Modern Physics: Physics 305, Section 1

NAME:

Homework 6: 1-Dimensional Applications of Non-Relativistic Quantum Mechanics: Homeworks are due as posted on the course web site. They are **NOT** handed in. The student reports that it is completed and receives one point for this. Solutions are already posted, but students are only permitted to look at the solutions after completion. The solutions are intended to be (but not necessarily are) super-perfect and go beyond a complete answer expected on a test.

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	16.	O	O	O	O	O
2.	O	O	O	O	O	17.	O	O	O	O	O
3.	O	O	O	O	O	18.	O	O	O	O	O
4.	O	O	O	O	O	19.	O	O	O	O	O
5.	O	O	O	O	O	20.	O	O	O	O	O
6.	O	O	O	O	O	21.	O	O	O	O	O
7.	O	O	O	O	O	22.	O	O	O	O	O
8.	O	O	O	O	O	23.	O	O	O	O	O
9.	O	O	O	O	O	24.	O	O	O	O	O
10.	O	O	O	O	O	25.	O	O	O	O	O
11.	O	O	O	O	O	26.	O	O	O	O	O
12.	O	O	O	O	O	27.	O	O	O	O	O
13.	O	O	O	O	O	28.	O	O	O	O	O
14.	O	O	O	O	O	29.	O	O	O	O	O
15.	O	O	O	O	O	30.	O	O	O	O	O

1. In quantum mechanics, the infinite square well can be regarded as the prototype of:

   a) all bound systems. b) all unbound systems. c) both bound and unbound systems.
   d) neither bound nor unbound systems. e) Prometheus unbound.

2. In the infinite square well problem, the wave function and its first spatial derivative are:

   a) both continuous at the boundaries.
   b) continuous and discontinuous at the boundaries, respectively.
   c) both discontinuous at the boundaries.
   d) discontinuous and continuous at the boundaries, respectively.
   e) both infinite at the boundaries.

3. Meeting the boundary conditions of bound quantum mechanical systems imposes:

   a) Heisenberg's uncertainty principle. b) Schrödinger's equation. c) quantization.
   d) a vector potential. e) a time-dependent potential.

4. At energies higher than the bound stationary states there:

   a) are between one and several tens of unbound states. b) are only two unbound states.
   c) is a single unbound state. d) are no states. e) is a continuum of unbound states.

5. "Let's play *Jeopardy*! For \$100, the answer is: This effect occurs because wave functions can extend (in an exponentially decreasing way albeit) into the classically forbidden region: i.e., the region where a classical particle would have negative kinetic energy."

   What is ____________, Alex?

   a) stimulated radiative emission b) quantum mechanical tunneling c) quantization
   d) symmetrization e) normalization

6. A simple model of the outer electronic structure of a benzene molecule is a 1-dimensional infinite square well with:

   a) vanishing boundary conditions. b) periodic boundary conditions.
   c) aperiodic boundary conditions. d) no boundary conditions.
   e) incorrect boundary conditions.

7. You are given the time-independent Schrödinger equation

$$H\psi(x) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x) = E\psi(x)$$

and the infinite square well potential

$$V(x) = \begin{cases} 0\,, & x \in [0, a]; \\ \infty & \text{otherwise.} \end{cases}$$

a) What must the wave function be outside of the well (i.e., outside of the region $[0, a]$) in order to satisfy the Schrödinger equation? Why?

b) What boundary conditions must the wave function satisfy? Why must it satisfy these boundary conditions?

c) Reduce Schrödinger's equation inside the well to an equation of the same form as the **CLASSICAL** simple harmonic oscillator equation with all the constants combined into a factor of $-k^2$, where k is newly defined constant. What is k's definition?

d) Solve for the general solution for a **SINGLE** k value, but don't impose boundary conditions or normalization yet. A solution by inspection is adequate. Why can't we allow solutions with $E \leq 0$? Think carefully: it's not because k is imaginary when $E < 0$.

e) Use the boundary conditions to eliminate most of the solutions with $E > 0$ and to impose quantization on the allowed set of distinct solutions (i.e., on the allowed k values). Give the general wave function with the boundary conditions imposed and give the quantization rule for k in terms of a dimensionless quantum number n. Note that the multiplication of a wave function by an arbitrary global phase factor $e^{i\phi}$ (where ϕ is arbitrary) does not create a physically distinct wave function (i.e., does not create a new wave function as recognized by nature.) (Note the orthogonality relation used in expanding general functions in eigenfunctions also does not distinguish eigenfunctions that differ by global phase factors either: i.e., it gives the expansion coefficients only for distinct eigenfunctions. So the idea of distinct eigenfunctions arises in pure mathematics as well as in physics.)

f) Normalize the solutions.

g) Determine the general formula for the eigenenergies in terms of the quantum number n.

8. The one-dimensional infinite square well with a symmetric potential and width a is

$$V = \begin{cases} 0 & \text{for } |x| \leq a/2; \\ \infty & \text{for } |x| > a/2. \end{cases}$$

The eigenstates for infinite square well are given by

$$\psi_n(x) = \sqrt{\frac{2}{a}} \times \begin{cases} \cos(kx) & \text{for } n = 1, 3, 5\ldots; \\ \sin(kx) & \text{for } n = 2, 4, 6\ldots, \end{cases}$$

where

$$\frac{ka}{2} = \frac{n\pi}{2} \qquad \text{and} \qquad k = \frac{n\pi}{a} \ .$$

The n is the quantum number for eigenstates. The eigenstates have been normalized and are guaranteed orthogonal by the mathematics of Hermitian operators of the which the Hamiltonian is one. A quantum number is a dimensionless index (usually integer or half-integer) that specifies the eigenstates and eigenvalues somehow. The eigen-energies are given by

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 n^2 \ .$$

a) Verify the normalization of eigenstates.

b) Determine $\langle x \rangle$ for the eigenstates.

c) Determine $\langle p_{\text{op}} \rangle$ for the eigenstates. **HINT:** Recall

$$p_{\text{op}} = \frac{\hbar}{i}\frac{\partial}{\partial x} \ .$$

d) Determine $\langle p_{\text{op}}^2 \rangle$ and the momentum standard deviation σ_{p} for the eigenstates.

e) Determine $\langle x^2 \rangle$ and the position standard deviation σ_x in the large n limit. **HINT:** Assume x^2 can be approximated constant over one complete cycle of the probability density $\psi_n^* \psi_n$

f) Now for the boring part. Determine $\langle x^2 \rangle$ and the position standard deviation σ_x exactly now. **HINT:** There probably are several different ways of doing this, but there seem to be no quick tricks to the answer. The indefinite integral

$$\int x^2 \cos(bx)\, dx = \frac{x^2}{b}\sin(bx) + \frac{2}{b^2}x\cos(bx) - \frac{2}{b^3}\sin(bx)$$

might be helpful.

g) Verify that the Heisenberg uncertainty principle

$$\Delta x \Delta p = \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

is satisfied for the infinite square well case.

Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

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1 Geometrical Formulae

$$C_{\rm cir} = 2\pi r \qquad A_{\rm cir} = \pi r^2 \qquad A_{\rm sph} = 4\pi r^2 \qquad V_{\rm sph} = \frac{4}{3}\pi r^3$$

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2 Trigonometry

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right] \qquad \sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]$$

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3 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\left[e^{h\nu/(kT)}-1\right]} \qquad B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{\left[e^{hc/(kT\lambda)}-1\right]}$$

$$B_\lambda\,d\lambda = B_\nu\,d\nu \qquad \nu\lambda = c \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$k = 1.3806505(24)\times 10^{-23}\,\mathrm{J/K} \qquad c = 2.99792458\times 10^{8}\,\mathrm{m}$$

$$h = 6.6260693(11)\times 10^{-34}\,\mathrm{J\,s} = 4.13566743(35)\times 10^{-15}\,\mathrm{eV\,s}$$

$$\hbar = \frac{h}{2\pi} = 1.05457168(18)\times 10^{-34}\,\mathrm{J\,s}$$

$$hc = 12398.419\,\mathrm{eV\,Å} \approx 10^4\,\mathrm{eV\,Å} \qquad E = h\nu = \frac{hc}{\lambda} \qquad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \qquad \sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670400(40) \times 10^{-8}\,\mathrm{W/m^2/K^4}$$

$$\lambda_{\max} T = \text{constant} = \frac{hc}{kx_{\max}} \approx \frac{1.4387751 \times 10^{-2}}{x_{\max}}$$

$$B_{\lambda,\mathrm{Wien}} = \frac{2hc^2}{\lambda^5} e^{-hc/(kT\lambda)} \qquad B_{\lambda,\mathrm{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu = \frac{\omega}{c} \qquad k_i = \frac{\pi}{L} n_i \quad \text{standing wave BCs} \qquad k_i = \frac{2\pi}{L} n_i \quad \text{periodic BCs}$$

$$n(k)\,dk = \frac{k^2}{\pi^2}\,dk = \pi\left(\frac{2}{c}\right)\nu^2\,d\nu = n(\nu)\,d\nu$$

$$\ln(z!) \approx \left(z + \frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots$$

$$\ln(N!) \approx N\ln(N) - N$$

$$\rho(E)\,dE = \frac{e^{-E/(kT)}}{kT}\,dE \qquad P(n) = (1 - e^{-\alpha})e^{-n\alpha} \qquad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad f(x - vt) \qquad f(kx - \omega t)$$

4 Photons

$$KE = h\nu - w \qquad \Delta\lambda = \lambda_{\mathrm{scat}} - \lambda_{\mathrm{inc}} = \lambda_{\mathrm{C}}(1 - \cos\theta)$$

$$\lambda_{\mathrm{C}} = \frac{h}{m_e c} = 2.426310238(16) \times 10^{-12}\,\mathrm{m} \qquad e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$m_e = 9.1093826(16) \times 10^{-31}\,\mathrm{kg} = 0.510998918(44)\,\mathrm{MeV}$$

$$m_p = 1.67262171(29) \times 10^{-27}\,\mathrm{kg} = 938.272029(80)\,\mathrm{MeV}$$

$$\ell = \frac{1}{n\sigma} \qquad \rho = \frac{e^{-s/\ell}}{\ell} \qquad \langle s^m \rangle = \ell^m m!$$

5 Matter Waves

$$\lambda = \frac{h}{p} \qquad p = \hbar k \qquad \Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k)\Psi_k(x,t)\,dk \qquad \phi(k) = \int_{-\infty}^{\infty} \Psi(x,0)\frac{e^{-ikx}}{\sqrt{2\pi}}\,dk$$

$$v_{\mathrm{g}} = \left.\frac{d\omega}{dk}\right|_{k_0} = \frac{\hbar k_0}{m} = \frac{p_0}{m} = v_{\mathrm{clas},0}$$

6 Non-Relativistic Quantum Mechanics

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \qquad T = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \qquad H\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$$\rho = \Psi^*\Psi \qquad \rho\,dx = \Psi^*\Psi\,dx$$

$$A\phi_i = a_i\phi_i \qquad f(x) = \sum_i c_i\phi_i \qquad \int_a^b \phi_i^*\phi_j\,dx = \delta_{ij} \qquad c_j = \int_a^b \phi_j^* f(x)\,dx \qquad [A,B] = AB - BA$$

$$P_i = |c_i|^2 \qquad \langle A\rangle = \int_{-\infty}^{\infty} \Psi^* A\Psi\,dx = \sum_i |c_i|^2 a_i \qquad H\psi = E\psi \qquad \Psi(x,t) = \psi(x)e^{-i\omega t}$$

$$p_{\mathrm{op}}\phi = \frac{\hbar}{i}\frac{\partial\phi}{\partial x} = p\phi \qquad \phi = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad \frac{\partial^2\psi}{\partial x^2} = \frac{2m}{\hbar^2}(V-E)\psi$$

$$|\Psi\rangle \qquad \langle\Psi| \qquad \langle x|\Psi\rangle = \Psi(x) \qquad \langle\vec{r}|\Psi\rangle = \Psi(\vec{r}) \qquad \langle k|\Psi\rangle = \Psi(k) \qquad \langle\Psi_i|\Psi_j\rangle = \langle\Psi_j|\Psi_i\rangle^*$$

$$\langle\phi_i|\Psi\rangle = c_i \qquad 1_{\mathrm{op}} = \sum_i |\phi_i\rangle\langle\phi_i| \qquad |\Psi\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\Psi\rangle = \sum_i c_i|\phi_i\rangle$$

$$1_{\mathrm{op}} = \int_{-\infty}^{\infty} dx\,|x\rangle\langle x| \qquad \langle\Psi_i|\Psi_j\rangle = \int_{-\infty}^{\infty} dx\,\langle\Psi|x\rangle\langle x|\Psi\rangle \qquad A_{ij} = \langle\phi_i|A|\phi_j\rangle$$

$$Pf(x) = f(-x) \qquad P\frac{df(x)}{dx} = \frac{df(-x)}{d(-x)} = -\frac{df(-x)}{dx} \qquad Pf_{\mathrm{e/o}}(x) = \pm f_{\mathrm{e/o}}(x) \qquad P\frac{df_{\mathrm{e/o}}(x)}{dx} = \mp\frac{df_{\mathrm{e/o}}(x)}{dx}$$

7 Special Relativity

$$c = 2.99792458\times10^8\,\mathrm{m/s} \approx 2.998\times10^8\,\mathrm{m/s} \approx 3\times10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \gamma(\beta << 1) = 1 + \frac{1}{2}\beta^2 \qquad \tau = ct$$

Galilean Transformations	Lorentz Transformations
$x' = x - \beta\tau$	$x' = \gamma(x - \beta\tau)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$\tau' = \tau$	$\tau' = \gamma(\tau - \beta x)$
$\beta'_{\text{obj}} = \beta_{\text{obj}} - \beta$	$\beta'_{\text{obj}} = \dfrac{\beta_{\text{obj}} - \beta}{1 - \beta\beta_{\text{obj}}}$

$$\ell = \ell_{\text{proper}}\sqrt{1-\beta^2} \qquad \Delta\tau_{\text{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$m = \gamma m_0 \qquad p = mv = \gamma m_0 c\beta \qquad E_0 = m_0c^2 \qquad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \qquad E = \sqrt{(pc)^2 + (m_0c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2 = (\gamma - 1)m_0c^2$$

$$f = f_{\text{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating}$$

$$f(\beta << 1) = f_{\text{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\text{trans}} = f_{\text{proper}}\sqrt{1-\beta^2} \qquad f_{\text{trans}}(\beta << 1) = f_{\text{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1}\tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1}x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\text{intersection}}}{\gamma} = x'_{x\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \tau' = \frac{\tau_{\text{intersection}}}{\gamma} = \tau'_{\tau\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

$$\theta_{\text{Mink}} = \tan^{-1}(\beta)$$