Modern Physics: Physics 305, Section 1

NAME:

Homework 5: Non-Relativistic Quantum Mechanics Homeworks are due as posted on the course web site. They are **NOT** handed in. The student reports that it is completed and receives one point for this. Solutions are already posted, but students are only permitted to look at the solutions after completion. The solutions are intended to be (but not necessarily are) super-perfect and go beyond a complete answer expected on a test.

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	16.	O	O	O	O	O
2.	O	O	O	O	O	17.	O	O	O	O	O
3.	O	O	O	O	O	18.	O	O	O	O	O
4.	O	O	O	O	O	19.	O	O	O	O	O
5.	O	O	O	O	O	20.	O	O	O	O	O
6.	O	O	O	O	O	21.	O	O	O	O	O
7.	O	O	O	O	O	22.	O	O	O	O	O
8.	O	O	O	O	O	23.	O	O	O	O	O
9.	O	O	O	O	O	24.	O	O	O	O	O
10.	O	O	O	O	O	25.	O	O	O	O	O
11.	O	O	O	O	O	26.	O	O	O	O	O
12.	O	O	O	O	O	27.	O	O	O	O	O
13.	O	O	O	O	O	28.	O	O	O	O	O
14.	O	O	O	O	O	29.	O	O	O	O	O
15.	O	O	O	O	O	30.	O	O	O	O	O

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002 qmult 00080 1 1 2 easy memory: wave-particle duality

1. The principle that all microscopic physical entities have both wave and particle properties is called the wave-particle:

   a) singularity. b) duality. c) triality. d) infinality. e) nullility.

   SUGGESTED ANSWER: (b)

   Wrong answers:
   e) Oh, c'mon.

   Redaction: Jeffery, 2001jan01

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002 qmult 00090 1 4 5 easy deducto-memory: Sch eqn

2. "Let's play *Jeopardy*! For \$100, the answer is: The equation that governs (or equations that govern) the time evolution of quantum mechanical systems in the non-relativistic approximation."

   What is/are ____________, Alex?

   a) $\vec{F}_{\text{net}} = m\vec{a}$ b) Maxwell's equations c) Einstein's field equations of general relativity
   d) Dirac's equation e) Schrödinger's equation

   SUGGESTED ANSWER: (e)

   Wrong answers:
   d) The Dirac equation for electrons includes relativistic effects.

   Redaction: Jeffery, 2001jan01

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002 qmult 00100 1 1 1 easy memory: Sch eqn compact form

3. The full Schrödinger's equation in compact form is:

   a) $H\Psi = i\hbar\dfrac{\partial\Psi}{\partial t}$. b) $H\Psi = \hbar\dfrac{\partial\Psi}{\partial t}$. c) $H\Psi = i\dfrac{\partial\Psi}{\partial t}$. d) $H\Psi = i\hbar\dfrac{\partial\Psi}{\partial x}$.
   e) $H^{-1}\Psi = i\hbar\dfrac{\partial\Psi}{\partial t}$.

   SUGGESTED ANSWER: (a)

   Wrong Answers:
   b) The i is missing.
   c) The $\hbar$ is missing.

   Redaction: Jeffery, 2001jan01

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002 qmult 00210 1 1 1 easy memory: QM probability density

4. In the probabistic interpretation of wave function Ψ, the quantity $|\Psi|^2$ is:

   a) a probability density. b) a probability amplitude. c) 1. d) 0.
   e) a negative probability.

   SUGGESTED ANSWER: (a)

   Wrong answers:
   b) The probability amplitude is Ψ itself.
   e) A nonsense answer.

   Redaction: Jeffery, 2001jan01

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002 qmult 00220 1 1 5 easy memory: probability of finding particle in dx

5. The probability of finding a particle in differential region dx is:

   a) $\Psi(x,t)\,dx$. b) $\Psi(x,t)^*\,dx$. c) $[\Psi(x,t)^*/\Psi(x,t)]\,dx$. d) $\Psi(x,t)^2\,dx$.
   e) $\Psi(x,t)^*\Psi(x,t)\,dx = |\Psi(x,t)|^2\,dx$.

   SUGGESTED ANSWER: (e)

   Wrong Answers:

a) I'm always making this mistake when the wave function is pure real.

Redaction: Jeffery, 2001jan01

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002 qmult 00300 1 4 5 easy deducto-memory: observable defined

Extra keywords: See CT-137, Gr-104

6. "Let's play *Jeopardy*! For \$100, the answer is: It is an Hermitian operator that governs an dynamical variable in quantum mechanics."

What is an ____________, Alex?

a) intangible b) intaglio c) obtainable d) oblivion e) observable

SUGGESTED ANSWER: (e)

Wrong answers:

a) As Lurch would say AAAARGH.

Redaction: Jeffery, 2008jan01

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002 qmult 00310 1 1 3 easy memory: expectation value defined

7. In quantum mechanics, a dynamical variable is governed by a Hermitian operator called an observable that has an expectation value that is:

a) the most likely value of the quantity given by the probability density: i.e., the mode of the probability density.
b) the median value of the quantity given by the probability density.
c) the mean value of the quantity given by the probability density.
d) any value you happen to measure.
e) the time average of the quantity.

SUGGESTED ANSWER: (c): Why do we use this funny jargon term expectation value in quantum mechanics? Who knows. We're stuck with it though.

Wrong Answers:

e) No. The probability density is for an ensemble of identical states all at one time.

Redaction: Jeffery, 2001jan01

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002 qmult 00320 1 1 3 easy memory: expectation value notation

8. The expectation value of operator Q for some wave function is often written:

a) Q. b) $\rangle Q \langle$. c) $\langle Q \rangle$. d) $\langle f(Q) \rangle$. e) $f(Q)$.

SUGGESTED ANSWER: (c)

Wrong Answers:

d) This is expectation value of the operator f(Q).
e) This is the operator f(Q).

Redaction: Jeffery, 2001jan01

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002 qmult 00400 1 1 1 easy memory: physical requirments

Extra keywords: Gr-11

9. These quantum mechanical entities (with some exceptions) must be:

i) Single-valued (and their derivatives too).
ii) finite (and their derivatives too).
iii) continuous (and their derivatives too).
iv) normalizable or square-integrable.

They are:

a) wave functions. b) observables. c) expectation values. d) wavelengths. e) wavenumbers.

SUGGESTED ANSWER: (a)

Wrong answers:
b) So-so guess.

Redaction: Jeffery, 2008jan01

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002 qmult 00500 1 1 2 easy memory: the momentum operator defined

10. The momentum operator in one-dimension is:

a) $\hbar \frac{\partial}{\partial x}$. b) $\frac{\hbar}{i} \frac{\partial}{\partial x}$. c) $\frac{i}{\hbar} \frac{\partial}{\partial x}$. d) $\frac{i}{\hbar} \frac{\partial}{\partial t}$. e) $\hbar \frac{\partial}{\partial t}$.

SUGGESTED ANSWER: (b)

Wrong Answers:
e) C'mon.

Redaction: Jeffery, 2001jan01

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002 qmult 00530 1 4 5 easy deducto-memory: Heisenberg uncertainty

11. "Let's play *Jeopardy*! For $100, the answer is: It describes a fundamental limitation on the accuracy with which we can know position and momentum simultaneously."

What is ____________, Alex?

a) Tarkovsky's doubtful thesis b) Rublev's ambiguous postulate
c) Kelvin's vague zeroth law d) Schrödinger's wild hypothesis
e) Heisenberg's uncertainty principle

SUGGESTED ANSWER: (e)

Wrong answers:
a) Tarkovsky, you should be living in this hour.

Redaction: Jeffery, 2001jan01

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002 qmult 00700 1 1 4 easy memory: Schr. eqn. separation of variables

12. The time-independent Schrödinger equation is obtained from the full Schrödinger equation by:

a) colloquialism. b) solution for eigenfunctions. c) separation of the x and y variables.
d) separation of the space and time variables. e) expansion.

SUGGESTED ANSWER: (d)

Wrong Answers:
a) Huh?

Redaction: Jeffery, 2001jan01

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002 qmult 00720 1 1 1 easy memory: stationary state

13. A system in a stationary state will:

a) not evolve in time. b) evolve in time. c) both evolve and not evolve in time.
d) occasionally evolve in time. e) violate the Heisenberg uncertainty principle.

SUGGESTED ANSWER: (a) The wave function itself will have the time oscillation factor $e^{i\omega t}$, but that is not considered time evolution of the state.

Wrong answers:
b) Exactly wrong.

Redaction: Jeffery, 2001jan01

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002 qmult 00800 1 4 2 easy deducto-memory: orthogonality property

14. For a Hermitian operator eigenproblem, one can always find (subject to some qualitifications perhaps—but which are just mathemtical hemming and hawwing) a complete set (or basis) of eigenfunctions that are:

a) independent of the x-coordinate. b) orthonormal. c) collinear. d) pathological. e) righteous.

SUGGESTED ANSWER: (b)

Wrong Answers:

e) Not the best answer in this context anyway.

Redaction: Jeffery, 2001jan01

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002 qmult 00810 1 4 2 easy deducto-memory: basis expansion

Extra keywords: mathematical physics

15. "Let's play *Jeopardy*! For \$100, the answer is: If it shares the same boundary conditions as a basis set of functions and is at least piecewise continuous, then it can be expanded in the basis with a vanishing limit of the mean square error between it and the expansion."

What is a/an ____________, Alex?

a) equation b) function c) triangle d) deduction e) tax deduction

SUGGESTED ANSWER: (b) See WA-510.

Wrong answers:

e) Sounds plausible.

Redaction: Jeffery, 2008jan01

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002 qmult 00820 1 4 5 easy deducto-memory: general Born postulate

Extra keywords: mathematical physics

16. "Let's play *Jeopardy*! For \$100, the answer is: The postulate that expansion coefficients of a wave function in the eigenstates of an observable are the probability amplitudes for wave function collapse to eigenstates of that observable."

What is ____________, Alex?

a) the special Born postulate b) the very special Born postulate c) normalizability d) the mass-energy equivalence e) the general Born postulate

SUGGESTED ANSWER: (e)

Wrong answers:

b) As Lurch would say AAAARGH.

Redaction: Jeffery, 2008jan01

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002 qmult 00830 1 1 4 easy memory: basis expansion physics

17. The expansion of a wave function in an observable's basis (or complete set of eigenstates) is

a) just a mathematical decomposition. b) useless in quantum mechanics. c) irrelevant in quantum mechanics. d) not just a mathematical decomposition since the expansion coefficients are probability amplitudes. e) just.

SUGGESTED ANSWER: (c)

Wrong answers:

a) A nonsense answer.

Redaction: Jeffery, 2008jan01

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020 qmult 00840 1 4 5 easy deducto-memory: wave function collapse

Extra keywords: mathematical physics

18. "Let's play *Jeopardy*! For \$100, the answer is: It is a process in quantum mechanics that some decline to mention, some believe to be unspeakable, some believe does not exist (though they got some explaining to do about how one ever measures anything), some believe should not exist, and that some call the fundamental perturbation (but just once per textbook)."

What is ____________, Alex?

a) the Holy b) the Unholy c) the Unnameable d) the 4th secret of the inner circle
e) wave function collapse

SUGGESTED ANSWER: (e) The massive non-relativistic quantum mechanics textbook Cohen-Tannoudji on p. 226 mentions the "fundamental" perturbation (their quotation marks) and that's all that I can find: it's not in the index either. Cohen-Tannoudji says they will not consider problems associated with the "fundamental" perturbation and they never do.

Wrong answers:
b) Some would say so.
d) Well I'm not really sure, but sometimes when certain physicists meet there seems to be a bit of wink and a nod. I've even caught a glimpse of what may be a secret handshake. Odd words or emphases some time turn up in articles. Just around the edges of the known and the canny, things seem to get done, people promoted—or eliminated. Maybe they're always watching. If I vanish without a trace some day, best not to make a point of saying anything.

Redaction: Jeffery, 2008jan01

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002 qmult 00900 1 4 1 easy deducto-memory: macro object in stationary state

19. "Let's play *Jeopardy*! For $100, the answer is: A state that no macroscopic system can be in except arguably for states of Bose-Einstein condensates, superconductors, superfluids and maybe others sort of."

What is a/an ____________, Alex?

a) stationary state b) accelerating state c) state of the Union d) state of being
e) state of mind

SUGGESTED ANSWER: (a)

Wrong answers:
b) Clearly wrong.
c) Well we-all are in a state of the Union.
d) Lots of macroscopic objects are real things without being Bose-Einstein condensates either.
e) I'm a macroscopic object and I'm occasionally in a state of mind, but never noticeably in a Bose-Einstein condensate.

Redaction: Jeffery, 2001jan01

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002 qmult 01000 1 1 5 easy memory: stationary state is radical

20. A stationary state is:

a) just a special kind of classical state. b) more or less a kind of classical state.
c) voluntarily a classical state. d) was originally not a classical state, but grew into one.
e) radically unlike a classical state.

SUGGESTED ANSWER: (e)

Wrong Answers:
c) Nonsense answer.
d) Nonsense answer.

Redaction: Jeffery, 2001jan01

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002 qmult 01400 1 4 4 easy deducto-memory: operators and Sch. eqn.

21. "Let's play *Jeopardy*! For $100, the answer is: An equation that must hold in order for the non-relativistic Hamiltonian operator and the operator $i\hbar\partial/\partial t$ to both represent energy in the evaluation of the energy expectation value for a wave function $\Psi(x,t)$."

What is ____________, Alex?

a) the continuity equation b) the Laplace equation c) Newton's 2nd law
d) Schrödinger's equation e) Hamiton's equation

SUGGESTED ANSWER: (d)

Wrong answers:

c) Schrödinger's equation is the analog to Newton's law for quantum mechanics.
e) The two Hamilton's equations together are equations of motion in classical mechanics that can be used instead of Newton's law in advanced treatments.

Redaction: Jeffery, 2001jan01

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002 qmult 02000 2 1 4 moderate memory: does gravity quantize
Extra keywords: reference: Nesvizhevsky et al. 2002, Nature, 413, 297

22. Can the gravitational potential cause quantization of energy states?

a) No. b) It is completely uncertain. c) Theoretically yes, but experimentally no.
d) Experimental evidence to date (post-2001) suggests it can.
e) In principle there is no way of telling.

SUGGESTED ANSWER: (d)

Wrong Answers:

b) This used to be the right answer.
c) If so then either theory or experiment is wrong.
e) Experiments can address the issue.

Redaction: Jeffery, 2001jan01

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002 qfull 00100 1 3 0 easy math: probability and age distribution
Extra keywords: (Gr-10:1.1)

23. Given the following age distribution, compute its the normalization (i.e., the factor that normalizes the distribution), mean, variance, and standard deviation. Also give the mode (i.e., the age with highest frequency) and median.

Table: Age Distribution

Age (years)	Frequency
14	2
15	1
16	6
22	2
24	2
25	5

SUGGESTED ANSWER: The normalization is 1/18, the mean 19.56, the variance 18.36, and the standard deviation 4.28. The mode is 16. Because of the sparseness of the data, the median is somewhat ill-defined. One could put it anywhere from 16 to 22. The middle of this range 19 is probably most sensible.

Fortran Code

```
      program ages
      parameter (nage=6)
      dimension age(2,nage)
      data age/14.,2.,  15.,1.,  16.,6.,  22.,2.,
     &         22.,2.,  25.,5./
*
      sum0=0.
      sum1=0.
      sum2=0.
      do i=1,nage
       sum0=sum0+age(2,i)
       sum1=sum1+age(1,i)*age(2,i)
       sum2=sum2+age(1,i)**2*age(2,i)
      end do
```

```
      xmean=sum1/sum0
      var=sum2/sum0-xmean**2
      stdev=sqrt(var)
      print*,'sum0,xmean,var,stdev'
      print*,sum0,xmean,var,stdev
*            18.  19.5555553  18.3580322  4.28462744
*
      end
```

Redaction: Jeffery, 2001jan01

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002 qfull 00400 1 3 0 easy math: orthonormality leads to mean energy
Extra keywords: (Gr-30:2.10)

24. You are given a complete set of orthonormal stationary states (i.e., energy eigenfunctions) $\{\psi_n\}$ and a general wave equation $\Phi(x,t)$ for the same system: i.e., $\Phi(x,t)$ is determined by the same Hamiltonian as the complete set. Find the general expression, simplified as far as possible, for expectation value $\langle H^\ell\rangle$ where ℓ is any positive (or zero) integer. Give the special cases for $\ell = 0$, 1, and 2, and the expression for σ_E. **HINTS:** Use expansion and orthonormality. This should be a very short answer: 3 or 4 lines.

SUGGESTED ANSWER: Behold

$$
\begin{aligned}
\langle H^\ell\rangle &= \int_{-\infty}^{\infty} \Phi(x,t)^* H^\ell \Phi(x,t)\,dx \\
&= \sum_{m,n} c_m^* c_n \int_{-\infty}^{\infty} \Psi_m(x)^* H^\ell \Psi_n(x)\,dx \\
&= \sum_{m,n} c_m^* c_n E_n^\ell \int_{-\infty}^{\infty} \Psi_m(x)^* \Psi_n(x)\,dx \\
&= \sum_{m,n} c_m^* c_n E_n^\ell \delta_{m,n} \\
&= \sum_n |c_n|^2 E_n^\ell \ ,
\end{aligned}
$$

where we used expansion and orthonormality. The most interesting special cases are for normalization, energy expectation value, and second moment of the distribution:

$$
\begin{aligned}
1 &= \sum_n |c_n|^2 \\
\langle H\rangle &= \sum_n |c_n|^2 E_n
\end{aligned}
$$

and

$$
\langle H^2\rangle = \sum_n |c_n|^2 E_n^2 \ .
$$

The energy standard deviation is given by

$$
\sigma_E = \sqrt{\langle H^2\rangle - \langle H\rangle^2} \ .
$$

In fact on any measurement of energy (or ℓth power of energy) one obtains a value E_n (or E_n^ℓ) with a probability of $|c_n|^2$. Sometimes one hears $|c_n|^2$ called the probability of the system being in stationary state n. This is actually a bit careless. In the standard interpretation the system isn't in any particular stationary state (unless the expansion contains only one term) before the measurement: the measurement projects the system into (or collapses the wave function to) a particular stationary state. This is sensible. Take an ensemble and measure any dynamical quantity (represented by some operator) and you project the system into an eigenstate for that quantity's operator. But energy eigenstates (stationary states) and other kinds of eigenstates do not necessarily form the same set. So how can the system be in a stationary state and an eigenstate for some other operator at the same time. It can, of course, if the Hamiltonian and the other operator commute. But in general they won't.

Redaction: Jeffery, 2001jan01

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002 qfull 01000 2 5 0 moderate thinking: energy and normalization
Extra keywords: (Gr-24:2.2) zero-point energy

25. Classically $E \geq V_{\text{min}}$ for a particle in a conservative system.

a) Show that this classical result must be so. **HINT:** This shouldn't be a from-first-principles proof: it should be about one line.

b) The quantum mechanical analog is almost the same: $\bar{E} = \langle H \rangle > V_{\text{min}}$ for any state of the system considered. Note the equality $\bar{E} = \langle H \rangle = V_{\text{min}}$ never holds quantum mechanically. (There is an over-idealized exception, which we consider in part (e).) Prove the inequality. **HINTS:** The key point is to show that $\langle T \rangle > 0$ for all physically allowed states. Use integration by parts.

c) Now show that result $\bar{E} > V_{\text{min}}$ implies $E > V_{\text{min}}$, where E is any eigen-energy of the system considered. Note the equality $E = V_{\text{min}}$ never holds quantum mechanically (except for the over-idealized system considered in part (e)). In a sense, there is no rest state for quantum mechanical particle. This lowest energy is called the zero-point energy.

d) The $E > V_{\text{min}}$ result for an eigen-energy in turn implies a 3rd result: any ideal measurement always yields an energy greater than V_{min} Prove this by reference to a quantum mechanical postulate.

e) There is actually an exception to $E > V_{\text{min}}$ result for an eigen-energy where $E = V_{\text{min}}$ occurs. The exception is for quantum mechanical systems with periodic boundary conditions and a constant potential. In ordinary 3-dimensional Euclidean space, the periodic boundary conditions can only occur for rings (1-dimensional systems) and sphere surfaces (2-dimensional systems) I believe. Since any real system must have a finite size in all 3 spatial dimensions, one cannot have real systems with only periodic boundary conditions. Thus, the exception to the $E > V_{\text{min}}$ result is for unrealistic over-idealized systems. Let us consider the idealized ring system as an example case. The Hamiltonian for a 1-dimensional ring with a constant potential is

$$H = -\frac{\hbar^2}{2mr^2}\frac{\partial^2}{\partial\phi^2} + V ,$$

where r is the ring radius, ϕ is the azimuthal angle, and V is the constant potential. Find the eigen-functions and eigen-energies for the Schrödinger equation for the ring system with periodic boundary conditions imposed. Why must one impose periodic boundary conditions on the solutions? What solution has eigen-energy $E = V_{\text{min}}$?

SUGGESTED ANSWER:

a) Classically, for a conservative system $E = T + V$ is a constant: T is kinetic energy and V is potential energy: and $T = (1/2)mv^2 \geq 0$ always, of course. Thus $E \geq V \geq V_{\text{min}}$. If $E = V_{\text{min}}$, the system is in static equilibrium since $T = 0$ and $F = -dV/dx = 0$ for $V = V_{\text{min}}$. Of course, a system with $T > 0$ can be instantaneously at the point where $V = V_{\text{min}}$: the particle in this case is just passing through the equilibrium point where its acceleration is instantaneously zero.

b) In quantum mechanics the expectation value of the kinetic energy for any wave function is given by

$$\begin{aligned}\langle T \rangle &= \int_{-\infty}^{\infty} \Psi^* T \Psi \, dx = \frac{1}{2m}\int_{-\infty}^{\infty} \Psi^* p^2 \Psi \, dx = \frac{-1}{2m}\int_{-\infty}^{\infty} p\Psi^* p\Psi \, dx \\ &= \frac{1}{2m}\int_{-\infty}^{\infty} (p\Psi)^* p\Psi \, dx \\ &\geq 0 ,\end{aligned}$$

where we have used integration by parts, the vanishing of the boundary terms, and the fact that integrand of the penultimate line is evidently always greater than or equal to zero. Note the above derivation demands that the boundary terms vanish and that the function is sufficiently differentiable. The requirement of normalization and the fact that in reality all physical systems show no discontinuities or infinities guarantees conditions of the derivation. If discontinuities are introduced as a mathemetical idealization, then they must be treated so as to yield the same derivation in order to be valid. For instance, the infinite square well has a discontinuity in $\partial\psi/\partial x$ at the well walls. However, since the wave function is zero outside of the well and at the well walls, we derive the same result as above: $\langle T\rangle \geq 0$.

The only way $\langle T\rangle$ can be zero is if the derivative of the wave function were everywhere zero which requires that the wave function be a constant. A constant wave function cannot be normalized, and so is not a physically allowed wave function (except for the over-idealized systems considered in part (e).) Ergo

$$\langle T\rangle > 0$$

always. Ergo all over again

$$\bar{E} = \langle H\rangle > \langle V\rangle \geq V_{\rm min}$$

or

$$\bar{E} = \langle H\rangle > V_{\rm min}$$

which is the result we wanted to show.

c) For a stationary state, the expectation value of the Hamiltonian is just the eigen-energy:

$$E = \langle H\rangle \ .$$

Thus for a stationary state of eigen-energy E it follows that

$$E > V_{\rm min} \ .$$

d) Now for an Aristotelian syllogism:

Major premise: By quantum mechanical postulate the result of any ideal measurment of an observable is an eigenvalue of the observable for that system. Well if the eigenvalues form a continuum then it seems that an even ideal measurement must always be some average of a finite range of eigenvalues. But if the measurement is ideal it must be a very small range. The point is not cleared up in my sources.

Minor premise: All eigenvalues E obey $E > V_{\rm min}$.

Conclusion: All ideal measurements of energy of a quantum mechanical system yield an energy $E_{\rm measured} > V_{\rm min}$.

Further Considerations:

What if the potential is time-varying? I guess the argument then is that at any instant the potential can be treated as a constant with instant-existing eigenstates. Then again any expectation or eigen-energy is greater than $V_{\rm min}$ always.

Further insight into our results can from a reasoning argument. Physical states are described by wave functions. To be physical, a wave function must be normalizable. To be normalizable function must have some curvature. Curvature gives rise to a kinetic energy contribution to energy expectation value unless somehow a pathological wave function can be found where the kinetic energy contributions all cancel out. The proof done above shows that no such pathological wave function can be found. We can further note that the classical state of rest doesn't exist in quantum mechanics. A stationary state may correspond to rest in some respects, but it is radically different in other respects.

Yet more further insight into our results can come from considering the time-independent Schrödinger equation in the form

$$\psi'' = \frac{2m}{\hbar^2}[V(x) - E]\psi \ .$$

For simplicity let ψ be a pure real stationary state as we are always free to arrange. Any normalizable wave function must "turn over" at least once somewhere: i.e., it's absolute value must have a global maximum (or maybe more than one equal global maxima). Now if ψ'' and

ψ always have the same sign and are well defined, then for $\psi > 0$ there can only be a minimum and for $\psi < 0$ there can only be a maximum. Thus the absolute value can never have a global maximum and the wave function cannot be normalized.

This argument gives insight, but I don't think it is a fully convincing proof. One has to wonder couldn't $\psi' = \psi'' = \psi''' = 0$ and the nature of the stationary point be determined by the fourth order derivative $\psi^{(4)}$? Also what if ψ'' is undefined at maximum or minimum which is a cusp: this happens for the wave function of a Dirac Delta function (see Gr-54). Clearly, a wave function that somehow was normalizable with $E \leq V_{\text{min}}$ would be pathological, but more intricate argument is needed to show that it was impossible.

And also one can always imagine a normalizable wave function that is made of piecewise regions that have ψ and ψ'' always of the same sign. Maybe such wave functions can't be physical, but it would be tedious to argue generally enough to exclude them.

To investigate just a bit further let us consider a concrete example where $V(x)$ goes to a asymptotic constant V_{asy} for $|x|$ large. If $E < V_{\text{asy}}$ in this asymptotic limit, then

$$\psi(x)_{\text{asy}} = Ce^{\pm kx} \ ,$$

where C is some constant and

$$k = \sqrt{\frac{2m}{\hbar^2}[V_{\text{asy}} - E]} \ .$$

The asymptotic wave functions if they applied everywhere are clearly not normalizable. They are allowed as the wave functions in some regimes: e.g., for x positive, Ce^{-kx} is allowed and for x negative, Ce^{kx}.

e) The time-independent Schrödinger equation in this case is

$$-\frac{\hbar^2}{2mr^2}\frac{\partial^2\psi}{\partial\phi^2} + V\psi = E\psi$$

which we can rewrite as

$$\frac{\partial^2}{\partial\phi^2} = -k^2\psi \ ,$$

where

$$k = \pm\sqrt{\frac{2mr^2}{\hbar^2}(E - V)}$$

with $E \geq V$ asssumed. The normalized solutions are

$$\psi = \frac{e^{ik\phi}}{\sqrt{2\pi}} \ ,$$

where we've just taken the azimuthal angle ϕ as the coordinate and not $r\phi$ (which would require a normalization constant $1/\sqrt{2\pi r}$. In order to be single-valued (which is a necessary condition on wave functions), we must have an integer k. Let us write k as m since n is more integerish: n is the quantum number for the eigenstates and eigenvalues. Thus

$$n = 0, \pm1, \pm2, \pm3, \ldots$$

and the quantized eigen-energies are given by

$$E = V + \frac{\hbar^2}{2mr^2}n^2 \ .$$

Requiring single-valuedness amounts to the same thing as requiring periodic boundary conditions for the ring since one can choose any point on the ring as a conventional boundary. Thus, periodic boundary conditions are required.

In the ring system, nothing forbids the $n = 0$. This means that we have valid eigenstate $\psi = 1/\sqrt{2\pi}$ which is a constant and the lowest eigen-energy is

$$E = V \ .$$

Since the potential is a constant, $V = V_{\rm min}$, and thus the lowest eigen-energy E equals $V_{\rm min}$. This is an exception to our usual rule that the eigen-energies obey $E > V_{\rm min}$, but it is for an over-idealized case.

Note that there are solutions that are ruled out. If $E - V = 0$, we have the linear solution

$$\psi = a\phi + b \ ,$$

where a and b are constants. The solution with $a = 0$ is just the $n = 0$ solution which is allowed. But if $a \neq 0$, the linear solution is not single-valued nor normalizable and must be ruled out.

If $E - V < 0$, we have exponential solutions

$$\psi = e^{\pm\kappa x} \ ,$$

where

$$\kappa = \sqrt{\frac{2mr^2}{\hbar^2}(V - E)} \ .$$

These solutions are not single-valued nor normalizable and must be ruled out.

Redaction: Jeffery, 2001jan01

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002 qfull 01100 3 5 0 tough thinking: 1-d non-degeneracy

26. If there are no internal degrees of freedom (e.g., spin) and they are normalizable, then one-particle, 1-dimensional energy eigenstates are non-degenerate. We (that is to say you) will prove this.

a) Assume you have two degenerate 1-dimensional energy eigenstates for Hamiltonian H: ψ_1 and ψ_2. Prove that $\psi_1\psi_2' - \psi_2\psi_1'$ equals a constant where the primes indicate derivative with respect to x the spatial variable. **HINT:** Write down the eigenproblem for both ψ_1 and ψ_2 and do some multiplying and subtraction and integration.

b) Prove that the constant in part (a) result must be zero. **HINT:** To be physically allowable eigenstates, the eigenstates must be normalizable.

c) Show for all x that

$$\psi_2(x) = C\psi_1(x) \ ,$$

where C is a constant. **HINT:** The eigenproblem is a linear, homogeneous differential equation.

SUGGESTED ANSWER:

a) Assume you have two degenerate 1-d energy eigenstates for Hamiltonian H: ψ_1 and ψ_2. Then

$$H\psi_1 = E\psi_1 \qquad \text{and} \qquad H\psi_2 = E\psi_2 \ ,$$

and so

$$\psi 2H\psi_1 = \psi_2 E\psi_1 \qquad \text{and} \qquad \psi_1 H\psi_2 = \psi_1 E\psi_2 \ .$$

If we subtract first from the second of these, we get

$$\psi_1 H\psi_2 - \psi 2H\psi_1 = \psi_1 T\psi_2 - \psi 2T\psi_1 = 0 \ ,$$

where T is the kinetic energy operator. Thus,

$$\psi_1\psi_2'' - \psi 2\psi_1'' = 0$$

which integrates to

$$\psi_1\psi_2' - \psi 2\psi_1' = \text{Constant} \ .$$

b) To be normalizable, the energy eigenstates must be zero at infinity, and thus the constant form the part (a) result must be zero. Thus,

$$\psi_1\psi_2' - \psi 2\psi_1' = 0 \ .$$

Note it doesn't matter what the derivatives are doing at infinity. If they are zero, then that's just another reason for the constant to be zero.

Note that the free particle (i.e., potential zero or constant) eigenstates are not normalizable at infinity and are degenerate. There are two independent eigenstates for each energy:

$$psi_+ = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad \text{and} \qquad psi_- = \frac{e^{-ikx}}{\sqrt{2\pi}} \ ,$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}} \ .$$

c) Well if $\psi_2(x) = C\psi_1(x)$, then

$$\psi_1\psi_2' - \psi 2\psi_1\psi_1 C\psi_1' - C\psi 1\psi_1' = 0$$

and our condition is satisfied. Is there any other way to satisfy the condition? Well if $\psi_2(x) = C\psi_1(x)$ holds over any extended region of x, then by the linear, homogeneous nature of the eigenproblem (or so I believe), the equality must hold everywhere and our guessed solution is unique (aside from the multiplicative constant C which is physically irrelevant). Remember for a 2nd order linear differential equation (without any pathologies in its coefficient functions anyway) if you specify the solution at any two points or the solution and its derivative at any one point, then the whole solution follows. Thus, we merely have to prove that $\psi_2(x) = C\psi_1(x)$ holds over an extented region.

Note

$$\psi_1\psi_2' = \psi_2\psi_1' \ ,$$

$$\frac{\psi_2'}{\psi_2} = \frac{\psi_1'}{\psi_1} \ ,$$

$$\frac{d\psi_2}{\psi_2} = \frac{d\psi_1}{\psi_1} \ ,$$

$$\ln|\psi_2| = \ln|\psi_1| + \text{Constant} \ ,$$

and finally

$$|\psi_2| = C|\psi_1| \ ,$$

where C is a constant and we have used

$$\frac{d\ln|x|}{dx} = \frac{1}{|x|}(\pm 1) = \frac{1}{x} \ ,$$

where the upper case is for $x > 0$ and the lower for $x < 0$. Now if there is any region where both ψ_1 and ψ_2 have the same sign, then by our uniqueness argument above

$$\psi_2 = C\psi_1$$

must hold everywhere. If everywhere the two functions have a different sign, then everywhere

$$\psi_2 = -C\psi_1 \ ,$$

but this just to say that the two functions differ only by a multiplicative constant which is what we wanted to prove.

The upshot is that single-particle, 1-dimensional eigenstates (with no internal degrees of freedom) are non-degenerate. Any difference by a constant between two eigenstates is not a physical difference. Both functions must be normalized and any difference by a global phase factor has no physical effect.

Admittedly, one might be able to imagine some pathological potential that gives a degenerate eigenstate, but such a case is unlikely to turn up in nature.

Of course, if you have internal deqrees of freedom with no energy distinction in principle like spin, then degeneracy in 1-dimension is easily obtained. In nature, I believe some interaction always breaks spin or angular momentum degeneracy to some degree.

Redaction: Jeffery, 2001jan01

Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

1 Geometrical Formulae

$$C_{\rm cir} = 2\pi r \qquad A_{\rm cir} = \pi r^2 \qquad A_{\rm sph} = 4\pi r^2 \qquad V_{\rm sph} = \frac{4}{3}\pi r^3$$

2 Trigonometry

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right] \qquad \sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]$$

3 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\left[e^{h\nu/(kT)}-1\right]} \qquad B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{\left[e^{hc/(kT\lambda)}-1\right]}$$

$$B_\lambda\,d\lambda = B_\nu\,d\nu \qquad \nu\lambda = c \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$k = 1.3806505(24)\times 10^{-23}\,\mathrm{J/K} \qquad c = 2.99792458\times 10^{8}\,\mathrm{m}$$

$$h = 6.6260693(11)\times 10^{-34}\,\mathrm{J\,s} = 4.13566743(35)\times 10^{-15}\,\mathrm{eV\,s}$$

$$\hbar = \frac{h}{2\pi} = 1.05457168(18)\times 10^{-34}\,\mathrm{J\,s}$$

$$hc = 12398.419\,\mathrm{eV\,Å} \approx 10^4\,\mathrm{eV\,Å} \qquad E = h\nu = \frac{hc}{\lambda} \qquad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \qquad \sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670400(40) \times 10^{-8}\,\mathrm{W/m^2/K^4}$$

$$\lambda_{\max} T = \text{constant} = \frac{hc}{kx_{\max}} \approx \frac{1.4387751 \times 10^{-2}}{x_{\max}}$$

$$B_{\lambda,\mathrm{Wien}} = \frac{2hc^2}{\lambda^5} e^{-hc/(kT\lambda)} \qquad B_{\lambda,\mathrm{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu = \frac{\omega}{c} \qquad k_i = \frac{\pi}{L} n_i \quad \text{standing wave BCs} \qquad k_i = \frac{2\pi}{L} n_i \quad \text{periodic BCs}$$

$$n(k)\,dk = \frac{k^2}{\pi^2}\,dk = \pi\left(\frac{2}{c}\right)\nu^2\,d\nu = n(\nu)\,d\nu$$

$$\ln(z!) \approx \left(z + \frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots$$

$$\ln(N!) \approx N\ln(N) - N$$

$$\rho(E)\,dE = \frac{e^{-E/(kT)}}{kT}\,dE \qquad P(n) = (1 - e^{-\alpha})e^{-n\alpha} \qquad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad f(x - vt) \qquad f(kx - \omega t)$$

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4 Photons

$$KE = h\nu - w \qquad \Delta\lambda = \lambda_{\mathrm{scat}} - \lambda_{\mathrm{inc}} = \lambda_{\mathrm{C}}(1 - \cos\theta)$$

$$\lambda_{\mathrm{C}} = \frac{h}{m_e c} = 2.426310238(16) \times 10^{-12}\,\mathrm{m} \qquad e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$m_e = 9.1093826(16) \times 10^{-31}\,\mathrm{kg} = 0.510998918(44)\,\mathrm{MeV}$$

$$m_p = 1.67262171(29) \times 10^{-27}\,\mathrm{kg} = 938.272029(80)\,\mathrm{MeV}$$

$$\ell = \frac{1}{n\sigma} \qquad \rho = \frac{e^{-s/\ell}}{\ell} \qquad \langle s^m \rangle = \ell^m m!$$

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5 Matter Waves

$$\lambda = \frac{h}{p} \qquad p = \hbar k \qquad \Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k)\Psi_k(x,t)\,dk \qquad \phi(k) = \int_{-\infty}^{\infty} \Psi(x,0)\frac{e^{-ikx}}{\sqrt{2\pi}}\,dk$$

$$v_{\rm g} = \left.\frac{d\omega}{dk}\right|_{k_0} = \frac{\hbar k_0}{m} = \frac{p_0}{m} = v_{\rm clas,0}$$

6 Non-Relativistic Quantum Mechanics

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \qquad T = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \qquad H\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$$\rho = \Psi^*\Psi \qquad \rho\,dx = \Psi^*\Psi\,dx$$

$$A\phi_i = a_i\phi_i \qquad f(x) = \sum_i c_i\phi_i \qquad \int_a^b \phi_i^*\phi_j\,dx = \delta_{ij} \qquad c_j = \int_a^b \phi_j^* f(x)\,dx \qquad [A,B] = AB - BA$$

$$P_i = |c_i|^2 \qquad \langle A\rangle = \int_{-\infty}^{\infty} \Psi^* A\Psi\,dx = \sum_i |c_i|^2 a_i \qquad H\psi = E\psi \qquad \Psi(x,t) = \psi(x)e^{-i\omega t}$$

$$p_{\rm op}\phi = \frac{\hbar}{i}\frac{\partial\phi}{\partial x} = p\phi \qquad \phi = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad \frac{\partial^2\psi}{\partial x^2} = \frac{2m}{\hbar^2}(V-E)\psi$$

$$|\Psi\rangle \qquad \langle\Psi| \qquad \langle x|\Psi\rangle = \Psi(x) \qquad \langle\vec{r}|\Psi\rangle = \Psi(\vec{r}) \qquad \langle k|\Psi\rangle = \Psi(k) \qquad \langle\Psi_i|\Psi_j\rangle = \langle\Psi_j|\Psi_i\rangle^*$$

$$\langle\phi_i|\Psi\rangle = c_i \qquad 1_{\rm op} = \sum_i |\phi_i\rangle\langle\phi_i| \qquad |\Psi\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\Psi\rangle = \sum_i c_i|\phi_i\rangle$$

$$1_{\rm op} = \int_{-\infty}^{\infty} dx\,|x\rangle\langle x| \qquad \langle\Psi_i|\Psi_j\rangle = \int_{-\infty}^{\infty} dx\,\langle\Psi|x\rangle\langle x|\Psi\rangle \qquad A_{ij} = \langle\phi_i|A|\phi_j\rangle$$

$$Pf(x) = f(-x) \qquad P\frac{df(x)}{dx} = \frac{df(-x)}{d(-x)} = -\frac{df(-x)}{dx} \qquad Pf_{\rm e/o}(x) = \pm f_{\rm e/o}(x) \qquad P\frac{df_{\rm e/o}(x)}{dx} = \mp\frac{df_{\rm e/o}(x)}{dx}$$

7 Special Relativity

$$c = 2.99792458 \times 10^8\,{\rm m/s} \approx 2.998 \times 10^8\,{\rm m/s} \approx 3 \times 10^8\,{\rm m/s} \approx 1\,{\rm lyr/yr} \approx 1\,{\rm ft/ns}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \gamma(\beta << 1) = 1 + \frac{1}{2}\beta^2 \qquad \tau = ct$$

Galilean Transformations	Lorentz Transformations
$x' = x - \beta\tau$	$x' = \gamma(x - \beta\tau)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$\tau' = \tau$	$\tau' = \gamma(\tau - \beta x)$
$\beta'_{\mathrm{obj}} = \beta_{\mathrm{obj}} - \beta$	$\beta'_{\mathrm{obj}} = \dfrac{\beta_{\mathrm{obj}} - \beta}{1 - \beta\beta_{\mathrm{obj}}}$

$$\ell = \ell_{\mathrm{proper}}\sqrt{1-\beta^2} \qquad \Delta\tau_{\mathrm{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$m = \gamma m_0 \qquad p = mv = \gamma m_0 c\beta \qquad E_0 = m_0c^2 \qquad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \qquad E = \sqrt{(pc)^2 + (m_0c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2 = (\gamma - 1)m_0c^2$$

$$f = f_{\mathrm{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating}$$

$$f(\beta << 1) = f_{\mathrm{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\mathrm{trans}} = f_{\mathrm{proper}}\sqrt{1-\beta^2} \qquad f_{\mathrm{trans}}(\beta << 1) = f_{\mathrm{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1}\tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1}x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\mathrm{intersection}}}{\gamma} = x'_{x\ \mathrm{scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \tau' = \frac{\tau_{\mathrm{intersection}}}{\gamma} = \tau'_{\tau\ \mathrm{scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

$$\theta_{\mathrm{Mink}} = \tan^{-1}(\beta)$$