Modern Physics: Physics 305, Section 1

NAME:

Homework 5: Non-Relativistic Quantum Mechanics Homeworks are due as posted on the course web site. They are **NOT** handed in. The student reports that it is completed and receives one point for this. Solutions are already posted, but students are only permitted to look at the solutions after completion. The solutions are intended to be (but not necessarily are) super-perfect and go beyond a complete answer expected on a test.

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	16.	O	O	O	O	O
2.	O	O	O	O	O	17.	O	O	O	O	O
3.	O	O	O	O	O	18.	O	O	O	O	O
4.	O	O	O	O	O	19.	O	O	O	O	O
5.	O	O	O	O	O	20.	O	O	O	O	O
6.	O	O	O	O	O	21.	O	O	O	O	O
7.	O	O	O	O	O	22.	O	O	O	O	O
8.	O	O	O	O	O	23.	O	O	O	O	O
9.	O	O	O	O	O	24.	O	O	O	O	O
10.	O	O	O	O	O	25.	O	O	O	O	O
11.	O	O	O	O	O	26.	O	O	O	O	O
12.	O	O	O	O	O	27.	O	O	O	O	O
13.	O	O	O	O	O	28.	O	O	O	O	O
14.	O	O	O	O	O	29.	O	O	O	O	O
15.	O	O	O	O	O	30.	O	O	O	O	O

1. The principle that all microscopic physical entities have both wave and particle properties is called the wave-particle:

   a) singularity. b) duality. c) triality. d) infinality. e) nullility.

2. "Let's play *Jeopardy*! For \$100, the answer is: The equation that governs (or equations that govern) the time evolution of quantum mechanical systems in the non-relativistic approximation."

   What is/are ____________, Alex?

   a) $\vec{F}_{\rm net} = m\vec{a}$ b) Maxwell's equations c) Einstein's field equations of general relativity
   d) Dirac's equation e) Schrödinger's equation

3. The full Schrödinger's equation in compact form is:

   a) $H\Psi = i\hbar\frac{\partial\Psi}{\partial t}$. b) $H\Psi = \hbar\frac{\partial\Psi}{\partial t}$. c) $H\Psi = i\frac{\partial\Psi}{\partial t}$. d) $H\Psi = i\hbar\frac{\partial\Psi}{\partial x}$.
   e) $H^{-1}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$.

4. In the probabistic interpretation of wave function Ψ, the quantity $|\Psi|^2$ is:

   a) a probability density. b) a probability amplitude. c) 1. d) 0.
   e) a negative probability.

5. The probability of finding a particle in differential region dx is:

   a) $\Psi(x,t)\,dx$. b) $\Psi(x,t)^*\,dx$. c) $[\Psi(x,t)^*/\Psi(x,t)]\,dx$. d) $\Psi(x,t)^2\,dx$.
   e) $\Psi(x,t)^*\Psi(x,t)\,dx = |\Psi(x,t)|^2\,dx$.

6. "Let's play *Jeopardy*! For \$100, the answer is: It is an Hermitian operator that governs an dynamical variable in quantum mechanics."

   What is an ____________, Alex?

   a) intangible b) intaglio c) obtainable d) oblivion e) observable

7. In quantum mechanics, a dynamical variable is governed by a Hermitian operator called an observable that has an expectation value that is:

   a) the most likely value of the quantity given by the probability density: i.e., the mode of the probability density.
   b) the median value of the quantity given by the probability density.
   c) the mean value of the quantity given by the probability density.
   d) any value you happen to measure.
   e) the time average of the quantity.

8. The expectation value of operator Q for some wave function is often written:

   a) Q. b) $\rangle Q\langle$. c) $\langle Q\rangle$. d) $\langle f(Q)\rangle$. e) $f(Q)$.

9. These quantum mechanical entities (with some exceptions) must be:

   i) Single-valued (and their derivatives too).
   ii) finite (and their derivatives too).
   iii) continuous (and their derivatives too).
   iv) normalizable or square-integrable.

   They are:
   a) wave functions. b) observables. c) expectation values. d) wavelengths.
   e) wavenumbers.

10. The momentum operator in one-dimension is:

    a) $\hbar\frac{\partial}{\partial x}$. b) $\frac{\hbar}{i}\frac{\partial}{\partial x}$. c) $\frac{i}{\hbar}\frac{\partial}{\partial x}$. d) $\frac{i}{\hbar}\frac{\partial}{\partial t}$. e) $\hbar\frac{\partial}{\partial t}$.

11. "Let's play *Jeopardy*! For \$100, the answer is: It describes a fundamental limitation on the accuracy with which we can know position and momentum simultaneously."

What is ____________, Alex?

a) Tarkovsky's doubtful thesis b) Rublev's ambiguous postulate
c) Kelvin's vague zeroth law d) Schrödinger's wild hypothesis
e) Heisenberg's uncertainty principle

12. The time-independent Schrödinger equation is obtained from the full Schrödinger equation by:

a) colloquialism. b) solution for eigenfunctions. c) separation of the x and y variables.
d) separation of the space and time variables. e) expansion.

13. A system in a stationary state will:

a) not evolve in time. b) evolve in time. c) both evolve and not evolve in time.
d) occasionally evolve in time. e) violate the Heisenberg uncertainty principle.

14. For a Hermitian operator eigenproblem, one can always find (subject to some qualitifications perhaps—but which are just mathemtical hemming and hawwing) a complete set (or basis) of eigenfunctions that are:

a) independent of the x-coordinate. b) orthonormal. c) collinear. d) pathological.
e) righteous.

15. "Let's play *Jeopardy*! For $100, the answer is: If it shares the same boundary conditions as a basis set of functions and is at least piecewise continuous, then it can be expanded in the basis with a vanishing limit of the mean square error between it and the expansion."

What is a/an ____________, Alex?

a) equation b) function c) triangle d) deduction e) tax deduction

16. "Let's play *Jeopardy*! For $100, the answer is: The postulate that expansion coefficients of a wave function in the eigenstates of an observable are the probability amplitudes for wave function collapse to eigenstates of that observable."

What is ____________, Alex?

a) the special Born postulate b) the very special Born postulate c) normalizability
d) the mass-energy equivalence e) the general Born postulate

17. The expansion of a wave function in an observable's basis (or complete set of eigenstates) is

a) just a mathematical decomposition. b) useless in quantum mechanics.
c) irrelevant in quantum mechanics. d) not just a mathematical decomposition since the expansion coefficients are probability amplitudes. e) just.

18. "Let's play *Jeopardy*! For $100, the answer is: It is a process in quantum mechanics that some decline to mention, some believe to be unspeakable, some believe does not exist (though they got some explaining to do about how one ever measures anything), some believe should not exist, and that some call the fundamental perturbation (but just once per textbook)."

What is ____________, Alex?

a) the Holy b) the Unholy c) the Unnameable d) the 4th secret of the inner circle
e) wave function collapse

19. "Let's play *Jeopardy*! For $100, the answer is: A state that no macroscopic system can be in except arguably for states of Bose-Einstein condensates, superconductors, superfluids and maybe others sort of."

What is a/an ____________, Alex?

a) stationary state b) accelerating state c) state of the Union d) state of being
e) state of mind

20. A stationary state is:

a) just a special kind of classical state. b) more or less a kind of classical state.
c) voluntarily a classical state. d) was originally not a classical state, but grew into one.

e) radically unlike a classical state.

21. "Let's play *Jeopardy*! For \$100, the answer is: An equation that must hold in order for the non-relativistic Hamiltonian operator and the operator $i\hbar\partial/\partial t$ to both represent energy in the evaluation of the energy expectation value for a wave function $\Psi(x,t)$."

    What is ____________, Alex?

    a) the continuity equation b) the Laplace equation c) Newton's 2nd law
    d) Schrödinger's equation e) Hamiton's equation

22. Can the gravitational potential cause quantization of energy states?

    a) No. b) It is completely uncertain. c) Theoretically yes, but experimentally no.
    d) Experimental evidence to date (post-2001) suggests it can.
    e) In principle there is no way of telling.

23. Given the following age distribution, compute its the normalization (i.e., the factor that normalizes the distribution), mean, variance, and standard deviation. Also give the mode (i.e., the age with highest frequency) and median.

    Table: Age Distribution

Age (years)	Frequency
14	2
15	1
16	6
22	2
24	2
25	5

24. You are given a complete set of orthonormal stationary states (i.e., energy eigenfunctions) $\{\psi_n\}$ and a general wave equation $\Phi(x,t)$ for the same system: i.e., $\Phi(x,t)$ is determined by the same Hamiltonian as the complete set. Find the general expression, simplified as far as possible, for expectation value $\langle H^\ell \rangle$ where ℓ is any positive (or zero) integer. Give the special cases for $\ell = 0$, 1, and 2, and the expression for σ_E. **HINTS:** Use expansion and orthonormality. This should be a very short answer: 3 or 4 lines.

25. Classically $E \geq V_{\text{min}}$ for a particle in a conservative system.

    a) Show that this classical result must be so. **HINT:** This shouldn't be a from-first-principles proof: it should be about one line.

    b) The quantum mechanical analog is almost the same: $\bar{E} = \langle H \rangle > V_{\text{min}}$ for any state of the system considered. Note the equality $\bar{E} = \langle H \rangle = V_{\text{min}}$ never holds quantum mechanically. (There is an over-idealized exception, which we consider in part (e).) Prove the inequality. **HINTS:** The key point is to show that $\langle T \rangle > 0$ for all physically allowed states. Use integration by parts.

    c) Now show that result $\bar{E} > V_{\text{min}}$ implies $E > V_{\text{min}}$, where E is any eigen-energy of the system considered. Note the equality $E = V_{\text{min}}$ never holds quantum mechanically (except for the over-idealized system considered in part (e)). In a sense, there is no rest state for quantum mechanical particle. This lowest energy is called the zero-point energy.

    d) The $E > V_{\text{min}}$ result for an eigen-energy in turn implies a 3rd result: any ideal measurement always yields an energy greater than V_{min} Prove this by reference to a quantum mechanical postulate.

    e) There is actually an exception to $E > V_{\text{min}}$ result for an eigen-energy where $E = V_{\text{min}}$ occurs. The exception is for quantum mechanical systems with periodic boundary conditions and a constant potential. In ordinary 3-dimensional Euclidean space, the periodic boundary conditions can only occur for rings (1-dimensional systems) and sphere surfaces (2-dimensional systems) I believe. Since any real system must have a finite size in all 3 spatial dimensions, one cannot have real systems with only periodic boundary conditions. Thus, the exception to the $E > V_{\text{min}}$ result is for unrealistic over-idealized systems. Let us consider the idealized ring system as an example case. The Hamiltonian for a 1-dimensional ring with a constant potential is

$$H = -\frac{\hbar^2}{2mr^2}\frac{\partial^2}{\partial\phi^2} + V \ ,$$

where r is the ring radius, ϕ is the azimuthal angle, and V is the constant potential. Find the eigen-functions and eigen-energies for the Schrödinger equation for the ring system with periodic boundary conditions imposed. Why must one impose periodic boundary conditions on the solutions? What solution has eigen-energy $E = V_{\rm min}$?

26. If there are no internal degrees of freedom (e.g., spin) and they are normalizable, then one-particle, 1-dimensional energy eigenstates are non-degenerate. We (that is to say you) will prove this.

   a) Assume you have two degenerate 1-dimensional energy eigenstates for Hamiltonian H: ψ_1 and ψ_2. Prove that $\psi_1\psi_2' - \psi_2\psi_1'$ equals a constant where the primes indicate derivative with respect to x the spatial variable. **HINT:** Write down the eigenproblem for both ψ_1 and ψ_2 and do some multiplying and subtraction and integration.

   b) Prove that the constant in part (a) result must be zero. **HINT:** To be physically allowable eigenstates, the eigenstates must be normalizable.

   c) Show for all x that

   $$\psi_2(x) = C\psi_1(x) \ ,$$

   where C is a constant. **HINT:** The eigenproblem is a linear, homogeneous differential equation.

Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

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1 Geometrical Formulae

$$C_{\rm cir} = 2\pi r \qquad A_{\rm cir} = \pi r^2 \qquad A_{\rm sph} = 4\pi r^2 \qquad V_{\rm sph} = \frac{4}{3}\pi r^3$$

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2 Trigonometry

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right] \qquad \sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]$$

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3 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{[e^{h\nu/(kT)}-1]} \qquad B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{[e^{hc/(kT\lambda)}-1]}$$

$$B_\lambda\,d\lambda = B_\nu\,d\nu \qquad \nu\lambda = c \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$k = 1.3806505(24)\times10^{-23}\,\mathrm{J/K} \qquad c = 2.99792458\times10^{8}\,\mathrm{m}$$

$$h = 6.6260693(11)\times10^{-34}\,\mathrm{J\,s} = 4.13566743(35)\times10^{-15}\,\mathrm{eV\,s}$$

$$\hbar = \frac{h}{2\pi} = 1.05457168(18)\times10^{-34}\,\mathrm{J\,s}$$

$$hc = 12398.419\,\mathrm{eV\,Å} \approx 10^4\,\mathrm{eV\,Å} \qquad E = h\nu = \frac{hc}{\lambda} \qquad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \qquad \sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670400(40)\times 10^{-8}\,\mathrm{W/m^2/K^4}$$

$$\lambda_{\max} T = \text{constant} = \frac{hc}{kx_{\max}} \approx \frac{1.4387751\times 10^{-2}}{x_{\max}}$$

$$B_{\lambda,\mathrm{Wien}} = \frac{2hc^2}{\lambda^5}e^{-hc/(kT\lambda)} \qquad B_{\lambda,\mathrm{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu = \frac{\omega}{c} \qquad k_i = \frac{\pi}{L}n_i \quad \text{standing wave BCs} \qquad k_i = \frac{2\pi}{L}n_i \quad \text{periodic BCs}$$

$$n(k)\,dk = \frac{k^2}{\pi^2}\,dk = \pi\left(\frac{2}{c}\right)\nu^2\,d\nu = n(\nu)\,d\nu$$

$$\ln(z!) \approx \left(z+\frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots$$

$$\ln(N!) \approx N\ln(N) - N$$

$$\rho(E)\,dE = \frac{e^{-E/(kT)}}{kT}\,dE \qquad P(n) = (1-e^{-\alpha})e^{-n\alpha} \qquad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad f(x-vt) \qquad f(kx-\omega t)$$

4 Photons

$$KE = h\nu - w \qquad \Delta\lambda = \lambda_{\mathrm{scat}} - \lambda_{\mathrm{inc}} = \lambda_{\mathrm{C}}(1-\cos\theta)$$

$$\lambda_{\mathrm{C}} = \frac{h}{m_e c} = 2.426310238(16)\times 10^{-12}\,\mathrm{m} \qquad e = 1.602176487(40)\times 10^{-19}\,\mathrm{C}$$

$$m_e = 9.1093826(16)\times 10^{-31}\,\mathrm{kg} = 0.510998918(44)\,\mathrm{MeV}$$

$$m_p = 1.67262171(29)\times 10^{-27}\,\mathrm{kg} = 938.272029(80)\,\mathrm{MeV}$$

$$\ell = \frac{1}{n\sigma} \qquad \rho = \frac{e^{-s/\ell}}{\ell} \qquad \langle s^m\rangle = \ell^m m!$$

5 Matter Waves

$$\lambda = \frac{h}{p} \qquad p = \hbar k \qquad \Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k)\Psi_k(x,t)\,dk \qquad \phi(k) = \int_{-\infty}^{\infty} \Psi(x,0)\frac{e^{-ikx}}{\sqrt{2\pi}}\,dk$$

$$v_{\rm g} = \left.\frac{d\omega}{dk}\right|_{k_0} = \frac{\hbar k_0}{m} = \frac{p_0}{m} = v_{\rm clas,0}$$

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6 Non-Relativistic Quantum Mechanics

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \qquad T = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \qquad H\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$$\rho = \Psi^*\Psi \qquad \rho\,dx = \Psi^*\Psi\,dx$$

$$A\phi_i = a_i\phi_i \qquad f(x) = \sum_i c_i\phi_i \qquad \int_a^b \phi_i^*\phi_j\,dx = \delta_{ij} \qquad c_j = \int_a^b \phi_j^* f(x)\,dx \qquad [A,B] = AB - BA$$

$$P_i = |c_i|^2 \qquad \langle A\rangle = \int_{-\infty}^{\infty} \Psi^* A\Psi\,dx = \sum_i |c_i|^2 a_i \qquad H\psi = E\psi \qquad \Psi(x,t) = \psi(x)e^{-i\omega t}$$

$$p_{\rm op}\phi = \frac{\hbar}{i}\frac{\partial\phi}{\partial x} = p\phi \qquad \phi = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad \frac{\partial^2\psi}{\partial x^2} = \frac{2m}{\hbar^2}(V-E)\psi$$

$$|\Psi\rangle \qquad \langle\Psi| \qquad \langle x|\Psi\rangle = \Psi(x) \qquad \langle\vec{r}|\Psi\rangle = \Psi(\vec{r}) \qquad \langle k|\Psi\rangle = \Psi(k) \qquad \langle\Psi_i|\Psi_j\rangle = \langle\Psi_j|\Psi_i\rangle^*$$

$$\langle\phi_i|\Psi\rangle = c_i \qquad 1_{\rm op} = \sum_i |\phi_i\rangle\langle\phi_i| \qquad |\Psi\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\Psi\rangle = \sum_i c_i|\phi_i\rangle$$

$$1_{\rm op} = \int_{-\infty}^{\infty} dx\,|x\rangle\langle x| \qquad \langle\Psi_i|\Psi_j\rangle = \int_{-\infty}^{\infty} dx\,\langle\Psi|x\rangle\langle x|\Psi\rangle \qquad A_{ij} = \langle\phi_i|A|\phi_j\rangle$$

$$Pf(x) = f(-x) \qquad P\frac{df(x)}{dx} = \frac{df(-x)}{d(-x)} = -\frac{df(-x)}{dx} \qquad Pf_{\rm e/o}(x) = \pm f_{\rm e/o}(x) \qquad P\frac{df_{\rm e/o}(x)}{dx} = \mp\frac{df_{\rm e/o}(x)}{dx}$$

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7 Special Relativity

$$c = 2.99792458 \times 10^8\,\text{m/s} \approx 2.998 \times 10^8\,\text{m/s} \approx 3 \times 10^8\,\text{m/s} \approx 1\,\text{lyr/yr} \approx 1\,\text{ft/ns}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \gamma(\beta << 1) = 1 + \frac{1}{2}\beta^2 \qquad \tau = ct$$

Galilean Transformations	Lorentz Transformations
$x' = x - \beta\tau$	$x' = \gamma(x - \beta\tau)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$\tau' = \tau$	$\tau' = \gamma(\tau - \beta x)$
$\beta'_{\text{obj}} = \beta_{\text{obj}} - \beta$	$\beta'_{\text{obj}} = \dfrac{\beta_{\text{obj}} - \beta}{1 - \beta\beta_{\text{obj}}}$

$$\ell = \ell_{\text{proper}}\sqrt{1-\beta^2} \qquad \Delta\tau_{\text{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$m = \gamma m_0 \qquad p = mv = \gamma m_0 c\beta \qquad E_0 = m_0c^2 \qquad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \qquad E = \sqrt{(pc)^2 + (m_0c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2 = (\gamma - 1)m_0c^2$$

$$f = f_{\text{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating}$$

$$f(\beta << 1) = f_{\text{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\text{trans}} = f_{\text{proper}}\sqrt{1-\beta^2} \qquad f_{\text{trans}}(\beta << 1) = f_{\text{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1}\tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1}x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\text{intersection}}}{\gamma} = x'_{x\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \tau' = \frac{\tau_{\text{intersection}}}{\gamma} = \tau'_{\tau\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

$$\theta_{\text{Mink}} = \tan^{-1}(\beta)$$