot \text{m/A} \qquad 1\,\text{tesla (T)} = 10^4\,\text{gauss (G)}$$

$$\vec{F} = q\vec{v} \times \vec{B} \qquad \vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \qquad d\vec{F} = I\,d\vec{s} \times \vec{B}$$

$$\vec{r}_{\text{cyclotron}} = \frac{mv}{|q|B} \qquad \omega_{\text{cyclotron}} = \frac{|q|B}{m} \qquad f_{\text{cyclotron}} = \frac{|q|B}{2\pi m} \qquad P = \frac{2\pi m}{|q|B}$$

$$\vec{\mu} = NIA\hat{n} \qquad \tau = \vec{\mu} \times \vec{B} \qquad U = -\vec{\mu} \cdot \vec{B}$$

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{s} \times \hat{r}}{r^2} \qquad B_{\text{wire}} = \frac{\mu_0 I}{2\pi r} \qquad B_{\text{arc}} = \frac{\mu_0 I\theta}{4\pi r} \qquad \vec{F}_{1\text{ on }2} = \frac{\mu_0 \ell I_1 I_2}{2\pi r}\hat{r}_{2\text{ to }1}$$

$$B_{\text{solenoid}} = \mu_0 n I \qquad B_{\text{toroid}} = \frac{\mu_0 N I}{2\pi r} \qquad \vec{B}_{\text{circ. loop}} = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\hat{z} \qquad \vec{B}_{\text{dipole}} = \frac{\mu_0 \vec{\mu}}{2\pi z^3}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I \qquad \Phi_B = \int_{\text{linked area}} \vec{B} \cdot d\vec{A} \qquad \oint \vec{B} \cdot d\vec{A} = 0$$

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \qquad \mathcal{E} = \oint (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{s} = -\frac{d}{dt}\int_{\text{linked area}} \vec{B} \cdot d\vec{A} = -\frac{d\Phi_B}{dt}$$

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28 LR, LC, and RLC Circuits

$$L = \frac{\Phi_B}{I} \qquad L_{\text{solenoid}} = \mu_0 n^2 V_{\text{vol}} \qquad \mathcal{E} = -L\frac{dI}{dt} \qquad U_L = \frac{1}{2}LI^2 \qquad u_B = \frac{1}{2\mu_0}B^2$$

$$L_{\text{series}} = \sum_i L_i \qquad \frac{1}{L_{\text{parallel}}} = \sum_i \frac{1}{L}$$

$$\mathcal{E}_2 = -M\frac{dI_1}{dt} \qquad M = M_{12} = M_{21} \qquad M_{\text{con-axial solenoids}} = \mu_0 n_1 n_2 V_{\text{inner}}$$

$$\tau_L = \frac{L}{R} \qquad I = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau_L}\right) = I_\infty\left(1 - e^{-t/\tau_L}\right) \qquad I = I_0 e^{-t/\tau_L}$$

$$\omega = \frac{1}{\sqrt{LC}} \qquad q = A\cos(\omega t) + B\sin(\omega t) \qquad q = q_0\cos(\omega t) \qquad U = \frac{q_0^2}{2C}$$

$$\alpha_\pm = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad R_{\text{critical}} = 2\sqrt{\frac{L}{C}}$$

$$q = \left[q_0\cos(\omega t) + \frac{q_0}{\tau\omega}\sin(\omega t)\right]e^{-t/\tau} \qquad \tau = \frac{2L}{R}$$

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29 Alternating Current, Driven RLC Circuits, and Transformers

$$V = V_0\cos(\omega t) \qquad I = I_0\cos(\omega t - \phi) \qquad Z = \frac{V_0}{I_0}$$

$$Z = \sqrt{R^2 + [\omega L - 1/(\omega C)]^2} \qquad \phi = \tan^{-1}\left[\frac{\omega L - 1/(\omega C)}{R}\right] \qquad \omega_{\text{res}} = \frac{1}{\sqrt{LC}}$$

$$P = IV \qquad V_{\text{rms}} = \frac{V_0}{\sqrt{2}} \qquad I_{\text{rms}} = \frac{I_0}{\sqrt{2}} \qquad P_{\text{avg}} = \frac{1}{2}I_0V_0\cos\phi = I_{\text{rms}}V_{\text{rms}}\cos\phi$$

$$\frac{\mathcal{E}_{\text{p}}}{N_{\text{p}}} = \frac{\mathcal{E}_{\text{s}}}{N_{\text{s}}} \qquad I_{\text{p}}N_{\text{p}} = I_{\text{s}}N_{\text{s}} \qquad I_{\text{p}}\mathcal{E}_{\text{p}} = I_{\text{s}}\mathcal{E}_{\text{s}} \qquad \frac{\mathcal{E}_{\text{p}}}{\mathcal{E}_{\text{s}}} = \frac{N_{\text{p}}}{N_{\text{s}}} = \frac{I_{\text{s}}}{I_{\text{p}}}$$

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30 Electromagnetic Radiation (EMR)

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0} \qquad \oint \vec{B}\cdot d\vec{A} = 0 \qquad \oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} \qquad \oint \vec{B}\cdot d\vec{s} = \mu_0\oint \vec{J}\cdot d\vec{A} + \varepsilon_0\mu_0\frac{d\Phi_E}{dt}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad f(x,t) = f(x - vt)$$

$$\text{Period} = \frac{1}{f} \qquad c = f\lambda = \frac{\omega}{|k|} \qquad \lambda = \frac{2\pi}{|k|} \qquad k = \frac{2\pi}{\lambda} \qquad c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$$\vec{E} = \vec{E}_0 \sin(kx - wt) \qquad \vec{B} = \vec{B}_0 \sin(kx - wt) \qquad \frac{B}{E} = \frac{1}{c}$$

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \qquad S = \frac{EB}{\mu_0} = \frac{E^2}{\mu_0 c} = \frac{cB^2}{\mu_0} \qquad I = S_{\text{avg}} = \frac{E_{\max} B_{\max}}{2\mu_0} = \frac{E_{\max}^2}{2\mu_0 c} = \frac{cB_{\max}^2}{2\mu_0}$$

$$u_{\text{avg}} = \frac{I}{c} = \frac{\varepsilon E_{\max}^2}{2} = \frac{B_{\max}^2}{2\mu_0} \qquad I_{\text{point}} = \frac{L}{4\pi r^2}$$

$$\lambda = \frac{C_{\text{wien}}}{T} = \frac{2897.7685\,\mu\text{m-K}}{T} \approx \frac{3000\,\mu\text{m-K}}{T}$$

$$I_{\text{trans. polarized}} = I_{\text{inc}} \cos^2 \theta \qquad I_{\text{trans. non-polarized}} = \frac{1}{2} I_{\text{inc}}$$