Intro Physics Semester I **Name:**

Homework 11: Rotational Dynamics: One or two full answer questions will be marked. There will also be a mark for completeness. Homeworks are due usually the day after the chapter they are for is finished. Solutions will be posted soon thereafter. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

Answer Table

Name:

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	31.	O	O	O	O	O
2.	O	O	O	O	O	32.	O	O	O	O	O
3.	O	O	O	O	O	33.	O	O	O	O	O
4.	O	O	O	O	O	34.	O	O	O	O	O
5.	O	O	O	O	O	35.	O	O	O	O	O
6.	O	O	O	O	O	36.	O	O	O	O	O
7.	O	O	O	O	O	37.	O	O	O	O	O
8.	O	O	O	O	O	38.	O	O	O	O	O
9.	O	O	O	O	O	39.	O	O	O	O	O
10.	O	O	O	O	O	40.	O	O	O	O	O
11.	O	O	O	O	O	41.	O	O	O	O	O
12.	O	O	O	O	O	42.	O	O	O	O	O
13.	O	O	O	O	O	43.	O	O	O	O	O
14.	O	O	O	O	O	44.	O	O	O	O	O
15.	O	O	O	O	O	45.	O	O	O	O	O
16.	O	O	O	O	O	46.	O	O	O	O	O
17.	O	O	O	O	O	47.	O	O	O	O	O
18.	O	O	O	O	O	48.	O	O	O	O	O
19.	O	O	O	O	O	49.	O	O	O	O	O
20.	O	O	O	O	O	50.	O	O	O	O	O
21.	O	O	O	O	O	51.	O	O	O	O	O
22.	O	O	O	O	O	52.	O	O	O	O	O
23.	O	O	O	O	O	53.	O	O	O	O	O
24.	O	O	O	O	O	54.	O	O	O	O	O
25.	O	O	O	O	O	55.	O	O	O	O	O
26.	O	O	O	O	O	56.	O	O	O	O	O
27.	O	O	O	O	O	57.	O	O	O	O	O
28.	O	O	O	O	O	58.	O	O	O	O	O
29.	O	O	O	O	O	59.	O	O	O	O	O
30.	O	O	O	O	O	60.	O	O	O	O	O

1. Classical rotational dynamics principles are:

   a) secondary principles derived from the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, energy, etc.).
   b) independent postulates completely unrelated to the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, the energy concept, etc.).
   c) secondary principles derived from quantum mechanics.
   d) all gross approximations derived from the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, the energy concept, etc.).
   e) independent, but very approximate, postulates completely unrelated to the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, the energy concept, etc.).

2. The mathematical operation

$$\vec{a}\times\vec{b} = ab\sin\theta\hat{n} \ ,$$

   where $\vec{a}$ and $\vec{b}$ are general vectors, θ is the angle between them (with their tails joined), and $\hat{n}$ is a unit vector normal to the plane defined by $\vec{a}$ and $\vec{b}$ and with sense determined by a right-hand rule (sweep fingers of the right hand from $\vec{a}$ to $\vec{b}$ and the thumb gives the sense), is the:

   a) outer product. b) dot product. c) cross product. d) inner product.
   e) angry product.

3. Behold:

$$\vec{a}\times\vec{b} = \begin{cases} ab\sin\theta\hat{n} & \text{in general;} \\ 0 & \text{for } \theta = 0^\circ \text{ or } 180^\circ\text{;} \\ ab\hat{n} & \text{for } \theta = 90^\circ\text{;} \\ \rule{4cm}{0.4pt} & \text{in general.} \end{cases}$$

   a) $\vec{b}\times\vec{a}$. b) $-\vec{b}\times\vec{a}$. c) $-\vec{a}\times\vec{b}$. d) $-\vec{b}\cdot\vec{a}$. e) $\vec{a}\cdot\vec{b}$.

4. The rotational 2nd law for a particle or a system of particles is:

   a) $\vec{A}\times\vec{B}$. b) $\dfrac{d\vec{\tau}_{\rm net}}{dt} = \dfrac{1}{2}I\vec{L}$. c) $\dfrac{d\vec{L}}{dt} = \dfrac{1}{2}I\tau_{\rm net}$. d) $\dfrac{d\vec{\tau}_{\rm net}}{dt} = \vec{L}$. e) $\dfrac{d\vec{L}}{dt} = \vec{\tau}_{\rm net}$.

5. Torque is:

   a) that thing (for lack of a better word) needed for an angular **VELOCITY** to exist.
   b) that thing (for lack of a better word) that can cause angular **ACCELERATION**.
   c) the same as angular acceleration.
   d) the same as angular velocity.
   e) a minor rock star of the 1960's: the fairhaired Monkee.

6. Angular momentum $\vec{L}$ and torque $\vec{\tau}$ are actually pseudovectors (AKA axial vectors), but our purposes they can be treated just like ordinary vectors—and so we will just call them vectors. In this class, we usually deal only with the z-components of angular momentum and torque which are scalars. The direction of these vectors can be determined from their definitions

$$\vec{L} = \vec{r}\times\vec{p} \qquad \text{and} \qquad \vec{\tau} = \vec{r}\times\vec{F} \ .$$

   a) Vectors add like vectors. This can be done geometrically or using components in some coordinate system.
   b) Vectors add just like scalars.
   c) Vectors cannot be added at all. The concept of adding vectors is undefined.
   d) Vectors add just like scalars **MULTIPLY**.
   e) Vectors add just like scalars **DIVIDE**.

7. You are Katherina Witt (or Dorothy Hamill, Karen Magnussen, or even Barbara Ann Scott) at the Winter Olympics. After executing a flawless quad followed by a physically impossible horizontal leap, you torque yourself (using heal? toe?) into a spin. You now decide to speed up your spin (increase your ω). Remembering that without a net external torque, angular momentum ($L = I\omega$) is conserved and assuming that the ice friction torque on your blades is negligible, you do what? **HINT:** The more spread out from the axis a fixed amount of mass is, the greater the rotational inertia.

a) **PULL** your arms in to **INCREASE** your rotational inertia.
b) **PULL** your arms in to **DECREASE** your rotational inertia.
c) **FLING** your arms out to **DECREASE** your rotational inertia.
d) **FLING** your arms out to **INCREASE** your rotational inertia.
e) Belly flop.

8. There is a net torque of 3.0 N m on an object about a particular axis of the object. The object's rotational inertia for this axis is $10^2\,\mathrm{kg\,m^2}$. What is the angular acceleration?

   a) $\alpha = 3 \times 10^2\,\mathrm{rad/s^2}$. b) $\alpha = \frac{1}{3} \times 10^2\,\mathrm{rad/s^2}$. c) $\alpha = \frac{1}{3} \times 10^{-2}\,\mathrm{rad/s^2}$.
   d) $\alpha = 3 \times 10^{-2}\,\mathrm{rad/s^2}$. e) $\alpha = 10^{-2}\,\mathrm{rad/s^2}$.

9. Archimedes (287?–212 BC) (the discoverer of the law of the lever) said something like "give me a lever long enough, a fulcrum (a support for a lever), and a place to stand, and I will move:

   a) flea". b) the Earth". c) a horse to drink (after leading it to water)".
   d) a man to drink". e) a minor rock star of the 1960's: the fairhaired Monkee."

10. "Let's play *Jeopardy*! For $100, the answer is: Its torque about any origin can be calculated as if all the object's mass were located at the center of mass."

    What is ____________, Alex?

    a) a contact force b) friction c) a tension force d) a normal force
    e) gravity near the Earth's surface

11. The rotational inertia of a uniform solid ball of mass M and radius r is:

    a) Mr^2. b) Mr. c) $\frac{2}{5}Mr$. d) $\frac{2}{5}Mr^2$. e) Mr^3.

12. For rigid body rotation about a fixed axis, the rotational analogs of inertial mass (m), momentum ($\vec{P} = m\vec{v}$), Newton's 2nd law ($\vec{F}_{\text{ext}} = m\vec{a}$), and translational (or center-of-mass) kinetic energy ($KE = (1/2)mv^2$) are, respectively:

    a) rotational inertia (I), angular momentum ($L = I\omega$), the rotational Newton's 2nd law ($\tau = I\alpha$), and rotational kinetic energy ($KE = (1/2)I\omega^2$).
    b) rotational inertia (I), the rotational Newton's 2nd law ($\tau_{\text{ext}} = I\alpha$), and angular momentum ($L = I\omega$).
    c) the rotational Newton's 2nd law ($\tau_{\text{ext}} = I\alpha$), angular momentum ($L = I\omega$), and rotational kinetic energy ($KE = (1/2)I\omega^2$).
    d) mass (m), angular velocity (ω), the static Newton's 2nd law ($0 = 0$), and rotational kinetic energy ($KE = (1/2)I\omega^2$).
    e) without clear definitions.

13. In rolling motion (e.g., a ball rolling along the ground), there is in general both rotational and:

    a) angular motion. b) centripetal motion. c) center-of-mass or translational motion.
    d) slow motion. e) blocked motion.

14. "Let's play *Jeopardy*! For $100, the answer is: The condition that is required for wheels in most ordinary circumstances: e.g., for car wheels."

    What is the ____________ condition, Alex?

    a) no-trip b) no-rip c) no-slip d) no-grip e) no-blip

15. For a roller of radius r and defining the center-of-mass or translation direction as the positive s direction and the counterclockwise as the positive rotational direction, the relationships between translational and rotational kinematic variables with the no-slip condition are

    $$\Delta\theta = -\frac{\Delta s}{r}\ , \qquad \omega = -\frac{v}{r}\ ,$$

    and:

    a) $\alpha = -\frac{a}{r}$. b) $\alpha = -ar$. c) $\alpha = -\omega r$. d) $\alpha = \frac{a}{r}$. e) $\alpha = ar$.

16. A roller (i.e., a sphere or cylinder) needs static friction to enforce the no-slip condition in general. But it doesn't need static friction to keep rolling at constant velocity as it rolls along a level surface in the limit of no rolling friction. You know this (if for no other reason) because if the roller rolls off the table, it:

    a) keeps spinning. b) stops spinning. c) goes into a parabolic trajectory.
    d) goes into a straight line trajectory. e) drops straight down.

17. In the idealized object-rolling-down-incline system, the rolling object is perfectly round (a cylinder or sphere), the object and incline are perfectly rigid, and there is a static frictional force that causes there to be no slipping. (The frictional force is a static frictional force since ideally at the point of contact there is no relative motion between object and incline surfaces.) Mechanical energy (the sum of potential energy, center-of-mass (translational) kinetic energy, rotational kinetic energy) is conserved. And yet there is a frictional force is doing work on the object. Why doesn't the frictional force cause loss of mechanical energy to heat energy (i.e., dissipation as it is called)?

    a) There is never any frictional heating with round objects.
    b) The frictional force does no net work. Going down the incline, the friction force (which points uphill) does negative work lowering the object's center-of-mass kinetic energy (which being increased by gravitational potential energy), but positive work to increase its rotational kinetic energy. The friction work contributions cancel as a calculation would show. Going up the incline, the frictional force is still uphill, but now converts rotational kinetic energy into center-of mass kinetic energy (which is disappearing into gravitational potential energy). Again the friction work contributions can cancel out. The static friction, in fact, acts as channel between center-of-mass and rotational kinetic energies. Kinetic friction dissipates energy by turning it into random microscopic kinetic and potential energy. The ideal static friction force does not do this, and so does not dissipate energy. Of course, in reality there is always some slipping and some non-elastic deformation of the bodies, and so some energy gets dissipated. This slipping and deformation is often not obvious to the eye.
    c) Actually heat energy is being turned into macroscopic kinetic energy by friction in this case without rejection of any heat energy to a lower temperature thermal reservoir. This violates the 2nd law of thermodynamics, but that doesn't matter in a mechanics problem.
    d) The frictional force **CAUSES** a torque and forces that cause torques **NEVER** cause dissipation.
    e) The frictional force **DOES NOT CAUSE** a torque and forces that **DO NOT** cause torques **NEVER** cause dissipation.

18. We define the angular momentum and torque on a particle i by, respectively, the formulae

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i \qquad \text{and} \qquad \vec{\tau}_i = \vec{r}_i \times \vec{F}_i \ ,$$

where $\vec{r}_i$ is the particle position relative to a general origin, $\vec{p}_i$ is the particle momentum, and $\vec{F}_i$ is an external force on the paricle (but not necessarily the net external force on the particle). The term particle is used in the sense of a sample of matter sufficiently small that its extent in space can be neglected. Usually by particle in classical physics, one means a sample of matter that it macroscopically tiny—and so it doesn't matter exactly where you locate it's center of mass and other other properties as long as they are within the particle's actual extent—but still larger a microscopic particle (which requires quantum mechanics). Both angular momentum and torque are pseudovectors (AKA axial vectors) since they are formed by cross products of ordinary vectors. For our purposes, we can treat them just as ordinary vectors. As vectors, they have a direction in space space, but their extent is in their own abstract space. Where are they located in real space? Well nowhere really—it adds nothing to their utility to explicitly locate them. But for mental convenience, one often locates them at the origin.

From its definition, one can see that angular momentum is a sort of mass-weighted measure of rotation about an axis passing through the origin and perpendicular to $\vec{r}_i$ and $\vec{p}_i$. The particle with angular momentum does not have to be in rotation at all though: it could be going in a straight line and still have angular momentum about any origin. From its definition, one can see that torque is a sort of measure of the rotational acceleration effect of force $\vec{F}_i$ about the axis passing through the origin and perpendicular to $\vec{r}_i$ and $\vec{F}_i$. Since the definitions of angular momentum and torque are precise, they are precise measures and instead of saying "sort of something", we just say they are precisely what they are: angular momentum and torque.

Angular momentum and torque turn out to be quantitatively useful dynamical variables for analyzing rotational motion—and that's why we have defined them. In this, question we show the essentials of angular dynamics using angular momentum and torque by deriving, among other things, the rotational 2nd law. The rotational 2nd law is just called a law by convention since it is a derived result. It is called the rotational 2nd law since in rotationaly dynamics it plays a role analogous to Newton's 2nd law—which really is a law: it is a true axiom of classical mechanics.

a) What is the total angular momentum $\vec{L}$ of a system made up of particles i?

b) The dynamics of rotational systems is studied by studying the time variation of $\vec{L}$. To this end, we need formula for $d\vec{L}/dt$. Derive this formula and eliminate any zero terms. **HINT:** The product rule does apply to the cross product and the cross product of a vector with itself is zero.

c) Newton's 2nd all in general form is

$$\frac{d\vec{p}}{dt} = \vec{F}_{\text{ext}} ,$$

where $\vec{p}$ is the momentum of a system and $\vec{F}_{\text{ext}}$ is the net external force on the system. Use this law to replace the momentum derivative in the part (b) answer formula by the net external force on particle i

$$\vec{F}'_{\text{i}} = \vec{F}_i + \sum_j \vec{F}_{ji} ,$$

where $\vec{F}_i$ net force on particle i from force sources outside the system of particles, F_{ji} is the force of particle j on particle i, and F_{ii} is zero the external force of particle i on itself logically zero.

d) Now show that

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i ,$$

making use of Newton's 3rd law and using the assumption that all inter-particle forces are central forces. A central force is one where the force of one particle/system on another is aligned with the line joining the particles/particle centers of mass. For solid objects and many others, the assumption of central forces is valid valid.

e) Now use the definition of torque to show that

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{ext}} .$$

where $\vec{\tau}_{\text{ext}}$ net torque on the system due to external forces on the system. This result is the general form of the rotational 2nd law. **HINT:** There is almost nothing to show now.

f) The general rotational 2nd law is generally useful. But we can specialize it to a form which is useful for rigid body rotation about a fixed axis which conventionally is the z-axis of 3-dimensional Cartesian coordinate. As first step to this form, prove that the z component of

$$\vec{C} = \vec{A} \times \vec{B}$$

is given by

$$C_z = (\vec{A}_{xy} \times \vec{B}_{xy}) \cdot \hat{z} ,$$

where $\vec{A}_{xy}$ is the component of $\vec{A}$ in the xy plane $\vec{B}_{xy}$ is the component of $\vec{B}$ in the xy plane, and $\hat{z}$ is the unit vector in the z direction. The last formula show that the z component of z is independent of the z components of $\vec{A}$ and $\vec{B}$ and depends only on there xy plane componets.

g) Using the part (f) result, show that

$$L_z = \sum_i (\vec{r}_{xy,i} \times \vec{p}_{xy,i}) \cdot \hat{z} = \sum_i r_{xy,i} p_{xy,i} \sin\theta_i , \qquad \tau_z = \sum_i (\vec{r}_{xy,i} \times \vec{F}_{xy,i}) \cdot \hat{z} = \sum_i r_{xy,i} p_{xy,i} \sin\theta_i .$$

where θ_i is the angular coordinate of the second vector relative to the first measured positive in the counterclockwise direction. **HINT:** The proof is by inspection.

h) For rigid body rotation about the z axis, at least in simple analysis, you do not need consider the angular momentum or torque components for the x and y axes. In general, they will not be zero, but you just do not need them to describe the motion—in a simple analysis. For rigid body rotation about the z axis, show that

$$L_z = I\omega \ ,$$

where

$$I = \sum_i m_i r_{xy,i}^2$$

is the rotational inertia (or moment of inertai) of the body about the z axis and ω is the common angular acceleration of the body. For a rigid body, I is a constant, of course.

i) Now proof the special case rotational 2nd law

$$I\alpha = \tau_z \ .$$

where α is the angular acceleration. **HINT:** The proof is by inspection.

j) Moment arm is the name for the length of a perpendicular from the z axis of rotation to a line aligned with $\vec{F}_{xy,i}$. A signed moment arm is one where multiple the moment arm by ± 1, where the upper case is for a counterclockwise torque and the lower case for a clockwise torque. Prove that

$$r_{xy,i}F_{xy,i}\sin\theta_i = d_i F_{xy,i} \ ,$$

where d_i is the signed moment arm for $\vec{F}_{xy,i}$. What if $\theta_i = 0$? What is τ_z evaluated using signed moment arms? In many cases, using moment arms or signed moment arrms simplifies the calculation of τ_z. **HINT:** A diagram might help.

19. Let us consider some intereting features of rigid body rotation.

a) Starting from the general angular momentum definition for a system of particles

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i \ .$$

prove for a rigid body rotating around the z axis that

$$\vec{L} = I\omega\hat{z} - \left(\sum_i m_i \vec{r}_{xy,i} r_{z,i}\right)\omega \ ,$$

where I is the rotational inertia, $\vec{r}_{xy,i}$ is the xy component vector of $\vec{r}_i$ (which is a time-dependent quantity is the vector is rotating about the z axis with the particle), $r_{z,i}$ is the z component of $\vec{r}_i$, and ω is the angular velocity (which is not necessarily constant). The second term in the formula for $\vec{L}$ is not necessarily zero. This shows that rigid body rotators do not necessarily have the angular momentum vector aligned with the rotation z axis. However, if you are only studying the rotation, usually the studying the z component of angular momemtum suffices.

b) Give two simple cases where the second term in the part (a) result vanishes.

20. The general rule for torque is

$$\vec{\tau} = \vec{r} \times \vec{F} \ .$$

However, there is a simple rule that provides an alternative way of calculating the magnitude of the torque. This rule is particularly useful when calculating the torque for forces illustrated on 2-dimensional diagrams. The magnitude of the torque recall is

$$\tau = rF\sin\theta \ ,$$

where θ is the angle between the radius and force vectors. Draw a general radius $\vec{r}$ from an origin to a point where a force $\vec{F}$ is applied. Then find an obviously relevant displacement on the figure that has magnitude $r\sin\theta$. This displacement is called the moment arm (HRW-230). Finding the magnitude of the moment arm on a diagram often leads to simple calculation for the torque.

21. The shield door of the neutron test facility at Lawrence Livermore National Laboratory (where they design the bombs) is (or was), according to HRW-243, the world's most massive hinged door: mass

44,000 kg, rotational inertia about its hinge axis 8.7×10^4 kg m^2, door width 2.4 meters. Assuming zero friction, what constant force perpendicular to the door at its outer edge is needed to move the door 90° in 10 s starting from **REST**? Show your calculations. Could you open this door in the time given? Would you want to open this door?

22. Wee Pippa Passing runs up to a playground merry-go-round, initially at rest, and jumps radially onto the rim.

    a) What is the torque she exerts about the rotational axis of the merry-go-round? Does the merry-go-round start to rotate? Why or why not? Does the merry-go-round move at all? Why or why not?

    b) Pippa and the merry-go-round both can have angular momentum about the merry-go-round axis. She and the merry-go-round are coupled together by the static frictional force between her feet and the surface. But Pippa can directly control the relative velocity between herself and the merry-go-round by walking or running: thus she can change the coupling condition. When she is at rest on the merry-go-round, she and the merry-go-round constitute one rigid rotator. But when she moves they constitute two rigid rotators about the merry-go-round axis.

       Say Pippa starts running just on the rim of the merry-go-round just after jumping on. The merry-go-round axis is frictionless: thus the total angular momentum of the system about the axis cannot change. Using conservation of angular momentum for an isolated system find an expression for the merry-go-round angular frequency $\omega_{\rm m}$ in terms of Pippa's relative angular frequency $\omega_{\rm p\ rel}$ and the rotational inertias about the axis of Pippa $I_{\rm p}$ and the merry-go-round $I_{\rm m}$. Note that $\omega_{\rm p\ rel} = \omega_{\rm p} - \omega_{\rm m}$. Show your derivation.

    c) Give the expression for Pippa's final angular velocity relative to the ground using the part (b) results. What would Pippa's final angular velocity be in the limits that $I_{\rm m} \to \infty$ and $I_{\rm m} \to 0$. Show your derivation.

    d) Pippa runs on the rim at 3.0 m/s relative to the rim. The radius of the merry-go-round is 3 m. The tangential rim velocity of the merry-go-round is −2.0 m/s when Pippa is running. Pippa has a mass of 40 kg. Assuming the merry-go-round is a uniform disk, what is its mass. Show your calculation.

    e) Is it at all possible with Pippa and merry-go-round starting from rest relative to the ground that both Pippa and the merry-go-round could be made to spin in the same direction relative to the ground without external torques about the merry-go-round axis? Why or why not?

23. The rotational inertia of a solid uniform sphere about an axis through its center of mass is

$$I = \frac{2}{5}mr^2 \ ,$$

    where m is its mass and r is its radius.

    a) What is the mass of sphere of uniform density ρ and radius r?

    b) What is the rotational inertia of a uniform-density spherical shell of mass m and inner radius r_1 and outer radius r_2? Write the answer in terms of m, r_1, and r_2

    c) The binomial theorem is

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^n \ .$$

       Expand $(x + \Delta x)^n$ using the binomial theorem for $n = 3$ and $n = 5$ and truncate the series to 1st order in Δx. The truncated expansions are accurate to 1st order in small Δx.

    d) Making use of the parts (b) and (c) results what is the rotational inertia of a differentially thin hollow shell of mass m and radius r? **HINT:** Let $r_1 = r$ and $r_2 = r + \Delta r$ in the part (b) result.

24. Do the following problems.

    a) The perpendicular axis theorem applies exactly to infinitely thin planar objects and approximately to just thin planar objects. You have a planar object with area density σ. You choose an origin and define a set of coordinates with the x and y axes in the object plane and the z axis perpendicular to the object plane. Show that

$$I_z = I_x + I_y \ ,$$

where I_x, I_y, and I_z are the rotational inertias about, respectively, the x, y, and z axes.

b) The rotational inertia of an infinitely thin, uniform rod about axis through its center and perpendicular to the rod is

$$\frac{1}{12}Ma^2 ,$$

where M is the rod mass and a is the rod length. Now consider an infinitely thin, uniform rectangular plate. The plate has mass M. Let the z axis pass through the center of the plate and be perpendicular to the plate. The x and y axes are parallel to the sides of the plate. The length of the plate in the y direction is a. Derive the rotational inertia about the x axis? **HINT:** A diagram might help.

c) The length of the length of the plate from part (b) in the x direction is length b. Derive the rotational inertia about the y axis?

d) What is the rotational inertia of the plate about the z axis?

e) Now let's extend the plate in the z direction to create a rectangular parallelepiped. A rectangular box has 6 sides all of which are rectangles and the sides all meet at right angles. The z length of our box is c and the mass is M: the x and y lengths are the same as for the plate. Derive the rotational inertia about the y axis?

25. A car crankshaft rotating at 1500 rpm transfers energy from the car engine to the wheel axle at 74.6 kW (or 100 hp). What's the average torque of the crankshaft? What's the average **TANGENTIAL FORCE** of the crankshaft on the drum (or whatever one calls that thing on the axle that the crankshaft turns) assuming it has radius 0.10 m? Show your calculations.

26. A uniform solid ball of mass 0.5 kg starting from rest rolls down an incline of angle $\theta = 30°$. There is no slipping between the ball and incline: i.e., the no-slip condition is imposed.

a) Calculate the ball's center-of-mass acceleration given the downhill roller acceleration formula

$$a = \frac{g\sin\theta}{1 + I/(mr^2)} .$$

Show your calculation.

b) What is its center-of-mass velocity after 30 s and how far has it gone down the slope? Show your calculation.

c) At the 30 s mark the ball completely inelastically collides with a blob of gooey stuff of mass 5.0 kg resting and stuck on a piece of wax paper of negligible mass. The ball-blob-paper then slides 20 m further down the incline and comes to a stop. What is the kinetic friction coefficient $\mu_{\rm k}$ of the paper with the incline? Show your calculation.

Why does the ball-blob-paper stay at rest after it stops? To answer that question, ask yourself this question have the forces on it changed?

d) How far would the ball-blob-paper have gone if $\mu_{\rm k} = 0.1$? Why does it go this far?

27. You have a triangular block which gives you two inclines (i.e., a double incline). Incline 1 is at θ_1 to the horizontal and incline 2 at θ_2. The kinetic friction coefficients are $\mu_{\rm k\,1}$ and $\mu_{\rm k\,2}$ for, respectively, inclines 1 and 2. You have a pulley at the apex and a ideal, massless rope connecting two blocks: block 1 (with mass m_1) on incline 1 and block 2 (with m_2 on incline 2. The rope is taut and is parallel to each incline. The rope over the pulley wheel does not slip and the pulley wheel has rotational inertia I about its axis and radius r. The axle of the pulley is frictionless. Block 1 slides up the slope and Block 2 slides down the slope. Call both these directions positive. You know all the variables except the acceleration of the blocks along the slope and the tension forces in the two regions of the rope.

a) Draw the free body diagrams for the blocks and write down their equations of motion (i.e., Newton's 2nd law applied to their special cases). Why can't you solve for acceleration now?

b) What is the rotational equation of motion for the pulley?

c) Now solve for the general acceleration formula in terms of the knowns: i.e., sans tension forces.

d) Specialize the general formula for the case where $\mu_{\rm k\,1} = \mu_{\rm k\,2} = 0$.

e) Specialize the general formula for the case where $\theta_1 = 0°$.

f) Specialize the general formula for the case where $\theta_1 = 0°$ and $\theta_2 = 90°$.

g) Specialize the general formula for the case where $\theta_1 = \theta_2 = 90°$.

h) Specialize the general formula for the case where $I = 0$.

i) Why is the general formula we derived wrong for the case when the blocks slide in the opposite directions? What simple demarche corrects the expression?

28. You have an Atwood's machine: a pulley with two dangling blocks. Block 1 has $m_1 = 0.500\,\mathrm{kg}$ and block 2 has $m_2 = 0.520\,\mathrm{kg}$. The pulley has mass $m_{\mathrm{pu}} = 2.0\,\mathrm{kg}$ and effective radius $r = 0.070\,\mathrm{m}$. The rope is massless.

    a) Assume the pulley wheel magically has zero rotational inertia and a frictionless axle. What is the acceleration of the block system? Starting from rest how long does it take for the blocks to move $1\,\mathrm{m}$?

    b) Now assume the pulley does have rotational inertia. Estimate the rotational inertia of the pulley assuming it is a uniform disk. Answer the questions from part (a) in this case.

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

---

1 Constants

$$
\begin{aligned}
c &= 2.99792458 \times 10^{8}\,\mathrm{m/s} \approx 2.998 \times 10^{8}\,\mathrm{m/s} \approx 3 \times 10^{8}\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition} \\
e &= 1.602176487(40) \times 10^{-19}\,\mathrm{C} \\
G &= 6.67428(67) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2006, CODATA)} \\
g &= 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value} \\
k &= \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^{9} \approx 8.99 \times 10^{9} \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition} \\
k_{\mathrm{Boltzmann}} &= 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K} \\
m_e &= 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV} \\
m_p &= 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV} \\
\varepsilon_0 &= \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)} \\
\mu_0 &= 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}
\end{aligned}
$$

---

2 Geometrical Formulae

$$
C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3
$$

$$
\Omega_{\mathrm{sphere}} = 4\pi \qquad d\Omega = \sin\theta\, d\theta\, d\phi
$$

---

3 Trigonometry Formulae

$$
\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1
$$

$$
\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}
$$

$$
c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}
$$

$$
f(\theta) = f(\theta + 360^\circ)
$$

$$
\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)
$$

$$
\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)
$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a - b) - \cos(a + b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a - b) + \sin(a + b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1 - x} \approx 1 + x \ : \ (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

5 Quadratic Formula

$$\text{If} \qquad 0 = ax^2 + bx + c\ , \qquad \text{then} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series
$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) \\ &= f(x_0) + (x-x_0)f^{(1)}(x_0) + \frac{(x-x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x-x_0)^3}{3!} f^{(3)}(x_0) + \ldots \end{aligned}$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0 \qquad v^2 = v_0^2 + 2a(x - x_0)$$

$$x = \frac{1}{2}(v_0 + v)t + x_0 \qquad x = -\frac{1}{2}at^2 + vt + x_0 \qquad g = 9.8\,\mathrm{m/s^2}$$

$$x_{\rm rel} = x_2 - x_1 \qquad v_{\rm rel} = v_2 - v_1 \qquad a_{\rm rel} = a_2 - a_1$$

$$x' = x - v_{\rm frame} t \qquad v' = v - v_{\rm frame} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\rm avg} = \frac{\Delta \vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\rm avg} = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0} t \qquad y = -\frac{1}{2} g t^2 + v_{y,0} t + y_0 \qquad v_{x,0} = v_0 \cos\theta \qquad v_{y,0} = v_0 \sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0 \cos\theta} \qquad y = y_0 + x \tan\theta - \frac{x^2 g}{2 v_0^2 \cos^2\theta}$$

$$x_{{\rm for}\ y\ {\rm max}} = \frac{v_0^2 \sin\theta \cos\theta}{g} \qquad y_{\rm max} = y_0 + \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2 v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g} \qquad \theta_{\rm for\ max} = \frac{\pi}{4} \qquad x_{\rm max}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0 \sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2 r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\rm tan} = r\alpha$$

$$\vec{a}_{\rm centripetal} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\rm centripetal} = \frac{v^2}{r} = r\omega^2 = v\omega$$

13 Very Basic Newtonian Physics

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i \vec{r}_i}{m_{\text{total}}} = \frac{\sum_{\text{sub}} m_{\text{sub}} \vec{r}_{\text{cm sub}}}{m_{\text{total}}} \qquad \vec{v}_{\text{cm}} = \frac{\sum_i m_i \vec{v}_i}{m_{\text{total}}} \qquad \vec{a}_{\text{cm}} = \frac{\sum_i m_i \vec{a}_i}{m_{\text{total}}}$$

$$\vec{r}_{\text{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\text{total}}}$$

$$\vec{F}_{\text{net}} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,\text{m/s}^2$$

$$\vec{F}_{\text{normal}} = -\vec{F}_{\text{applied}} \qquad F_{\text{linear}} = -kx$$

$$f_{\text{normal}} = \frac{T}{r} \qquad T = T_0 - F_{\text{parallel}}(s) \qquad T = T_0$$

$$F_{\text{f static}} = \min(F_{\text{applied}}, F_{\text{f static max}}) \qquad F_{\text{f static max}} = \mu_{\text{static}} F_{\text{N}} \qquad F_{\text{f kinetic}} = \mu_{\text{kinetic}} F_{\text{N}}$$

$$v_{\text{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\text{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\text{centripetal}} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\text{drag,lin}} = bv \qquad v_{\text{T}} = \frac{mg}{b} \qquad \tau = \frac{v_{\text{T}}}{g} = \frac{m}{b} \qquad v = v_{\text{T}}(1 - e^{-t/\tau})$$

$$F_{\text{drag,quad}} = bv^2 = \frac{1}{2}C\rho A v^2 \qquad v_{\text{T}} = \sqrt{\frac{mg}{b}}$$

14 Energy and Work

$$dW = \vec{F}\cdot d\vec{s} \qquad W = \int \vec{F}\cdot d\vec{s} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\text{mechanical}} = KE + PE$$

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F}\cdot\vec{v}$$

$$\Delta KE = W_{\text{net}} \quad \Delta PE_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \quad \Delta E = W_{\text{nonconservative}}$$

$$F = -\frac{dPE}{dx} \qquad \vec{F} = -\nabla PE \qquad PE = \frac{1}{2}kx^2 \qquad PE = mgy$$

15 Momentum

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta KE_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\text{cm}} = \vec{F}_{\text{net non-flux}} + (\vec{v}_{\text{flux}} - \vec{v}_{\text{cm}})\frac{dm}{dt} = \vec{F}_{\text{net non-flux}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_0 + v_{\text{ex}} \ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

16 Collisions

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\, dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\text{net}}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$KE_{\text{total } f} = KE_{\text{total } i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\text{rel}'} = -v_{\text{rel}} \qquad \text{1-d Elastic Collision Expressions}$$

17 Rotational Kinematics

$$2\pi = 6.2831853\ldots \qquad \frac{1}{2\pi} = 0.15915494\ldots$$

$$\frac{180^\circ}{\pi} = 57.295779\ldots \approx 60^\circ \qquad \frac{\pi}{180^\circ} = 0.017453292\ldots \approx \frac{1}{60^\circ}$$

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\omega = \alpha t + \omega_0 \qquad \Delta\theta = \frac{1}{2}\alpha t^2 + \omega_0 t \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t \qquad \Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$

18 Rotational Dynamics

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

$$L_z = RP_{xy} \sin\gamma_L \qquad \tau_z = RF_{xy} \sin\gamma_\tau \qquad L_z = I\omega \qquad \tau_{z,\text{net}} = I\alpha$$

$$I = \sum_i m_i R_i^2 \qquad I = \int R^2 \rho \, dV \qquad I_{\text{parallel axis}} = I_{\text{cm}} + mR_{\text{cm}}^2 \qquad I_z = I_x + I_y$$

$$I_{\text{cyl,shell,thin}} = MR^2 \qquad I_{\text{cyl}} = \frac{1}{2}MR^2 \qquad I_{\text{cyl,shell,thick}} = \frac{1}{2}M(R_1^2 + R_2^2)$$

$$I_{\text{rod,thin,cm}} = \frac{1}{12}ML^2 \qquad I_{\text{sph,solid}} = \frac{2}{5}MR^2 \qquad I_{\text{sph,shell,thin}} = \frac{2}{3}MR^2$$

$$a = \frac{g\sin\theta}{1 + I/(mr^2)}$$

$$KE_{\text{rot}} = \frac{1}{2}I\omega^2 \qquad dW = \tau_z \, d\theta \qquad P = \frac{dW}{dt} = \tau_z\omega$$

$$\Delta KE_{\text{rot}} = W_{\text{net}} = \int \tau_{z,\text{net}} \, d\theta \qquad \Delta PE_{\text{rot}} = -W = -\int \tau_{z,\text{con}} \, d\theta$$

$$\Delta E_{\text{rot}} = KE_{\text{rot}} + \Delta PE_{\text{rot}} = W_{\text{non,rot}} \qquad \Delta E = \Delta KE + KE_{\text{rot}} + \Delta PE = W_{\text{non}} + W_{\text{rot}}$$

---

19 Static Equilibrium

$$\vec{F}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = \tau'_{\text{ext,net}} \qquad \text{if } F_{\text{ext,net}} = 0$$

$$0 = F_{\text{net }x} = \sum F_x \qquad 0 = F_{\text{net }y} = \sum F_y \qquad 0 = \tau_{\text{net}} = \sum \tau$$