Modern Physics: Physics 305, Section 1

NAME:

Homework 4: Matter Waves Homeworks are due as posted on the course web site. They are **NOT** handed in. The student reports that it is completed and receives one point for this. Solutions are already posted, but students are only permitted to look at the solutions after completion. The solutions are intended to be (but not necessarily are) super-perfect and go beyond a complete answer expected on a test.

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003 qfull 00500 3 3 0 tough math: phase velocity

1. It's embarrassing thing in elemetary quantum mechanics to admit that the momentum eigenstates or wavenumber eigenstates cannot be normalized. The two eigenstates are the same thing since a momentum eigenvalue p is equal to $\hbar k$ where k is the wavenumber eigenvalue. This means that no particle can every actually be in a wavenumber eigenstate or have a definite wavenumber eigenvalue. A particle can only ever be in superpositions of eigenstates.

   a) The normalization condition for a wave function Ψ is that

   $$\int_{-\infty}^{\infty} \Psi^*\Psi \, dx$$

   be a finite, non-zero number. If this is the case, then one can normalize Ψ by multiplying it by a constant such that one obtains

   $$1 = \int_{-\infty}^{\infty} \Psi^*\Psi \, dx \ .$$

   Since $\Psi^*\Psi$ is a probability density, normnalizability means that the probability of finding a particle somewhere is 1 as logic dictates. The wavenumber eigenstates are given by

   $$\Psi_k(x,t) = \frac{e^{i(kx-\omega t)}}{\sqrt{2\pi}} \ ,$$

   where t is time, $\omega = E/\hbar$ is angular frequency, and the $1/\sqrt{2\pi}$ is a conventional factor. Show that these eigenstates cannot be normalized.

   b) An actual general wave function for a free particle $\Psi(x,t)$ can be expanded in a superposition of wavenumber eigenstates:

   $$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k)\Psi_k(x,t) \, dk \ ,$$

   where $\phi(k)$ is a function in k-space. This $\Psi(x,t)$ is called a wave packet. Now $\phi(k)$ is actually the Fourier transform of $\Psi(x,0)$. By Plancherel's theorem $\Psi(x,0)$ is the Fourier transform of $\phi(x,0)$:

   $$\phi(k) = \int_{-\infty}^{\infty} \Psi(x,0)\frac{e^{-ikx}}{\sqrt{2\pi}} \, dk \ ,$$

   Now the phase velocity of any wavenumber eigenstate is

   $$v = \frac{\omega}{k} = \frac{E}{p} \ .$$

   But the classical velocity for a particle with energy E and momentum p is

   $$v_{\rm clas} = \frac{2E}{p} = 2v \ .$$

   There is a strange paradox here. This can be resolved by considering the concept of group velocity. Assume $\phi(k)$ is sharply peaked around k_0 for a wave packet. This actually means that $\Psi(x,0)$ is broad about the mean value of x. But this not a limitation since it turns out that the idea of a group isn't well defined for sharply peaked $\Psi(x,0)$ since the wave packet spreads out so quickly. Since $\phi(k)$ is sharply peaked around k_0 we can Taylor's series expand $w(k)$ to first order in k. Do this and write the approximate expression for the wave packet in terms of the function

   $$e^{-i(\omega_0-\omega_0' k_0)t} \ .$$

c) Now one of the rules (i.e., micro-postulates) of quantum mechanics is that the physics cannot be changed by a global phase factor in the wave equation: i.e., a factor A of the whole wave function that satisfies
$$A^*A = 1 \ .$$
Use this rule to simplify the expression obtained in the part (b) answer and show that
$$\Psi(x,t) \approx \Psi(x - \omega_0' t, 0) \ .$$

d) What is the "phase velocity" of the wave packet result in part (c) answer. This group "phase velocity" is the group velocity $v_{\rm g}$ for the wave packet. Show that the group velocity is the classical velocity one would expect classically for a particle of momentum $p_0 = \hbar k_0$.

SUGGESTED ANSWER:

a) Behold:
$$\Psi_k(x,t)^*\Psi(x,t) = \frac{1}{2\pi} \ .$$
The integral
$$\int_{-\infty}^{\infty} \Psi^*\Psi\,dx = \int_{-\infty}^{\infty} \frac{1}{2\pi}\,dx$$
clearly diverges. The wavenumber eigenstates cannot be normalized.

b) Well
$$\omega(k) \approx \omega_0 + (k - k_0)\omega_0' \ ,$$
and so
$$\Psi(x,t) \approx \int_{-\infty}^{\infty} \phi(k)\frac{e^{i(kx-\omega_0' kt)}}{\sqrt{2\pi}} e^{-i(\omega_0-\omega_0' k_0)t}\,dk = e^{-i(\omega_0-\omega_0' k_0)t}\int_{-\infty}^{\infty} \phi(k)\frac{e^{i(kx-\omega_0' kt)}}{\sqrt{2\pi}}\,dk \ .$$
Actually, for non-relativistic particles we know the expression for ω in terms of k:
$$\omega = \frac{E}{\hbar} = \frac{\hbar k^2}{2m} \ .$$
The first two derivatives are
$$\frac{d\omega}{dx} = \frac{\hbar k}{m} \qquad \text{and} \qquad \frac{d\omega^2}{dx^2} = \frac{\hbar}{m} \ .$$
Higher derivatives are zero, and thus the expansion for ω to 2nd order in k is an exact expression for ω. But expanding to 2nd order would give us a k^2 term in the in exponential that could not be taken out of the k integral. This would spoil our nice result in the part (c) answer that allows us to identify the group velocity cleanly. So expanding to 2nd order does not offer any advantages here.

c) Well
$$e^{-i(\omega_0-\omega_0' k_0)t}$$
is clearly a global phase factor: i.e., a factor of magnitude 1 that is factor of the entire wave function. Global phase factors do not affect the probability density of the wave function or any expection value or any expansion in any complete set. The fact that this global phase factor is time-dependent does not make any difference—or so I think. Global phase factors are physically irrelevant. Thus we drop $e^{-i(\omega_0-\omega_0' k_0)t}$ and obtain for the wave packet the wave function
$$\Psi(x,t) \approx \int_{-\infty}^{\infty} \phi(k)\frac{e^{i(kx-\omega_0' kt)}}{\sqrt{2\pi}}\,dk = \int_{-\infty}^{\infty} \phi(k)\frac{e^{ik(x-\omega_0' t)}}{\sqrt{2\pi}}\,dk = \Psi(x-\omega_0' t, 0) \ .$$

d) The wave packet "phase velocity" or group velocity is clearly
$$v_{\rm g} = \omega_0' = \left.\frac{d\omega}{dk}\right|_0 = \left.\frac{d}{dk}\left(\frac{\hbar k^2}{2m}\right)\right|_0 = \frac{\hbar k_0}{m} = \frac{p_0}{m} = v_{{\rm clas},0} \ .$$

The group velocity is the classical velocity for a particle of momentum p_0. Thus, the group velocity corresponds to our classical expectations for the overall velocity of the wave packet that represents a particle.

Redaction: Jeffery, 2008jan01

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003 qfull 00510 2 3 0 moderate math: Dopper shift matter wave

2. Something that is never discussed in quantum texts (as far the instructor can tell) is the non-relativistic Doppler shift for matter waves. Perhaps this because one can always just work it for oneself.

a) Given the de Broglie law

$$\lambda = \frac{h}{p}$$

and that one makes frame transformation to a frame with velocity v_0 relative to the initial frame, find the transformation expressions for λ, k, and momentum. The problem is all 1-dimensional. Use prime symbols to indicate quantities in the new frame.

b) Now show that the group velocity transforms consistently: i.e.,

$$v'_{\rm g}|_{k'_1} = v_{\rm g}|_{k_1} - v_0$$

is obtained when one evaluates the group velocity in the primed frame. The subscript "1" denotes the central wavenumber of the wave packet in this case.

SUGGESTED ANSWER:

a) Well we should define

$$p_0 = mv_0 \qquad \text{and} \qquad k_0 = \frac{p_0}{\hbar} \ .$$

We find then

$$p' = p - p_0 \qquad \text{and} \qquad k' = k - k_0 \ ,$$

where the primes indicate the new frame. Well

$$\lambda' = \frac{h}{p'} = \lambda \frac{1}{1 - p_0/p} = \lambda \frac{1}{1 - k_0/k} = \lambda \frac{1}{1 - v_0/v} \ .$$

This Doppler shift formula is not the same as the classical sound Doppler shift formula nor the relativistic Doppler shift formula. In the classical sound formula, wavelength is actually an invariant between frame—which is something that introductory textbooks seldom/never make clear.

b) Behold

$$v'_{\rm g}|_{k_1} = \left.\frac{d\omega'}{dk'}\right|_{k'_1} = \frac{\hbar k'_1}{m} = v_{\rm g}|_{k_1} - v_0 \ .$$

Redaction: Jeffery, 2008jan01

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001 qmult 00500 1 4 3 easy deducto-memory: Bohr atom

3. "Let's play *Jeopardy*! For \$100, the answer is: This model of an atom is of historical and pedagogical interest, but it is of no use in modern practical calculations and from the modern standpoint is probably misleading rather than insight-giving."

What is __________, Alex?

a) Shrödinger's model of the hydrogen atom
b) the Thomas-Fermi model of a many electron atom
c) Bohr's model of the hydrogen atom
d) the liquid drop model of the atom
e) the model hydrogen atom of Leucippos and Democritos

SUGGESTED ANSWER: (c) That the Bohr model exists is sort of an accident of nature. As far as I can see, it's interest nowadays is only historical and pedagogical. Its picture of the atom seems to me to be somewhat misleading even.

Wrong answers:

a) Schrödinger's model of the hydrogen atom is the analytic solution of his equation for that atom: this model the basis of all modern atomic physics. We never call it the Schrödinger model though: all Schrödinger's glory is subsumed in his equation itself.
b) The Thomas-Fermi atom is still very useful.
d) The liquid drop model of the nucleus is still useful for insight.
e) The answer applies to the atoms of the ancient Greeks too, but not to this particular question. Leucippos and Democritos never heard of hydrogen.

Redaction: Jeffery, 2001jan01

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001 qmult 00600 1 1 5 easy memory: Greek atomists

4. The atomic theory was first proposed by the ancient Greeks Leucippos (5th century BCE) and Democritos (5th to 4th century BCE: he reputedly lived to be 100). The term atomos means uncut: e.g., the grass is atomos. The atomists started from a philosophical position that there had to be something to give stability to nature: obviously the macroscopic world was full of change: therefore what was imperishable or uncutable—atoms—must be below perception. The modern quantum theory does indeed bear out some of their thinking. Microscopic particles can be created and destroyed, of course, but the members of a class are much more identical than macroscopic objects can ever be: fundamental particles like electrons and quarks are thought to be absolutely identical. Thus the forms particles can take are apparently eternal: a hydrogen atom today is the same in theory as one at any time in universal history.

The atomists tried to work out an atomic understanding of existence in general. For instance they constructed a cosmology using atoms that bears some resemblance to modern eternal inflationary cosmology in which there are infinitely many universes that are born out of primordial space-time foam and perhaps return to that—foam to foam. Unfortunately, the atomists got off on the wrong foot on the shape of the Earth: they were still flat Earthers when the round Earth theory was being established. Quite obviously to us, the atomists were badly non-experimental. Much of their thinking can be called rational myth. To a degree they were lucky in happening to be attracted to an essentially right idea.

The atomists were eventually stigmatized as atheists: they did not deny that gods exist, but didn't leave anything for the gods to do. This may have been their downfall. The more orthodox and popular philosophies of Plato, Aristotle, and the Stoics rejected atomism probably, among other things, for its seeming atheism. Christianity followed suit in this regard. The writings of the atomists only exist in fragments—and Democritos seems to have been as famous as Plato in his day. The Epicurean philosophers adopted atomism, but also suffered the stigmatization as atheists—and also hedonists who are, of course, the worst. But the atom idea lingered on through the centuries: Leucippos and Democritos, Epicurus, Lucretius (his surviving poem *De Rerum Natura* [*On Nature*] expounds atomism), Gassendi (17th century), Newton, Dalton: the chain is unbroken: it is not true that modern atomism has no historical or essential connection to ancient atomism.

A good account of ancient atomism can be found in David Furley's *The Greek Cosmologists*.

Now, without recurring to the top of this preamble, atomism was invented in:

a) the early 19th century. b) the 17th century by Gassendi. c) the 10th century CE. d) the 5th century CE. e) the 5th century BCE.

SUGGESTED ANSWER: (e)

Wrong answers:

a) The modern chemical atomism of Dalton was invented then.
b) I think Gassendi was about the first to advocate a revival of the atom idea.
c) Ah, yes, the Dark Ages.
d) Wrong sign.

Redaction: Jeffery, 2001jan01

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001 qfull 00500 3 5 0 tough thinking: Rutherford's nucleus

Extra keywords: (HRW-977:62P)

5. Rutherford discovered the nucleus in 1911 by bombarding metal foils with alpha particles now known to be helium nuclei (atomic mass 4.0026). An alpha particle has positive charge $2e$. He expected the alpha particles to pass right through the foils with only small deviations. Most did, but some scattered off a very large angles. Using a classical particle picture of the alpha particles and the entities they were scattering off of he came to the conclusion that atoms contained most of their mass and positive charge inside a region with a size scale of $\sim 10^{-15}\,\text{m} = 1\,\text{fm}$: this 10^{-5} times smaller than the atomic size. (Note fm stands officially for femtometer, but physicists call this unit a fermi.) Rutherford concluded that there must be a dense little core to an atom: the nucleus.

   a) Why did the alpha particles scatter off the nucleus, but not off the electrons? **HINTS:** Think dense core and diffuse cloud. What is the force causing the scattering?

   b) If the alpha particles have kinetic energy 7.5 Mev, what is their de Broglie wavelength?

   c) The closest approach of the alpha particles to the nucleus was of order 30 fm. Would the wave nature of the alpha particles have had any effect? Note the wave-particle duality was not even suspected for massive particles in 1911.

SUGGESTED ANSWER:

a) The scattering force is just the electrostatic force between charges. The electrons in the foils form a rather diffuse medium of negative charge throughout materials. They have high and low density regions, but nothing as extreme as nuclei. Thus the alpha particle just sees a comparatively mildly varying attractive electrostatic field all about them due to the electrons. Thus the electrons will cause only smallish deviations from straight-line propagation. On the other hand the nuclei, if approached closely beneath their screen of electrons, are dense centers of highly repulsive positive charge. Thus coming near a nucleus can cause a strong deviation from straight-line propagation.

   In fact Rutherford rather expected the positive charge to be diffusely spread about too. The then current model of the electron had the electrons swarming about in a positive diffuse cloud of positive charge. This model was J.J. Thomson's plum pudding model (SWT-206): the electrons were the raisins and the positive charge was the pudding. There was no useful mathematical expression of this model so far as I know: it was just a picture. In the Thomson model, the alpha particles should have been only mildly deflected all the time. Thus Rutherford's surprise when he got large scattering angles.

b) Behold

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = 5.25 \times 10^{-15}\,\text{m} = 5.25\,\text{fm} \ .$$

c) Well given that size scale of the effective scattering diffracting is about 6 times larger than the wavelength of the projectile, I would say diffraction effects are likely to be smallish. The alpha particles would probably move mostly like classical particles. Of course, the exact nature of their wave packets is uncertain. But, in fact, the particle picture for such fast projectiles generally works well. Perhaps only experience tells us that though.

Fortran Code

```
print*
en=7.5e+6*1.602e-19
hemass=4.0026*1.66e-27
hplanck=6.63e-34
wave=hplanck/sqrt(2.*hemass*en)
print*,'wave=',wave    !  5.24700488E-15
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Redaction: Jeffery, 2001jan01

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001 qfull 01100 2 5 0 moderate thinking: Bohr atom

6. In 1913, Niels Bohr presented his model of the hydrogen atom which was quickly generalized to the hydrogenic atom (i.e., the one-electron atom of any nuclear charge Z). This model correctly gives the main hydrogenic atom energy levels and consists of a mixture of quantum mechanical and classical ideas. It is historically important for showing that quantization is somehow important in atomic structure and

pedagogically it is of interest since it shows how simple theorizing can be done. But the model is, in fact, incorrect and from the modern perspective probably even misleading about the quantum mechanical nature of the atom. It is partially an accident of nature that it exists to be found. Only partially an accident since it does contain correct ingredients. And it is no accident that Bohr found it.

Bohr knew what he wanted: a model that would successfully predict the hydrogen atom spectrum which is a line spectrum showing emission at fixed frequencies. He knew from Einstein's photoelectric effect theory that electromagnetic radiation energy was quantized in amounts $h\nu$ where $h = 6.626 \times 10^{-27}$ erg s was Planck's constant (which was introduced along with the quantization notion to explain black-body radiation in 1900) and ν was frequency of the quantum of radiation. He recognized that Planck's constant had units of angular momentum. He knew from Rutherford's nuclear model of the atom that the positive charge of an atom was concentrated in region that was much smaller than the atom size and that almost all the mass of the atom was in the nucleus. He knew that there were negative electrons in atoms and they were much less massive than the nucleus. He knew the structure of atoms was stable somehow. By a judicious mixture of classical electromagnetism, classical dynamics, and quantum ideas he found his model. A more sophisticated mixture of these concepts would lead to modern quantum mechanics.

Let's see if we can follow the steps of the ideal Bohr—not the Bohr of history. **NOTE:** This a semi-classical question: Bohr, ideal or otherwise, knew nothing of the Schrödinger equation in 1913. Also note that this question uses Gaussian CGS units not MKS units.

a) Bohr thought to build the electron system about the nucleus based on the electrostatic inverse square law with the electron system supported against collapse onto the nucleus by kinetic energy. The nucleus was known to be much more massive than the electon, and so could be considered an immobile center of force. The electron—there is only one in a hydrogenic atom—was taken to be in orbit about the nucleus. Circular orbits seemed the simplest way to proceed. The electrostatic force law (in Gaussian cgs units) in scalar form for a circular orbit is

$$F = -\frac{Ze^2}{r^2} \ ,$$

where Ze is the nuclear charge, e is the electron charge, and r is the radial distance between nucleus and electron. What is the potential energy of the electron with the zero of potential energy for the electron at infinity as usual? **HINT:** If the result isn't obvious, you can get it using the work-potential energy formula:

$$V = -\int \vec{F} \cdot d\vec{r} + \text{constant} \ .$$

b) Using the centripetal force law (which is really $f = ma$ for uniform circular motion)

$$F = -\frac{mv^2}{r} \ ,$$

find an expression for the classical kinetic energy T of the electron in terms of Z, e, and r alone.

c) What is the total energy of the electron in the orbit?

d) Classically an accelerating charge radiates. This seemed well established experimentally in Bohr's time. But an orbiting electron is accelerating, and so should lose energy continuously until it collapses into the nucleus: this catastrophe obviously doesn't happen. Electrons do not collapse into the nucleus. Also they radiate only at fixed frequencies which means fixed quantum energies by Einstein's photoelectric effect theory. So Bohr postulated that the electron could only be in certain orbits which he called stationary states and that the electron in a stationary state did not radiate. Only on transitions between stationary states was there an emission of radiation in a quantum or (to use an anachronism) a photon. To get the fixed energies of emission only certain energies were allowed for the stationary states. But the emitted photons didn't come out with equally spaced energies: ergo the orbits couldn't be equally spaced in energy. From the fact that Planck's constant h has units of angular momentum, Bohr hypothesized the orbits were quantized in equally spaced amounts of angular momentum. But h was not the spacing that worked. Probably after a bit of fooling around, Bohr found that $h/(2\pi)$ or, as we now write it, $\hbar$ was the spacing that gave the right answer. The allowed angular momenta were given by

$$L = n\hbar \ ,$$

where n is any positive non-zero integer. The n was the first quantum number: we now call it the principal quantum number. It indeed determines the main spacing of the hydrogenic energy levels. Rewrite kinetic energy T in terms of $n\hbar$ and solve for an expression for r in terms n, $\hbar$, Ze^2 and m only. **HINT:** Recall the classical expression for angular momentum of particle in a circular orbit is $L = mrv$.

e) Using the formula for r from the part (d) answer write an expression for the energy of a stationary state in terms of m, c, α, Z, and n only. The c is the speed of light and the α is the fine structure constant: in Gaussian cgs units

$$\alpha = \frac{e^2}{\hbar c} \ .$$

(Real physicists use Gaussian cgs units). This formula for orbit energy turns out to be correct for the spacing of the main energy levels or shells as we would now call them. But a shell doesn't, in fact, have angular momentum $n\hbar$: it consists of has orbitals (as we now call them) with angular momenta in the range $[0, n-1]$ in units of $\hbar$ (e.g., Gr-139).

SUGGESTED ANSWER:

a) Well

$$V = -\int_{\infty}^{r} F\,dr' = -\frac{Ze^2}{r'}\bigg|_{\infty}^{r} = -\frac{Ze^2}{r} \ ,$$

where we have used the work-potential energy formula.

b) The electrostatic force is the force of the centripetal force law. Thus

$$-\frac{Ze^2}{r^2} = -\frac{mv^2}{r}$$

which immediately yields

$$T = \frac{1}{2}mv^2 = \frac{1}{2}\frac{Ze^2}{r} \ .$$

c) The total energy is

$$E = T + V = -\frac{1}{2}\frac{Ze^2}{r} \ .$$

d) Well

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{L}{mr}\right)^2 = \frac{L^2}{2mr^2} = \frac{n^2\hbar^2}{2mr^2} = \frac{1}{2}\frac{Ze^2}{r} \ ,$$

and thus

$$r = \frac{n^2\hbar^2}{mZe^2} \ .$$

The smallest radius allowed for $Z = 1$ (i.e., for hydrogen itself) is the Bohr radius given by

$$a = \frac{\hbar^2}{me^2} = \frac{\hbar}{mc\alpha} = 0.529177 \times 10^{-8}\,\mathrm{cm} \ ,$$

where

$$\alpha = \frac{e^2}{\hbar c}$$

is the fine structure constant in Gaussian cgs units.

e) Behold

$$E = -\frac{1}{2}\frac{Ze^2}{r} = -\frac{1}{2}mc^2\alpha^2\frac{Z^2}{n^2} = -E_{\mathrm{ryd}}\frac{Z^2}{n^2} \approx= -13.6\,\mathrm{eV} \times \frac{Z^2}{n^2} \ .$$

where $E_{\rm ryd} \approx 13.6\,{\rm eV}$ is the Rydberg energy. I like the Rydberg expresssion

$$E_{\rm ryd} = \frac{1}{2}mc^2\alpha^2$$

since it clearly shows the $E_{\rm ryd}$ is an energy because of the mc^2 factor: the other factors are dimensionless. So this is the form I've committed to memory. It's much better than those clusters of obscure constants one often sees.

Redaction: Jeffery, 2001jan01

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001 qfull 00150 3 5 0 tough thinking: Einstein, Runyon

Extra keywords: Bosher

7. "God does not play dice"—Einstein. Discuss.

SUGGESTED ANSWER:

Oi and the boys were having a drink—it not being lunch yet see—in a handy on 37th street where there'd be no disturbances—John Law getting paid regular see. Then in walks Harry the Horse, and the Horse says unto us: "Werner the Greek Heisenberg and Erv (Ladykiller) Schrödinger have proposed new theories of quantum mechanics in which probabilism is fundamental: the theories posit a wave-particle duality in which the wave function is a probability amplitude that tells you everything you can know in principle about the behavior of the system—the system itself does not have the definitive evolution programmed into it. This indeterminacy is manifest in atomic, electronic, and presumably, nuclear systems. But the astute minds around this pitcher have no doubt already intuited that such microscopic phenomena—since they can be observed at all—must have amplified effects in the macroscopic world. Like it or not, the brave new world is one in which all our human concerns are inevitably afflicted by a fundamental noise—there is less meaning in all our ends, rough-hew them how we will."

Now Louis the Prince de Broglie had popped in from Paris—swell guy—met him when I was over there learning at Argonne that I'd had enough of soljering. Louis, swell guy—but you never know where you are with him—it gives me the willies the way he diffracts through doors—he says that the Greek and Ladykiller pummeled Al the Brain Einstein at the latest Solvay conference with Viking Neils Bohr doing some goring too. And then he says: "La lumière et la matière sont le meme chose et tout que fûrent solide, fonde dans la vent."

Not even from the Prince was the Bosher taking any of that. "Tray bong, tray, tray bong," says Oi and "Mais merde-moi, la realité hier, aujourd'hui, et demain: tout sont que fûrent." "Bocheur! Bocheur!" (this is the Prince.) "Merde-moi, je sais, monsieur le Prince est quelle erreur et je-ne-sais croissant." And that stopped him: I always get the last word when I speak with him en Français: he gets that deer-in-the-headlight look.

Jimmy Durante—he's on Broadway see—says "All of our journeyings end in this: we arrive at the beginning and know that place for the first time." And Franco Parmese—Italian, Camorra see—says "We live by the faith and duty that we learn in our homes." Then Nick Charles—swell guy Nick: he'd buy a man a drink or lend him a fin—or even a grand if you had to get out of town sudden-like see—then Nick—he's Greek really—not like Heisenberg—his Pa, old Nick Chrysos, came from Athens or Sparta or Delphi—then Nick said that Hem said—who probably heard it from his great aunt Hepzibah—that what a man needs is grace under pressure—which sadly, in the words of Father Pat, led me to harbor impure thoughts.

Breaking this amusing up, Horseface said

$$\text{“}\; H\psi(x) = \left[\frac{p^2}{2m} + V(x)\right]\psi(x) = E\psi(x)$$

is the time-independent Schrödinger equation whose solutions are the stationary states and these are the bedrock of atomic, molecular, solid state, and nuclear structure." Well I wasn't taking any of that guff neither: "Father Pat told me to salute the flag, honor the Holy Mother Church, and

$$\vec F_{\rm net} = m\vec a\ ,$$

and you can take your Hamiltonian and stuff it up your nose." Horseflesh came back with deprecatory remarks on the relative unacquaintance of my parents—so I lays him out—Dempsey couldn't have done it sweeter.

That fracas could have become obsequious—Horse-maneuver not being the only Marx brother in the den of inquiry—the Prince was already behind the Moon or maybe under the table—like I aforesaid, you never know with the Prince—but it was all grand for me: I'm the Bosher: seul contre tout.

But Nick nabbing my elbow suggested Chinatown for Chinese and maybe talk on fetalism in Proust—and after some thought, it seemed the better part of velour—and we decapitated pronto out of there and caught the line at the corner.

Trolleying we was on Broadway: "You know," Nick said "it's just this: you play square and take your chances and pay up in style at the end." Grand I said to myself: grand for a fine gent like Nick—I'm happy for his good luck—got a rich wife owns a timber mine in Organ. But for me, the game's been fixed—though I hang my hat on a star.

Redaction: Jeffery, 2001jan01

Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

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1 Geometrical Formulae

$$C_{\rm cir} = 2\pi r \qquad A_{\rm cir} = \pi r^2 \qquad A_{\rm sph} = 4\pi r^2 \qquad V_{\rm sph} = \frac{4}{3}\pi r^3$$

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2 Trigonometry

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos^2\theta = \frac{1}{2}[1+\cos(2\theta)] \qquad \sin^2\theta = \frac{1}{2}[1-\cos(2\theta)] \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right] \qquad \sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]$$

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3 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{[e^{h\nu/(kT)}-1]} \qquad B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{[e^{hc/(kT\lambda)}-1]}$$

$$B_\lambda\,d\lambda = B_\nu\,d\nu \qquad \nu\lambda = c \qquad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$k = 1.3806505(24)\times10^{-23}\,\mathrm{J/K} \qquad c = 2.99792458\times10^{8}\,\mathrm{m}$$

$$h = 6.6260693(11)\times10^{-34}\,\mathrm{J\,s} = 4.13566743(35)\times10^{-15}\,\mathrm{eV\,s}$$

$$\hbar = \frac{h}{2\pi} = 1.05457168(18)\times10^{-34}\,\mathrm{J\,s}$$

$$hc = 12398.419\,\mathrm{eV\,\AA} \approx 10^4\,\mathrm{eV\,\AA} \qquad E = h\nu = \frac{hc}{\lambda} \qquad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \qquad \sigma = \frac{2\pi^5}{15}\frac{k^4}{c^2h^3} = 5.670400(40) \times 10^{-8}\,\mathrm{W/m^2/K^4}$$

$$\lambda_{\max} T = \text{constant} = \frac{hc}{kx_{\max}} \approx \frac{1.4387751 \times 10^{-2}}{x_{\max}}$$

$$B_{\lambda,\mathrm{Wien}} = \frac{2hc^2}{\lambda^5} e^{-hc/(kT\lambda)} \qquad B_{\lambda,\mathrm{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c}\nu = \frac{\omega}{c} \qquad k_i = \frac{\pi}{L}n_i \quad \text{standing wave BCs} \qquad k_i = \frac{2\pi}{L}n_i \quad \text{periodic BCs}$$

$$n(k)\,dk = \frac{k^2}{\pi^2}\,dk = \pi\left(\frac{2}{c}\right)\nu^2\,d\nu = n(\nu)\,d\nu$$

$$\ln(z!) \approx \left(z + \frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \ldots$$

$$\ln(N!) \approx N\ln(N) - N$$

$$\rho(E)\,dE = \frac{e^{-E/(kT)}}{kT}\,dE \qquad P(n) = (1 - e^{-\alpha})e^{-n\alpha} \qquad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad f(x - vt) \qquad f(kx - \omega t)$$

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4 Photons

$$KE = h\nu - w \qquad \Delta\lambda = \lambda_{\mathrm{scat}} - \lambda_{\mathrm{inc}} = \lambda_{\mathrm{C}}(1 - \cos\theta)$$

$$\lambda_{\mathrm{C}} = \frac{h}{m_e c} = 2.426310238(16) \times 10^{-12}\,\mathrm{m} \qquad e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$m_e = 9.1093826(16) \times 10^{-31}\,\mathrm{kg} = 0.510998918(44)\,\mathrm{MeV}$$

$$m_p = 1.67262171(29) \times 10^{-27}\,\mathrm{kg} = 938.272029(80)\,\mathrm{MeV}$$

$$\ell = \frac{1}{n\sigma} \qquad \rho = \frac{e^{-s/\ell}}{\ell} \qquad \langle s^m \rangle = \ell^m m!$$

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5 Matter Waves

$$\lambda = \frac{h}{p} \qquad p = \hbar k \qquad \Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Psi(x,t) = \int_{-\infty}^{\infty} \phi(k)\Psi_k(x,t)\, dk \qquad \phi(k) = \int_{-\infty}^{\infty} \Psi(x,0)\frac{e^{-ikx}}{\sqrt{2\pi}}\, dk$$

$$v_{\mathrm{g}} = \left.\frac{d\omega}{dk}\right|_{k_0} = \frac{\hbar k_0}{m} = \frac{p_0}{m} = v_{\mathrm{clas},0}$$

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6 Non-Relativistic Quantum Mechanics

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \qquad T = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \qquad H\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$$\rho = \Psi^*\Psi \qquad \rho\, dx = \Psi^*\Psi\, dx$$

$$A\phi_i = a_i\phi_i \qquad f(x) = \sum_i c_i\phi_i \qquad \int_a^b \phi_i^*\phi_j\, dx = \delta_{ij} \qquad c_j = \int_a^b \phi_j^* f(x)\, dx \qquad [A,B] = AB - BA$$

$$P_i = |c_i|^2 \qquad \langle A\rangle = \int_{-\infty}^{\infty} \Psi^* A\Psi\, dx = \sum_i |c_i|^2 a_i \qquad H\psi = E\psi \qquad \Psi(x,t) = \psi(x)e^{-i\omega t}$$

$$p_{\mathrm{op}}\phi = \frac{\hbar}{i}\frac{\partial\phi}{\partial x} = p\phi \qquad \phi = \frac{e^{ikx}}{\sqrt{2\pi}} \qquad \frac{\partial^2\psi}{\partial x^2} = \frac{2m}{\hbar^2}(V-E)\psi$$

$$|\Psi\rangle \qquad \langle\Psi| \qquad \langle x|\Psi\rangle = \Psi(x) \qquad \langle\vec{r}|\Psi\rangle = \Psi(\vec{r}) \qquad \langle k|\Psi\rangle = \Psi(k) \qquad \langle\Psi_i|\Psi_j\rangle = \langle\Psi_j|\Psi_i\rangle^*$$

$$\langle\phi_i|\Psi\rangle = c_i \qquad 1_{\mathrm{op}} = \sum_i |\phi_i\rangle\langle\phi_i| \qquad |\Psi\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\Psi\rangle = \sum_i c_i|\phi_i\rangle$$

$$1_{\mathrm{op}} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x| \qquad \langle\Psi_i|\Psi_j\rangle = \int_{-\infty}^{\infty} dx\, \langle\Psi|x\rangle\langle x|\Psi\rangle \qquad A_{ij} = \langle\phi_i|A|\phi_j\rangle$$

$$Pf(x) = f(-x) \qquad P\frac{df(x)}{dx} = \frac{df(-x)}{d(-x)} = -\frac{df(-x)}{dx} \qquad Pf_{\mathrm{e/o}}(x) = \pm f_{\mathrm{e/o}}(x) \qquad P\frac{df_{\mathrm{e/o}}(x)}{dx} = \mp\frac{df_{\mathrm{e/o}}(x)}{dx}$$

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7 Special Relativity

$$c = 2.99792458\times 10^8\,\mathrm{m/s} \approx 2.998\times 10^8\,\mathrm{m/s} \approx 3\times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns}$$

$$\beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \gamma(\beta << 1) = 1 + \frac{1}{2}\beta^2 \qquad \tau = ct$$

Galilean Transformations	Lorentz Transformations
$x' = x - \beta\tau$	$x' = \gamma(x - \beta\tau)$
$y' = y$	$y' = y$
$z' = z$	$z' = z$
$\tau' = \tau$	$\tau' = \gamma(\tau - \beta x)$
$\beta'_{\text{obj}} = \beta_{\text{obj}} - \beta$	$\beta'_{\text{obj}} = \dfrac{\beta_{\text{obj}} - \beta}{1 - \beta\beta_{\text{obj}}}$

$$\ell = \ell_{\text{proper}}\sqrt{1-\beta^2} \qquad \Delta\tau_{\text{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$m = \gamma m_0 \qquad p = mv = \gamma m_0 c\beta \qquad E_0 = m_0c^2 \qquad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \qquad E = \sqrt{(pc)^2 + (m_0c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2 = (\gamma - 1)m_0c^2$$

$$f = f_{\text{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating}$$

$$f(\beta << 1) = f_{\text{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\text{trans}} = f_{\text{proper}}\sqrt{1-\beta^2} \qquad f_{\text{trans}}(\beta << 1) = f_{\text{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1}\tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1}x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\text{intersection}}}{\gamma} = x'_{x\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \qquad \tau' = \frac{\tau_{\text{intersection}}}{\gamma} = \tau'_{\tau\text{ scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

$$\theta_{\text{Mink}} = \tan^{-1}(\beta)$$