Modern Physics: Physics 305, Section 1

NAME:

Homework 8: Spin, Magnetic Dipole Moments, and the Spin-Orbit Effect: Homeworks are due as posted on the course web site. They are **NOT** handed in. The student reports that it is completed and receives one point for this. Solutions are already posted, but students are only permitted to look at the solutions after completion. The solutions are intended to be (but not necessarily are) super-perfect and go beyond a complete answer expected on a test.

1. "Let's play *Jeopardy*! For \$100, the answer is: It is the intrinsic angular momentum of a fundamental (or fundamental-for-most-purposes) particle. It is invariant and its quantum number s is always an integer or half-integer.

   What is ____________, Alex?

   a) rotation b) quantum number c) magnetic moment d) orbital angular momentum
   e) spin

2. "Let's play *Jeopardy*! For \$100, the answer is: Goudsmit and Ulhenbeck."

   a) Who are the original proposers of electron spin in 1925, Alex?
   b) Who performed the Stern-Gerlach experiment, Alex?
   c) Who are Wolfgang Pauli's evil triplet brothers, Alex?
   d) What are two delightful Dutch cheeses, Alex?
   e) What were Rosencrantz and Gildenstern's first names, Alex?

3. A spin s particle's angular momentum vector magnitude (in the vector model picture) is

   a) $\sqrt{s(s+1)}\hbar$. b) $s\hbar$ c) $\sqrt{s(s-1)}\hbar$ d) $-s\hbar$ e) $s(s+1)\hbar^2$

4. The eigenvalues of a component of the spin of a spin 1/2 particle are always:

   a) $\pm\hbar$. b) $\pm\frac{\hbar}{3}$. c) $\pm\frac{\hbar}{4}$. d) $\pm\frac{\hbar}{5}$. e) $\pm\frac{\hbar}{2}$.

5. The quantum numbers for the component of the spin of a spin s particle are always:

   a) ± 1. b) $s, s-1, s-2, \ldots, -s+1, -s$. c) $\pm\frac{1}{2}$. d) ± 2. e) $\pm\frac{1}{4}$.

6. Is the spin (not spin component) of an electron dependent on the electron's environment?

   a) Always.
   b) No. Spin is an intrinsic, unchanging property of a particle.
   c) In atomic systems, no, but when free, yes.
   d) Both yes and no.
   e) It depends on a recount in Palm Beach.

7. A spatial operator and a spin operator commute:

   never. b) sometimes. c) always. d) always and never. e) to the office.

8. What is

$$\mu_{\rm b} = \frac{e\hbar}{2m_e} = 9.27400915(26) \times 10^{-24}\,{\rm J/T} = 5.7883817555(79) \times 10^{-5}\,{\rm eV/T}\ ?$$

   a) The nuclear magneton, the characteristic magnetic moment of nuclear systems.
   b) The Bohr magneton, the characteristic magnetic moment of electronic systems.
   c) The intrinsic magnetic dipole moment of an electron.
   d) The coefficient of sliding friction.
   e) The zero-point energy of an electron.

9. The g factor in quantum mechanics is the dimensionless factor for some system that multiplied by the appropriate magneton (e.g., Bohr magneton for electron systems) times the angular momentum of

the system divided by $\hbar$ gives the magnetic moment of the system. For example for the electron, the intrinsic magnetic moment operator associated with intrinsic spin is given by

$$\vec{\mu}_{\rm op} = -g\mu_{\rm b}\frac{\vec{S}_{\rm op}}{\hbar} ,$$

where $\mu_{\rm b}$ is the Bohr magneton and $S_{\rm op}$ is the spin vector operator. What is g for the intrinsic magnetic moment operator of an electron to modern accuracy?

a) 1. b) 2. c) 2.0023193043622(15). d) 1/137. e) 137.

10. An object in a uniform magnetic field with magnetic moment due to the object's angular momentum will both classically and quantum mechanically:

    a) Lancy progress. b) Lorenzo regress. c) London recess. d) Larmor precess.
    e) Lamermoor transgress.

11. What is the main internal perturbation preventing the spinless hydrogenic eigenstates from being the actual ones?

    a) The Stark effect. b) The Zeeman effect. c) The Stern-Gerlach effect.
    d) The spin-orbit interaction. e) The Goldhaber interaction.

12. The hydrogen atom energy level energies corrected for the fine structure perturbations (i.e., the relativistic and spin-orbit perturbations) is

$$E(n, \ell, \pm 1/2, j) = -\frac{E_{\rm Ryd}}{n^2}\frac{m}{m_e}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right] ,$$

where n is the principal quantum number, ℓ is the orbital angular momentum quantum number, $\pm 1/2$ is allowed variations of j from ℓ, j (the total angular momentum quantum number) is a redundant parameter since $j = \max(\ell \pm 1/2, 1/2)$ (but it is a convenient one),

$$E_{\rm Ryd} = \frac{1}{2}m_e c^2\alpha^2$$

is the Rydberg energy, m_e is the electron mass, $\alpha \approx 1/137$ is the fine structure constant, and

$$m = \frac{m_e m_p}{m_e + m_p}$$

is the reduced mass with m_p being the proton mass. The bracketed perturbation correction term is

$$\frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)$$

which is of order $\alpha^2 \approx 10^{-4}$ times smaller than the unperturbed energy. Show that the perturbation term is always negative and reduces the energy from the unperturbed energy: i.e., show that

$$\frac{n}{j+1/2} - \frac{3}{4} > 0$$

in all cases.

Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

1 Constants

$$\begin{aligned}
c &= 2.99792458\times10^{8}\,\mathrm{m/s} \approx 2.998\times10^{8}\,\mathrm{m/s} \approx 3\times10^{8}\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \\
e &= 1.602176487(40)\times10^{-19}\,\mathrm{C} \\
E_{\mathrm{Rydberg}} &= 13.60569193(34)\,\mathrm{eV} \\
g_e &= 2.0023193043622 \qquad \text{(electron g-factor)} \\
h &= 6.62606896(33)\times10^{-34}\,\mathrm{J\,s} = 4.13566733(10)\times10^{-15}\,\mathrm{eV\,s} \\
hc &= 12398.419\,\mathrm{eV\,\mathring{A}} \approx 10^{4}\,\mathrm{eV\,\mathring{A}} \\
\hbar &= 1.054571628(53)\times10^{-34}\,\mathrm{J\,s} = 6.58211899(16)\times10^{-16}\,\mathrm{eV\,s} \\
k &= 1.3806504(24)\times10^{-23}\,\mathrm{J/K} = 0.8617343(15)\times10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,eV/K \\
m_e &= 9.10938215(45)\times10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV} \\
m_p &= 1.672621637(83)\times10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV} \\
\alpha &= e^2/(4\pi\epsilon_0\hbar c) = 7.2973525376(50)\times10^{-3} = 1/137.035999679(94) \approx 1/137 \\
\lambda_{\mathrm{C}} &= h/(m_e c) = 2.4263102175(33)\times10^{-12}\,\mathrm{m} = 0.0024263102175(33)\,\mathrm{\mathring{A}} \\
\mu_{\mathrm{B}} &= 5.7883817555(79)\times10^{-5}\,\mathrm{eV/T}
\end{aligned}$$

2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

3 Trigonometry

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right] \qquad \sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]$$

4 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{[e^{h\nu/(kT)}-1]} \qquad B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{[e^{hc/(kT\lambda)}-1]}$$

$$B_\lambda\,d\lambda = B_\nu\,d\nu \qquad \nu\lambda = c \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$E = h\nu = \frac{hc}{\lambda} \qquad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \qquad \sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670400(40)\times 10^{-8}\,\mathrm{W/m^2/K^4}$$

$$\lambda_{\mathrm{max}}T = \mathrm{constant} = \frac{hc}{kx_{\mathrm{max}}} \approx \frac{1.4387751\times 10^{-2}}{x_{\mathrm{max}}}$$

$$B_{\lambda,\mathrm{Wien}} = \frac{2hc^2}{\lambda^5}e^{-hc/(kT\lambda)} \qquad B_{\lambda,\mathrm{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu = \frac{\omega}{c} \qquad k_i = \frac{\pi}{L}n_i \quad \text{standing wave BCs} \qquad k_i = \frac{2\pi}{L}n_i \quad \text{periodic BCs}$$

$$n(k)\,dk = \frac{k^2}{\pi^2}\,dk = \pi\left(\frac{2}{c}\right)\nu^2\,d\nu = n(\nu)\,d\nu$$

$$\ln(z!) \approx \left(z+\frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots$$

$$\ln(N!) \approx N\ln(N) - N$$

$$\rho(E)\,dE = \frac{e^{-E/(kT)}}{kT}\,dE \qquad P(n) = (1-e^{-\alpha})e^{-n\alpha} \qquad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad f(x-vt) \qquad f(kx-\omega t)$$

5 Photons

$$KE = h\nu - w \qquad \Delta\lambda = \lambda_{\mathrm{scat}} - \lambda_{\mathrm{inc}} = \lambda_{\mathrm{C}}(1-\cos\theta)$$

$$\ell = \frac{1}{n\sigma} \qquad \rho = \frac{e^{-s/\ell}}{\ell} \qquad \langle s^m\rangle = \ell^m m!$$

6 Matter Waves

$$\lambda = \frac{h}{p} \qquad p = \hbar k \qquad \Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k) \Psi_k(x,t)\, dk \qquad \phi(k) = \int_{-\infty}^{\infty} \Psi(x,0) \frac{e^{-ikx}}{\sqrt{2\pi}}\, dk$$

$$v_{\mathrm{g}} = \left.\frac{d\omega}{dk}\right|_{k_0} = \frac{\hbar k_0}{m} = \frac{p_0}{m} = v_{\mathrm{clas},0}$$

7 Non-Relativistic Quantum Mechanics

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \qquad T = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \qquad H\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

$$\rho = \Psi^* \Psi \qquad \rho\, dx = \Psi^* \Psi\, dx$$

$$A\phi_i = a_i \phi_i \qquad f(x) = \sum_i c_i \phi_i \qquad \int_a^b \phi_i^* \phi_j\, dx = \delta_{ij} \qquad c_j = \int_a^b \phi_j^* f(x)\, dx$$

$$[A, B] = AB - BA$$

$$P_i = |c_i|^2 \qquad \langle A \rangle = \int_{-\infty}^{\infty} \Psi^* A \Psi\, dx = \sum_i |c_i|^2 a_i \qquad H\psi = E\psi \qquad \Psi(x,t) = \psi(x) e^{-i\omega t}$$

$$p_{\mathrm{op}}\phi = \frac{\hbar}{i}\frac{\partial \phi}{\partial x} = p\phi \qquad \phi = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad \frac{\partial^2 \psi}{\partial x^2} = \frac{2m}{\hbar^2}(V - E)\psi$$

$$|\Psi\rangle \qquad \langle\Psi| \qquad \langle x|\Psi\rangle = \Psi(x) \qquad \langle\vec{r}|\Psi\rangle = \Psi(\vec{r}) \qquad \langle k|\Psi\rangle = \Psi(k) \qquad \langle\Psi_i|\Psi_j\rangle = \langle\Psi_j|\Psi_i\rangle^*$$

$$\langle\phi_i|\Psi\rangle = c_i \qquad 1_{\mathrm{op}} = \sum_i |\phi_i\rangle\langle\phi_i| \qquad |\Psi\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\Psi\rangle = \sum_i c_i|\phi_i\rangle$$

$$1_{\mathrm{op}} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x| \qquad \langle\Psi_i|\Psi_j\rangle = \int_{-\infty}^{\infty} dx\, \langle\Psi_i|x\rangle\langle x|\Psi_j\rangle \qquad A_{ij} = \langle\phi_i|A|\phi_j\rangle$$

$$Pf(x) = f(-x) \qquad P\frac{df(x)}{dx} = \frac{df(-x)}{d(-x)} = -\frac{df(-x)}{dx} \qquad Pf_{\mathrm{e/o}}(x) = \pm f_{\mathrm{e/o}}(x)$$

$$P\frac{df_{\mathrm{e/o}}(x)}{dx} = \mp \frac{df_{\mathrm{e/o}}(x)}{dx}$$

8 Spherical Harmonics

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}} \qquad Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos(\theta) \qquad Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin(\theta) e^{\pm i\phi}$$

$$L^2 Y_{\ell m} = \ell(\ell+1)\hbar^2 Y_{\ell m} \qquad L_z Y_{\ell m} = m\hbar Y_{\ell m} \qquad |m| \leq \ell \qquad m = -\ell, -\ell+1, \ldots, \ell-1, \ell$$

0	1	2	3	4	5	6	...
s	p	d	f	g	h	i	...

9 Hydrogenic Atom

$$\psi_{n\ell m} = R_{n\ell}(r) Y_{\ell m}(\theta, \phi) \qquad \ell \leq n-1 \qquad \ell = 0, 1, 2, \ldots, n-1$$

$$a_z = \frac{a_0}{Z}\left(\frac{m_e}{m_{\text{reduced}}}\right) \qquad a_0 = \frac{\hbar}{m_e c\alpha} = \frac{\lambda_{\text{C}}}{2\pi\alpha} \qquad m_{\text{reduced}} = \frac{m_1 m_2}{m_1 + m_2}$$

$$R_{10} = 2a_Z^{-3/2} e^{-r/a_Z} \qquad R_{20} = \frac{1}{\sqrt{2}} a_Z^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a_Z}\right) e^{-r/(2a_Z)}$$

$$R_{21} = \frac{1}{\sqrt{24}} a_Z^{-3/2} \frac{r}{a_Z} e^{-r/(2a_Z)}$$

$$R_{n\ell} = -\left\{\left(\frac{2}{na_Z}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}\right\}^{1/2} e^{-\rho/2}\rho^\ell L_{n+\ell}^{2\ell+1}(\rho) \qquad \rho = \frac{2r}{nr_Z}$$

$$L_q(x) = e^x \left(\frac{d}{dx}\right)^q \left(e^{-x}x^q\right) \quad \text{Rodrigues's formula for the Laguerre polynomials}$$

$$L_q^j(x) = \left(\frac{d}{dx}\right)^j L_q(x) \qquad \text{Associated Laguerre polynomials}$$

$$\langle r\rangle_{n\ell m} = \frac{a_Z}{2}\left[3n^2 - \ell(\ell+1)\right]$$

$$\text{Nodes} = (n-1) - \ell \qquad \text{not counting zero or infinity}$$

$$E_n = -\frac{1}{2} m_e c^2 \alpha^2 \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} = -E_{\text{Ryd}} \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} \approx -13.606 \times \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} \text{ eV}$$

10 Spin, Magnetic Dipole Moment, Spin-Orbit Interaction

$$S_{\text{op}}^2 = \frac{3}{4}\hbar \begin{pmatrix} 1 & 0 \\ 01 & \end{pmatrix} \qquad s = \frac{1}{2} \qquad s(s+1) = \frac{3}{4} \qquad S = \sqrt{s(s+1)}\hbar = \frac{\sqrt{3}}{2}\hbar$$

$$S_{z,\mathrm{op}} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad m_s = \pm\frac{1}{2} \qquad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu_{\mathrm{b}} = \frac{e\hbar}{2m_e} = 9.27400915(26) \times 10^{-24}\,\mathrm{J/T} = 5.7883817555(79) \times 10^{-5}\,\mathrm{eV/T}$$

$$\mu_{\mathrm{nuclear}} = \frac{e\hbar}{2m_p} = 5.05078324(13) \times 10^{-27}\,\mathrm{J/T} = 3.1524512326(45) \times 10^{-8}\,\mathrm{eV/T}$$

$$\vec{\mu}_\ell = -g_\ell \mu_{\mathrm{b}} \frac{\vec{L}}{\hbar} \qquad \mu_\ell = g_\ell \mu_{\mathrm{b}} \ell(\ell+1) \qquad \mu_{\ell,z} = -g_\ell \mu_{\mathrm{b}} \frac{L_z}{\hbar} \qquad \mu_{\ell,z} = -g_\ell \mu_{\mathrm{b}} m_\ell$$

$$\vec{\tau} = \vec{\mu} \times \vec{B} \qquad PE = -\vec{\mu}\cdot\vec{B} \qquad \vec{F} = \Delta(\vec{\mu}\cdot\vec{B}) \qquad F_z = \sum_j \mu_j \frac{\partial B_j}{\partial z} \qquad \vec{\omega} = \frac{g_\ell \mu_{\mathrm{b}}}{\hbar}\vec{B}$$

$$\vec{J} = \vec{L} + \vec{S} \qquad J = \sqrt{j(j+1)}\hbar \qquad j = |\ell - s|, |\ell - s + 1|, \ldots, \ell + s \quad \text{triangle rule}$$

$$J_z = m_j \hbar \qquad m_j = -j, -j+1, \ldots, j-1, j$$

$$E(n, \ell, \pm 1/2, j) = -\frac{E_{\mathrm{Ryd}}}{n^2}\frac{m}{m_e}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right]$$

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11 **Special Relativity**

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \gamma(\beta << 1) = 1 + \frac{1}{2}\beta^2 \qquad \tau = ct$$

Galilean Transformations	Lorentz Transformations
$x' = x - \beta\tau$	$x' = \gamma(x - \beta\tau)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$\tau' = \tau$	$\tau' = \gamma(\tau - \beta x)$
$\beta'_{\mathrm{obj}} = \beta_{\mathrm{obj}} - \beta$	$\beta'_{\mathrm{obj}} = \dfrac{\beta_{\mathrm{obj}} - \beta}{1 - \beta\beta_{\mathrm{obj}}}$

$$\ell = \ell_{\mathrm{proper}}\sqrt{1-\beta^2} \qquad \Delta\tau_{\mathrm{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$m = \gamma m_0 \qquad p = mv = \gamma m_0 c\beta \qquad E_0 = m_0 c^2 \qquad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \qquad E = \sqrt{(pc)^2 + (m_0c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2 = (\gamma - 1)m_0c^2$$

$$f = f_{\text{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating}$$

$$f(\beta << 1) = f_{\text{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\text{trans}} = f_{\text{proper}}\sqrt{1-\beta^2} \qquad f_{\text{trans}}(\beta << 1) = f_{\text{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1}\tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1}x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\text{intersection}}}{\gamma} = x'_{x\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \tau' = \frac{\tau_{\text{intersection}}}{\gamma} = \tau'_{\tau\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

$$\theta_{\text{Mink}} = \tan^{-1}(\beta)$$