Intro Physics Semester I **Name:**

Homework 12: Equilibrium: One or two full answer questions will be marked. There will also be a mark for completeness. Homeworks are due usually the day after the chapter they are for is finished. Solutions will be posted soon thereafter. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

Answer Table

Name:

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	31.	O	O	O	O	O
2.	O	O	O	O	O	32.	O	O	O	O	O
3.	O	O	O	O	O	33.	O	O	O	O	O
4.	O	O	O	O	O	34.	O	O	O	O	O
5.	O	O	O	O	O	35.	O	O	O	O	O
6.	O	O	O	O	O	36.	O	O	O	O	O
7.	O	O	O	O	O	37.	O	O	O	O	O
8.	O	O	O	O	O	38.	O	O	O	O	O
9.	O	O	O	O	O	39.	O	O	O	O	O
10.	O	O	O	O	O	40.	O	O	O	O	O
11.	O	O	O	O	O	41.	O	O	O	O	O
12.	O	O	O	O	O	42.	O	O	O	O	O
13.	O	O	O	O	O	43.	O	O	O	O	O
14.	O	O	O	O	O	44.	O	O	O	O	O
15.	O	O	O	O	O	45.	O	O	O	O	O
16.	O	O	O	O	O	46.	O	O	O	O	O
17.	O	O	O	O	O	47.	O	O	O	O	O
18.	O	O	O	O	O	48.	O	O	O	O	O
19.	O	O	O	O	O	49.	O	O	O	O	O
20.	O	O	O	O	O	50.	O	O	O	O	O
21.	O	O	O	O	O	51.	O	O	O	O	O
22.	O	O	O	O	O	52.	O	O	O	O	O
23.	O	O	O	O	O	53.	O	O	O	O	O
24.	O	O	O	O	O	54.	O	O	O	O	O
25.	O	O	O	O	O	55.	O	O	O	O	O
26.	O	O	O	O	O	56.	O	O	O	O	O
27.	O	O	O	O	O	57.	O	O	O	O	O
28.	O	O	O	O	O	58.	O	O	O	O	O
29.	O	O	O	O	O	59.	O	O	O	O	O
30.	O	O	O	O	O	60.	O	O	O	O	O

013 qmult 00100 1 1 3 easy memory: rotational equilibrium

1. To be in rotational equilibrium relative to some origin in an inertial frame, an object must have (relative to that origin):

   a) zero angular momentum. b) non-zero angular momentum.
   c) constant angular momentum. d) non-constant angular momentum. e) no hair.

   SUGGESTED ANSWER: (c)

   Wrong answers:
   a) No. It can have non-zero angular momentum.
   b) No. It can have zero angular momentum.
   d) Exactly wrong.
   e) Black holes have no hair.

   Redaction: Jeffery, 2001jan01

013 qmult 00200 2 5 3 moderate thinking: static equilibrium

2. In **STATIC** equilibrium for a rigid body:

   a) there is no center-of-mass or rotational acceleration, but there can be **NONZERO** center-of-mass velocity and angular velocity.
   b) there is no center-of-mass or rotational acceleration, and **NO** center-of-mass or rotational velocity. If static equilibrium exists in a specific reference frame, it exists in **ALL** reference frames no matter how those reference frames may be moving.
   c) there is no center-of-mass or rotational acceleration, and **NO** center-of-mass or rotational velocity. If static equilibrium exists in a specific reference frame, it exists **ONLY** in reference frames **NOT** moving with respect to the specific reference frame.
   d) there are no forces at all.
   e) there are no torques at all.

   SUGGESTED ANSWER: (c) The students have to absorb the idea of moving frames of reference.

   Wrong answers:
   b) A table in static equilibrium on a train, is not in static equilibrium relative to the ground.
   d) There can be no net force.
   e) There can be no net torque.

   Redaction: Jeffery, 2001jan01

013 qmult 00300 1 4 1 easy deducto-memory: hanging center of mass

3. "Let's play *Jeopardy*! For \$100, the answer is: The net gravitational torque on an object about any origin at all is the same as if all the mass of the object were concentrated at the center of mass: i.e.,

$$\vec{\tau}_{\rm grav} = \vec{r}_{\rm cm} \times mg\hat{g} ,$$

   where $\vec{r}_{\rm cm}$ is measured from the origin and $\hat{g}$ is a unit vector in the direction of the gravitational force. Only when $\vec{r}_{\rm cm}$ and $\hat{g}$ are aligned or when $\vec{r}_{\rm cm} = 0$ does the gravitational torque vanish. Thus for an object hanging from a frictionless pivot, equilibrium only exists for the center of mass directly above, on, or directly below the pivot. Above is unstable because any perturbation causes a gravitational torque away from the equilibrium. On is a neutral equilibrium since there is no gravitational torque whatever the orientation of the object. Below is stable since the gravitational torque then tries to pull the center of mass back to the equilibrium point. With any damping to kill rotational kinetic energy (but insufficient static friction in the pivot point to oppose any gravitational torque), the object will come to a static stable equilibrium with the center of mass directly below the pivot."

   a) Why does the center of mass of an object tend to come to rest directly **BELOW** a freely turning pivot from which the object is hanging, Alex?
   b) Why does the center of mass of an object tend to come to rest directly **ABOVE** a freely turning pivot from which the object is hanging, Alex?
   c) Why is there a center of mass, Alex?

d) Why is there a center, Alex?
e) Why is there a universe, Alex?

SUGGESTED ANSWER: (a)

Wrong answers:
b) Above is a point of unstable equilibrium.
e) This is not the question to which the answer applies. The reason for the universe is—no, sorry, that would be telling.

Redaction: Jeffery, 2001jan01

---

013 qmult 00400 1 3 5 easy math: simple beam torque calculation

4. An object of mass 1 kg sits on a horizontal beam at 1 m from a point fulcrum. What is the torque about the fulcrum that the weight of the mass causes?

a) 1 N m. b) 2 N m. c) 3 N m. d) 4 N m. e) 9.8 N m.

SUGGESTED ANSWER: (e)

But the student does really have to know how to calculate a torque. Note that the units of torque are dimensionally the same as energy. But despite this dimensional likeness, torque and energy are different things.

Wrong answers:

Redaction: Jeffery, 2001jan01

---

013 qmult 00500 2 3 1 moderate math: torque calculation with a beam

5. Two objects are sitting on a horizontal beam. The beam rests on a point fulrum at its center of mass. The beam is free to rotate about the fulcrum. Object 1 sits on the left-hand side of the pivot at a distance ℓ_1 from the fulcrum. Object 2 sits on the right-hand side at a distance ℓ_2. Given $m_1 = Nm_2$, what is ℓ_2 in terms of ℓ_1? **HINT:** Draw a diagram.

a) $\ell_2 = N\ell_1$. b) $\ell_2 = \ell_1/N$. c) $\ell_2 = \ell_1$. d) $\ell_2 = 2\ell_1$. e) $\ell_2 = 0$.

SUGGESTED ANSWER: (a)

Sheer common sense and deductions should lead to the right answer. For equilibrium, $\tau_{\text{net}} = -m_1 g\ell_1 + m_2 g\ell_2 = 0$ taking the fulcrum as the origin. Now g cancels out. Masses determined by balancing are independent of g. Balances really measure mass, not weight. Thus, $\ell_2 = \ell_1 m_1/m_2 = \ell_1 N$.

Note that the beam gravity force and fulcrum normal force exert no torques since they both effectively act at the fulcrum point. Also note that the beam does not have to be uniform for it to have zero torque: it just have to have its center of mass at the fulcrum point. If the masses are removed the beam stays balance since the torques about the fulcrum point are still zero.

Wrong answers:
e) Not unless $N = 0$.

Redaction: Jeffery, 2001jan01

---

013 qmult 00600 1 1 3 easy memory: indeterminate equilibrium cases

6. In a planar or 2-dimensional case of static equilibrium, can you solve for four unknown normal and static friction forces assuming perfectly rigid objects?

a) No. The system is **INDETERMINATE**: you only have **FOUR** equilibrium equations.
b) Yes. The system is **DETERMINATE** since you have **FOUR** equilibrium equations.
c) No. The system is **INDETERMINATE**: you only have **THREE** equilibrium equations.
d) Yes. The system is **DETERMINATE**: you have **THREE** equilibrium equations.
e) No. The system is **INDETERMINATE**: you only have **TWO** equilibrium equations.

SUGGESTED ANSWER: (c)

Wrong answers:
e) Nah you have three: the x and y force equations and the z torque equation.

Redaction: Jeffery, 2001jan01

---

013 qfull 0110 1 3 0 easy math: origin-independent torque

7. There are some results that are useful to know in studying equilibrium for a rigid body. We will not specialize to a rigid system initially—we will say when we do so specialize. We only consider inertial frames in our derivations and discussions. Non-inertial frames can be treated, but they are trickier.

   a) The net external torque on a system about a first general origin O in an inertial frame is

$$\vec{\tau}_{\text{ext}} = \sum_i \vec{r}_i \times \vec{F}_i \ .$$

   The net external force on the system is

$$\vec{F}_{\text{ext}} = \sum_i \vec{F}_i \ .$$

   A second general origin O′ is located at $\Delta\vec{r}$ relative to the first origin. The vector $\Delta\vec{r}$ can in general depend linearly on time, and thus the second origin can define a second inertial frame. The displacements relative to the second origin are related to displacements relative to the first origin by

$$\vec{r}\,'_i = \vec{r}_i - \Delta\vec{r}$$

   —and yes, the minus sign is right: you should draw a diagram to see this. Derive the expression for the net external torque $\vec{\tau}\,'_{\text{ext}}$ about the second origin in terms of the first origin quantities. Note that forces are frame-invariant quantities in classical mechanics.

   b) If $\vec{F}_{\text{ext}} = 0$, what is the relationship of $\vec{\tau}\,'_{\text{ext}}$ and $\vec{\tau}_{\text{ext}}$. What is the relationship between the two angular momenta relative to the two origins in this case? **HINT:** You'll have to do a trivial integral to answer the second question.

   c) Now we specialize to a rigid body of constant mass m and consider conditions imposed one after another. Explain what each condition implies about the system and why it does so? First, we impose $\vec{F}_{\text{ext}} = 0$. Second, we impose that the system is examined in the rest frame of its center of mass. Third, we impose that $\tau_{\text{ext}} = 0$ using the center of mass as the origin. Fourth, we impose that system is not rotating in anyway about the center of mass.

   d) Are $\vec{F}_{\text{ext}} = 0$ and $\vec{\tau}_{\text{ext}} = 0$ sufficient or necessary conditions for overall equilibrium for a rigid body? Are they sufficient or necessary conditions for overall static equilibrium? Explain your answers.

SUGGESTED ANSWER:

   a) Behold:

$$\vec{\tau}\,'_{\text{ext}} = \sum_i \vec{r}\,'_i \times \vec{F}_i = \sum_i (\vec{r}_i - \Delta\vec{r}) \times \vec{F}_i = \sum_i \vec{r}_i \times \vec{F}_i - \Delta\vec{r} \times \vec{F}_{\text{ext}} = \vec{\tau}_{\text{ext}} - \Delta\vec{r} \times \vec{F}_{\text{ext}} \ .$$

   Thus, we find

$$\vec{\tau}\,'_{\text{ext}} = \vec{\tau}_{\text{ext}} - \Delta\vec{r} \times \vec{F}_{\text{ext}} \ .$$

   b) Clearly, if $F_{\text{ext}} = 0$,

$$\vec{\tau}\,'_{\text{ext}} = \vec{\tau}_{\text{ext}} \ .$$

   This means that

$$\frac{d\vec{L}'}{dt} = \frac{d\vec{L}}{dt}$$

   from which it follows by integration that

$$\vec{L}' = \vec{L} + \vec{C}$$

   where $\vec{C}$ is a constant vector.

It's almost obvious that $\vec{C}$ will not be zero in general. But for a proof, consider a particle of mass m moving on a straight line in the xy plane at constant velocity $\vec{v}$ in some inertial frame. Its z angular momentum for the two origins at rest in the inertial frame are

$$L_z = mrv\sin\theta = dmv \qquad L'_z = mr'v\sin\theta' = d'mv$$

where v is the particle speed, r and r' are the xy plane components of the displacements, θ and θ' are the angular coordinates of the velocities relative to the displacement vectors measured positive in the clockwise direction, and $d = r\sin\theta$ and $d' = r'\sin\theta'$ are signed moment arms (here for momenta rather than force). Since $d \neq d'$ in general, it's clear that L_z and L'_z are not equal in general.

c) Imposing $\vec{F}_{\text{ext}} = 0$, implies (according to Newton's 2nd law) that the center of mass velocity is constant (or equivalently that the momentum is constant). Imposing the rest of frame of the center of mass implies that the center of mass is at rest: it must be at rest in the frame it defines. Imposing that $\tau_{\text{ext}} = 0$ implies (according to the rotationa 2nd law) that the angular momentum of the system is constant. From the part (b) result, we know that $\tau_{\text{ext}} = 0$ for any inertial-frame origin since it is true for the center-of-mass origin. Thus, the angular momentum is constant for any inertial frame origin.

Imposing that the rigid body is not rotating about the center of mass implies that zero angular momentum about the center of mass. This would not be true for a non-rigid body where total angular momentum can be zero, but parts can have non-zero angular momentum. I do not think there is anyway to have zero angular momentum about the center of mass for a rigid body and still have some rotation. Now if there is no rotation about the center of mass, there can be rotation about any other fixed point. If a particle of the system were moving relative any fixed point, it would be moving relative to the center of mass.

Finally, we conclude that the system is in overall static equalibrium.

d) The conditions $\vec{F}_{\text{ext}} = 0$ and $\tau_{\text{ext}} = 0$ are necessary for overall equilibrium for without them one or both of momentum and angular momentum are changing and by definition of equilibrium that is not an equilibrium case. They are also both sufficient, since we define equilibrium for a rigid body by saying we have them. Note that if angular momentum is not zero, the rigid body could be rotating and not necessarily, I think, as about a single fixed axis.

The conditions are necessary for overall static equilibrium since it is an equilibrium case. They are not sufficient since one must also have zero momentum and zero angular momentum.

Redaction: Jeffery, 2008jan01

---

013 qfull 00200 1 3 0 easy math: general horizontal beam

8. The general formula for the net torque about a z axis is

$$\tau_z = \sum_i r_{xy,i} F_{xy,i} \sin\theta_i \ ,$$

where i indexes the applied forces, $r_{xy,i}$ is the xy component of radial vector from the origin to where a force is applied, $F_{xy,i}$ is the xy component of an applied force, and θ_i is the angular coordinate of $\vec{F}_{xy,i}$ relative to $r_{xy,i}$ measured positive in the counterclockwise direction usually.

a) Consider a horizontal beam with a supporting fulcrum and weights of mass m_i put along it with their center of masses at x_i. The beam itself is one of the weights. The beam system is aligned with the x direction and is symmetric about the xy plane through the origin. This means there are no x and y torques. This is a usual setup for a balance scale with the rotation axis assigned to be the z axis. Draw a diagram of the horizontal-beam system.

b) Specialize the general formula for the net torque about a z axis to the horizontal-beam system with the origin taken at the fulcrum point. The fulcrum is ideal and exerts no torque.

c) There are no forces in the x or z directions for the horizontal beam system. What is the general formula for the force in the y direction? **HINT:** Do not forget the normal force F_{N} of the fulcrum.

d) What are the conditions for static equilibrium for the horizontal beam system? We assume the beam and weights act as a rigid body.

e) Usually, we can only solve for $F_{\rm N}$ from the static equilibrium equations themselves. Thus, we can only really solve the two equations of equilibrium for one other unknown, either an x_j or an m_j. Using $\sum_{i,i\neq j}$ to mean sum over i excluding j, solve for unknowns x_j and $F_{\rm N}$ and then for unknowns m_j and $F_{\rm N}$.

SUGGESTED ANSWER:

a) You will have to imagine the diagram.

b) Behold:

$$\tau_z = -\sum_i x_i m_i g \ ,$$

where the negative sign accounts for the fact that torques on the right of the fulcrum are clockwise and those on the left counterclockwise. Since the fulcrum is at the origin, it exerts no torque.

c) Behold:

$$F_y = F_{\rm N} - \sum_i m_i g \ ,$$

where $F_{\rm N}$ is the normal force of the fulcrum which must turn out ot be a positive value in this case.

d) For equilibrium, we require

$$0 = F_{\rm N} - \sum_i m_i g \ , \qquad\qquad 0 = \sum_i x_i m_i \ .$$

To be static equilibrium, we also require that at some instant in time nothing is moving. Since the net force and net torque are zero, nothing will ever move.

e) For the unknown x_j and $F_{\rm N}$ case, the solutions by inspection are

$$x_j = -\frac{\sum_{i,i\neq j} x_i m_i}{m_j} \qquad\qquad F_{\rm N} = \sum_i m_i g \ .$$

For the unknown m_j and $F_{\rm N}$ case, the solutions by inspection are

$$m_j = -\frac{\sum_{i,i\neq j} x_i m_i}{x_j} \qquad\qquad F_{\rm N} = \sum_{i,i\neq j} m_i g - \frac{\sum_{i,i\neq j} x_i m_i g}{x_j} \ .$$

Redaction: Jeffery, 2008jan01

---

013 qfull 00210 1 3 0 easy math: horizontal balanced beam calculation

9. A horizontal beam is balanced on a point fulcrum at $x_{\rm fulcrum} = 0.20\,{\rm m}$. The beam is has mass $0.10\,{\rm kg}$ and its center of mass is at $x_{\rm beam\ cm} = 0.50\,{\rm m}$. There is also a single object of mass $2.0\,{\rm kg}$ on the beam. Where is its center of mass?

SUGGESTED ANSWER:

The equilibrium torque equation with g canceled and the origin shifted to the fulcrum is:

$$0 = \sum_i \ell_i m_i$$

where ℓ is the displacement from the fulcrum. Solving for ℓ_j gives

$$\ell_j = -\frac{\sum_i' \ell_i m_i}{m_j} \ ,$$

where the prime means we exclude j from the summation. In the present case,

$$\ell_j = -\frac{0.30\times 0.10}{0.50} = -0.015\,{\rm m} \ .$$

In the x coordinate, the object center of mass is at $0.185\,\mathrm{m}$.

Redaction: Jeffery, 2001jan01

---

013 qfull 00310 2 3 0 moderate math: equilibrium ladder

10. A ladder leans against a wall in static equilibrium. Ladder, wall, and ground are perfectly rigid. The ladder has mass m, length ℓ, and center of located at $\ell_{\rm cm}$ along its length measuring from its base. The problem is 2-dimensional: the ladder and wall are seen in the xy plane and a z axis is the only rotation axis.

   a) Draw a good diagram marking on all possible forces: gravity, ground normal force $F_{\rm N1}$ ground friction force $F_{\rm f1}$ wall normal force $F_{\rm N2}$ and. wall friction force $F_{\rm f2}$. Mark the forces where they act; in the case of gravity, the center of mass is the appropriate place. Draw the ladder leaning to the **RIGHT** so that we are all consistent. The angle between the ladder and the **VERTICAL** is θ. Make the diagram large enough to be easily read.

   b) Write out all the equations of equilibrium including all possible forces. Just so we are all on the same wavelength, take the origin for the torque equation to be the contact point between ladder and ground. Why is this a good choice? **HINT:** Using moment arms is a convenient way to determine the torques, but write them out in terms of ℓ, $\ell_{\rm cm}$, and trigonometric functions of θ. Also, in setting up the equations you must adopt some conventions about which directions are positive for which forces and what is the positive torque direction. As long as you are consistent everything works out the same physically no matter what conventions you adopt.

   c) In our idealized system, we have have no general formulae for normal forces or friction forces. We must must solve for them from the laws of motion or rotational motion. Given only m, g, θ as knowns, can we solve for all of the for the normal and wall forces? Explain your answer?

   d) Assuming the wall is frictionless, derive the formulae for $F_{\rm N1}$, $F_{\rm f1}$ and $F_{\rm N2}$. Are these general formulae for these forces? Explain your answer.

   e) Given ordinary static friction between the ladder and ground and still zero friction for the wall, what must happen as θ increases, but before it reaches 90°? Explain your answer.

   f) Again assume the wall is frictionless. Say the ladder is just on the verge of slipping at $\theta_{\rm slip}$. Derive the formula for the static friction coefficient of the ground.

   g) Assuming the floor is frictionless, derive the formulae for $F_{\rm N1}$, $F_{\rm N2}$, and $F_{\rm f2}$. What ordinary friction rule is violated in this case?

SUGGESTED ANSWER:

a) You will have to imagine the diagram.

b) The three equations of equilibrium are

$$
\begin{aligned}
0 &= F_{\rm f1} - F_{\rm N2} \ , \\
0 &= F_{\rm N1} + F_{\rm f2} - mg \ , \\
0 &= -mg\ell_{\rm cm}\sin\theta + F_{\rm f2}\ell\sin\theta + F_{\rm N2}\ell\cos\theta \ ,
\end{aligned}
$$

where we have taken the ladder base on the ground as the origin for the torque equation. The signed moment arms were easily determined geometrically. One could, of course, use the torque definition and the trigonometric identities

$$\sin(\pi-\theta)=\sin\theta \qquad \text{and} \qquad \sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta$$

to find the torques. In writing down the torque equation, we have taken counterclockwise as positive.

In principle, we could choose any point in the plane of the problem as the origin for the torque equation. For equilibrium, clearly the net torque about any origin at all must be zero: individual torques remain origin dependent, but forces friction and normal forces are, of course, origin independent. The choice of origin is a good one because the torque of the forces F_1 and $F_{\rm f1}$ are zero for this choice. Thus, the choice of origin simplifies the equations to be solved.

We cannot solve for normal and frictional forces since we only three equations of equilibrium and four unknowns. The problem is indeterminate using the idealized perfectly rigid objects that we have invoked. Nature has no problem though giving those forces definite values. This is because in reality all objects show some deformation under applied force (unless the applied force is uniform field force as gravity is for most small objects) and the deformation causes an equal and opposite restoring force. The force laws governing those deformations and restoring forces provide enough constraints that the realistic problem is always determinate. It is well beyond our scope, however, to go into elasticity theory.

c) No. There are four unknown forces and only three equations of equilibrium. We cannot solve for the unknowns without more information.

d) Given $F_{\rm f2} = 0$, we can solve the equations for the three unknown forces by inspection. We obtain

$$F_{\rm N1} = mg \ ,$$
$$F_{\rm f1} = mg\frac{\ell_{\rm cm}}{\ell}\tan\theta \ ,$$
$$F_{\rm N2} = F_{\rm f1} = mg\frac{\ell_{\rm cm}}{\ell}\tan\theta \ .$$

These are **NOT** general formulae for these forces. They are formulae for what the forces must be given our static system.

e) Given $F_{\rm f2} = 0$, we can solve the equations for the ground friction $F_{\rm f1}$ must be to maintain static equilibrium. We obtained this formula in the part (d) answer. However, static friction has an upper limit $\mu_{\rm st}|F_{\rm N}|$. If θ grows sufficiently large, then this limit will be exceeded since according to our formula for $F_{\rm f1}$ goes to infinity as $\theta \to 90^\circ$ while the $F_{\rm N1}$ stays constant. So static equilibrium will fail and the ladder will slide to the ground. I once saw this happen with a worker—who was not hurt to save suspense—on a ladder in the computer room of the astronomy department of the University of Barcelona.

f) Using the approximate law $|F_{\rm st,max}| = \mu_{\rm st}|F_{\rm N}|$ and the part (d) answer, we find

$$\mu_{\rm st} = \left|\frac{F_{\rm f1}}{F_{\rm N1}}\right| = \frac{\ell_{\rm cm}}{\ell}\tan\theta_{\rm slip} \ .$$

g) Given $F_{\rm f1} = 0$, it follows from the equations in the part (d) answer that

$$F_{\rm N1} = 0 \ ,$$
$$F_{\rm f2} = mg\frac{\ell_{\rm cm}}{\ell} \ ,$$
$$F_{\rm N2} = m_{\rm ladder}g\left(1 - \frac{\ell_{\rm cm}}{\ell}\right) \ .$$

It certainly violates our ordinary friction rules to have a wall frictional force without a wall normal force. But those rules don't actually account for all cases. Here the wall must be sticky—maybe with fresh paint.

Redaction: Jeffery, 2001jan01

---

013 qfull 00330 3 5 0 tough thinking: rolling a roller over a step

11. A roller (either a spherically symmetric ball or a cylindrically symmetric cylinder) is rolled up to a step and rests against it. The roller has radius R and mass m. The step has height y with $y \leq R$. A force $F_{\rm app}$ is applied horizontally to the roller at height R to try to push the roller over the step. A line drawn through the applied force through passes the roller's center of mass. Find the expression for $F_{\rm app}$ that just marginally lifts the roller. This is a static equilibrium situation where the normal force of the ground has just gone to zero, but the roller is still marginally touching the ground. What are the $F_{\rm app}$ values for $y = 0$ and $y = R$? **HINTS:** Draw a good diagram, use geometry, and identify the best pivot point for a torque calculation.

SUGGESTED ANSWER:

You will have to imagine the diagram. First note that since the roller has just lifted from the ground, the normal and friction forces of the ground have gone to zero. Thus, only three forces act: gravity, the applied force, and the corner force where the roller touches the corner. We must choose the origin for the torque calculation to be at the corner. The force at the corner cancels the applied force and gravity to maintain translational equilibrium, but until we find out what the applied force is we can't work out the corner force. If we don't know the corner force, we don't know its torque unless we take the origin so that its moment arm is zero and it has zero torque. After we find the applied force we could find out the corner force, but I confess to having no burning interest in it.

From the torque equilibrium equation and a bit of geometry to find the moment arms (or one could go back to the torque definition), we find

$$0 = F_{\text{app}}(R - y) - mg\sqrt{2Ry - y^2} \ .$$

Thus

$$F_{\text{app}} = mg \begin{cases} \dfrac{\sqrt{2Ry - y^2}}{R - y} & \text{in general;} \\ 0 & \text{for } y = 0; \\ \infty & \text{for } y = R. \end{cases}$$

Clearly, no applied force as described in the problem can lift the roller over a step of height equal to the roller's radius.

If $y \to 0$, then the applied force must turn off for there to be equilibrium. In this case the corner force becomes a normal force to counter gravity. The roller is just sitting in neutral equilibrium on the ground.

The formula actually works for $y > R$ too. A proof of this is really necessary though from a new diagram. For $y > R$, F_{app} becomes negative (i.e., points away from the roller). Also the corner is no longer a corner, but some sort of clamp that pulls up (to support against gravity) and away from the roller to cancel F_{app}.

Redaction: Jeffery, 2001jan01

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

---

1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67384(80) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2012, CODATA)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\mathrm{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23),\mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

---

2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\mathrm{sphere}} = 4\pi \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

---

3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

---

4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1-x} \approx 1 + x \ : \ (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

---

5 Quadratic Formula

$$\text{If} \qquad 0 = ax^2 + bx + c\,, \qquad \text{then} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) \\ &= f(x_0) + (x - x_0) f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots \end{aligned}$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$$

$$v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0 \qquad v^2 = v_0^2 + 2a(x - x_0)$$

$$x = \frac{1}{2}(v_0 + v)t + x_0 \qquad x = -\frac{1}{2}at^2 + vt + x_0 \qquad g = 9.8\,\mathrm{m/s^2}$$

$$x_{\text{rel}} = x_2 - x_1 \qquad v_{\text{rel}} = v_2 - v_1 \qquad a_{\text{rel}} = a_2 - a_1$$

$$x' = x - v_{\text{frame}} t \qquad v' = v - v_{\text{frame}} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\text{avg}} = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0} t \qquad y = -\frac{1}{2} g t^2 + v_{y,0} t + y_0 \qquad v_{x,0} = v_0 \cos\theta \qquad v_{y,0} = v_0 \sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0 \cos\theta} \qquad y = y_0 + x \tan\theta - \frac{x^2 g}{2 v_0^2 \cos^2\theta}$$

$$x_{\text{for } y \text{ max}} = \frac{v_0^2 \sin\theta \cos\theta}{g} \qquad y_{\max} = y_0 + \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2 v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g} \qquad \theta_{\text{for max}} = \frac{\pi}{4} \qquad x_{\max}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0 \sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2 r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\tan} = r\alpha$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = v\omega$$

13 Very Basic Newtonian Physics

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i \vec{r}_i}{m_{\text{total}}} = \frac{\sum_{\text{sub}} m_{\text{sub}} \vec{r}_{\text{cm sub}}}{m_{\text{total}}} \qquad \vec{v}_{\text{cm}} = \frac{\sum_i m_i \vec{v}_i}{m_{\text{total}}} \qquad \vec{a}_{\text{cm}} = \frac{\sum_i m_i \vec{a}_i}{m_{\text{total}}}$$

$$\vec{r}_{\text{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\text{total}}}$$

$$\vec{F}_{\text{net}} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,\text{m/s}^2$$

$$\vec{F}_{\text{normal}} = -\vec{F}_{\text{applied}} \qquad F_{\text{linear}} = -kx$$

$$f_{\text{normal}} = \frac{T}{r} \qquad T = T_0 - F_{\text{parallel}}(s) \qquad T = T_0$$

$$F_{\text{f static}} = \min(F_{\text{applied}}, F_{\text{f static max}}) \qquad F_{\text{f static max}} = \mu_{\text{static}} F_{\text{N}} \qquad F_{\text{f kinetic}} = \mu_{\text{kinetic}} F_{\text{N}}$$

$$v_{\text{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\text{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\text{centripetal}} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\text{drag,lin}} = bv \qquad v_{\text{T}} = \frac{mg}{b} \qquad \tau = \frac{v_{\text{T}}}{g} = \frac{m}{b} \qquad v = v_{\text{T}}(1 - e^{-t/\tau})$$

$$F_{\text{drag,quad}} = bv^2 = \frac{1}{2}C\rho A v^2 \qquad v_{\text{T}} = \sqrt{\frac{mg}{b}}$$

14 Energy and Work

$$dW = \vec{F} \cdot d\vec{s} \qquad W = \int \vec{F} \cdot d\vec{s} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\text{mechanical}} = KE + PE$$

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F} \cdot \vec{v}$$

$$\Delta KE = W_{\text{net}} \quad \Delta PE_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \quad \Delta E = W_{\text{nonconservative}}$$

$$F = -\frac{dPE}{dx} \qquad \vec{F} = -\nabla PE \qquad PE = \frac{1}{2}kx^2 \qquad PE = mgy$$

15 Momentum

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta KE_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\text{cm}} = \vec{F}_{\text{net non-flux}} + (\vec{v}_{\text{flux}} - \vec{v}_{\text{cm}})\frac{dm}{dt} = \vec{F}_{\text{net non-flux}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_0 + v_{\text{ex}} \ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

---

16 **Collisions**

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\,dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\text{net}}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$KE_{\text{total }f} = KE_{\text{total }i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2 v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\text{rel}'} = -v_{\text{rel}} \qquad \text{1-d Elastic Collision Expressions}$$

---

17 **Rotational Kinematics**

$$2\pi = 6.2831853\ldots \qquad \frac{1}{2\pi} = 0.15915494\ldots$$

$$\frac{180^\circ}{\pi} = 57.295779\ldots \approx 60^\circ \qquad \frac{\pi}{180^\circ} = 0.017453292\ldots \approx \frac{1}{60^\circ}$$

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\omega = \alpha t + \omega_0 \qquad \Delta\theta = \frac{1}{2}\alpha t^2 + \omega_0 t \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t \qquad \Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$

---

18 **Rotational Dynamics**

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

$$L_z = RP_{xy} \sin\gamma_L \qquad \tau_z = RF_{xy} \sin\gamma_\tau \qquad L_z = I\omega \qquad \tau_{z,\text{net}} = I\alpha$$

$$I = \sum_i m_i R_i^2 \qquad I = \int R^2 \rho \, dV \qquad I_{\text{parallel axis}} = I_{\text{cm}} + mR_{\text{cm}}^2 \qquad I_z = I_x + I_y$$

$$I_{\text{cyl,shell,thin}} = MR^2 \qquad I_{\text{cyl}} = \frac{1}{2}MR^2 \qquad I_{\text{cyl,shell,thick}} = \frac{1}{2}M(R_1^2 + R_2^2)$$

$$I_{\text{rod,thin,cm}} = \frac{1}{12}ML^2 \qquad I_{\text{sph,solid}} = \frac{2}{5}MR^2 \qquad I_{\text{sph,shell,thin}} = \frac{2}{3}MR^2$$

$$a = \frac{g\sin\theta}{1 + I/(mr^2)}$$

$$KE_{\text{rot}} = \frac{1}{2}I\omega^2 \qquad dW = \tau_z \, d\theta \qquad P = \frac{dW}{dt} = \tau_z\omega$$

$$\Delta KE_{\text{rot}} = W_{\text{net}} = \int \tau_{z,\text{net}} \, d\theta \qquad \Delta PE_{\text{rot}} = -W = -\int \tau_{z,\text{con}} \, d\theta$$

$$\Delta E_{\text{rot}} = KE_{\text{rot}} + \Delta PE_{\text{rot}} = W_{\text{non,rot}} \qquad \Delta E = \Delta KE + KE_{\text{rot}} + \Delta PE = W_{\text{non}} + W_{\text{rot}}$$

---

19 Static Equilibrium

$$\vec{F}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = \tau'_{\text{ext,net}} \quad \text{if } F_{\text{ext,net}} = 0$$

$$0 = F_{\text{net }x} = \sum F_x \qquad 0 = F_{\text{net }y} = \sum F_y \qquad 0 = \tau_{\text{net}} = \sum \tau$$

---

20 Gravity

$$\vec{F}_{1\text{ on }2} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \qquad \vec{g} = -\frac{GM}{r^2}\hat{r} \qquad \oint \vec{g} \cdot d\vec{A} = -4\pi GM$$

$$PE = -\frac{Gm_1m_2}{r_{12}} \qquad V = -\frac{GM}{r} \qquad v_{\text{escape}} = \sqrt{\frac{2GM}{r}} \qquad v_{\text{orbit}} = \sqrt{\frac{GM}{r}}$$

$$P^2 = \left(\frac{4\pi^2}{GM}\right)r^3 \qquad P = \left(\frac{2\pi}{\sqrt{GM}}\right)r^{3/2} \qquad \frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} = \text{Constant}$$

$$R_{\text{Earth,mean}} = 6371.0\,\text{km} \qquad R_{\text{Earth,equatorial}} = 6378.1\,\text{km} \qquad M_{\text{Earth}} = 5.9736\times 10^{24}\,\text{kg}$$

$$R_{\text{Earth mean orbital radius}} = 1.495978875\times 10^{11}\,\text{m} = 1.0000001124\,\text{AU} \approx 1.5\times 10^{11}\,\text{m} \approx 1\,\text{AU}$$

$$R_{\text{Sun,equatorial}} = 6.955\times 10^{8} \approx 109\times R_{\text{Earth,equatorial}} \qquad M_{\text{Sun}} = 1.9891\times 10^{30}\,\text{kg}$$

---

21 Fluids

$$\rho = \frac{\Delta m}{\Delta V} \qquad p = \frac{F}{A} \qquad p = p_0 + \rho g d_{\text{depth}}$$

$$\begin{aligned} \text{Pascal's principle} \quad & p = p_{\text{ext}} - \rho g(y - y_{\text{ext}}) \qquad \Delta p = \Delta p_{\text{ext}} \\ \text{Archimedes principle} \quad & F_{\text{buoy}} = m_{\text{fluid dis}} g = V_{\text{fluid dis}} \rho_{\text{fluid}} g \\ \text{equation of continuity for ideal fluid} \quad & R_V = Av = \text{Constant} \\ \text{Bernoulli's equation} \quad & p + \frac{1}{2}\rho v^2 + \rho g y = \text{Constant} \end{aligned}$$

---

22 Oscillation

$$P = f^{-1} \qquad \omega = 2\pi f \qquad F = -kx \qquad PE = \frac{1}{2}kx^2 \qquad a(t) = -\frac{k}{m}x(t) = -\omega^2 x(t)$$

$$\omega = \sqrt{\frac{k}{m}} \qquad P = 2\pi\sqrt{\frac{m}{k}} \qquad x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$E_{\text{mec total}} = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}kx_{\max}^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$P = 2\pi\sqrt{\frac{I}{mgr}} \qquad P = 2\pi\sqrt{\frac{r}{g}}$$

---

23 Waves

$$\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2} \qquad v = \sqrt{\frac{F_{\text{T}}}{\mu}} \qquad y = f(x \mp vt)$$

$$y = y_{\max}\sin[k(x \mp vt)] = y_{\max}\sin(kx \mp \omega t)$$

$$\text{Period} = \frac{1}{f} \qquad k = \frac{2\pi}{\lambda} \qquad v = f\lambda = \frac{\omega}{k} \qquad P \propto y_{\max}^2$$

$$y = 2y_{\max}\sin(kx)\cos(\omega t) \qquad n = \frac{L}{\lambda/2} \qquad L = n\frac{\lambda}{2} \qquad \lambda = \frac{2L}{n} \qquad f = n\frac{v}{2L}$$

$$v = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} \qquad n\lambda = d\sin(\theta) \qquad \left(n + \frac{1}{2}\right)\lambda = d\sin(\theta)$$

$$I = \frac{P}{4\pi r^2} \qquad \beta = (10\,\mathrm{dB}) \times \log\left(\frac{I}{I_0}\right)$$

$$f = n\frac{v}{4L}\ :\ n = 1, 3, 5, \ldots \qquad f_{\mathrm{medium}} = \frac{f_0}{1 - v_0/v_{\mathrm{medium}}}$$

$$f' = f\left(1 - \frac{v'}{v}\right) \qquad f = \frac{f'}{1 - v'/v}$$

---

24 Thermodynamics

$$dE = dQ - dW = T\,dS - p\,dV$$

$$T_{\mathrm{K}} = T_{\mathrm{C}} + 273.15\,\mathrm{K} \qquad T_{\mathrm{F}} = 1.8 \times T_{\mathrm{C}} + 32^\circ\mathrm{F}$$

$$Q = mC\Delta T \qquad Q = mL$$

$$PV = NkT \qquad P = \frac{2}{3}\frac{N}{V}KE_{\mathrm{avg}} = \frac{2}{3}\frac{N}{V}\left(\frac{1}{2}mv_{\mathrm{RMS}}^2\right)$$

$$v_{\mathrm{RMS}} = \sqrt{\frac{3kT}{m}} = 2735.51\ldots \times \sqrt{\frac{T/300}{A}}$$

$$PV^\gamma = \text{constant} \qquad 1 < \gamma \le \frac{5}{3} \qquad v_{\mathrm{sound}} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{-V(\partial P/\partial V)_S}{m(N/V)}} = \sqrt{\frac{\gamma kT}{m}}$$

$$\varepsilon = \frac{W}{Q_{\mathrm{H}}} = \frac{Q_{\mathrm{H}} - Q_{\mathrm{C}}}{W} = 1 - \frac{Q_{\mathrm{C}}}{Q_{\mathrm{H}}} \qquad \eta_{\mathrm{heating}} = \frac{Q_{\mathrm{H}}}{W} = \frac{Q_{\mathrm{H}}}{Q_{\mathrm{H}} - Q_{\mathrm{C}}} = \frac{1}{1 - Q_{\mathrm{C}}/Q_{\mathrm{H}}} = \frac{1}{\varepsilon}$$

$$\eta_{\mathrm{cooling}} = \frac{Q_{\mathrm{C}}}{W} = \frac{Q_{\mathrm{H}} - W}{W} = \frac{1}{\varepsilon} - 1 = \eta_{\mathrm{heating}} - 1$$

$$\varepsilon_{\mathrm{Carnot}} = 1 - \frac{T_{\mathrm{C}}}{T_{\mathrm{H}}} \qquad \eta_{\mathrm{heating,Carnot}} = \frac{1}{1 - T_{\mathrm{C}}/T_{\mathrm{H}}} \qquad \eta_{\mathrm{cooling,Carnot}} = \frac{T_{\mathrm{C}}/T_{\mathrm{H}}}{1 - T_{\mathrm{C}}/T_{\mathrm{H}}}$$