

## Modern Physics: Physics 305, Section 1

### NAME:

**Homework 4: Matter Waves** Homeworks are due as posted on the course web site. They are **NOT** handed in. The student reports that it is completed and receives one point for this. Solutions are already posted, but students are only permitted to look at the solutions after completion. The solutions are intended to be (but not necessarily are) super-perfect and go beyond a complete answer expected on a test.

1. It's embarrassing thing in elementary quantum mechanics to admit that the momentum eigenstates or wavenumber eigenstates cannot be normalized. The two eigenstates are the same thing since a momentum eigenvalue  $p$  is equal to  $\hbar k$  where  $k$  is the wavenumber eigenvalue. This means that no particle can ever actually be in a wavenumber eigenstate or have a definite wavenumber eigenvalue. A particle can only ever be in superpositions of eigenstates.

- a) The normalization condition for a wave function  $\Psi$  is that

$$\int_{-\infty}^{\infty} \Psi^* \Psi dx$$

be a finite, non-zero number. If this is the case, then one can normalize  $\Psi$  by multiplying it by a constant such that one obtains

$$1 = \int_{-\infty}^{\infty} \Psi^* \Psi dx .$$

Since  $\Psi^* \Psi$  is a probability density, normalizability means that the probability of finding a particle somewhere is 1 as logic dictates. The wavenumber eigenstates are given by

$$\Psi_k(x, t) = \frac{e^{i(kx - \omega t)}}{\sqrt{2\pi}} ,$$

where  $t$  is time,  $\omega = E/\hbar$  is angular frequency, and the  $1/\sqrt{2\pi}$  is a conventional factor. Show that these eigenstates cannot be normalized.

- b) An actual general wave function for a free particle  $\Psi(x, t)$  can be expanded in a superposition of wavenumber eigenstates:

$$\Psi(x, t) = \int_{-\infty}^{\infty} \phi(k) \Psi_k(x, t) dk ,$$

where  $\phi(k)$  is a function in  $k$ -space. This  $\Psi(x, t)$  is called a wave packet. Now  $\phi(k)$  is actually the Fourier transform of  $\Psi(x, 0)$ . By Plancherel's theorem  $\Psi(x, 0)$  is the Fourier transform of  $\phi(k, 0)$ :

$$\phi(k) = \int_{-\infty}^{\infty} \Psi(x, 0) \frac{e^{-ikx}}{\sqrt{2\pi}} dx ,$$

Now the phase velocity of any wavenumber eigenstate is

$$v = \frac{\omega}{k} = \frac{E}{p} .$$

But the classical velocity for a particle with energy  $E$  and momentum  $p$  is

$$v_{\text{clas}} = \frac{2E}{p} = 2v .$$

There is a strange paradox here. This can be resolved by considering the concept of group velocity. Assume  $\phi(k)$  is sharply peaked around  $k_0$  for a wave packet. This actually means that  $\Psi(x, 0)$  is broad about the mean value of  $x$ . But this not a limitation since it turns out that the idea of a group isn't well defined for sharply peaked  $\Psi(x, 0)$  since the wave packet spreads out so quickly. Since  $\phi(k)$  is sharply peaked around  $k_0$  we can Taylor's series expand  $w(k)$  to first order in  $k$ . Do this and write the approximate expression for the wave packet in terms of the function

$$e^{-i(\omega_0 - \omega'_0 k_0)t} .$$

- c) Now one of the rules (i.e., micro-postulates) of quantum mechanics is that the physics cannot be changed by a global phase factor in the wave equation: i.e., a factor  $A$  of the whole wave function that satisfies

$$A^* A = 1 .$$

Use this rule to simplify the expression obtained in the part (b) answer and show that

$$\Psi(x, t) \approx \Psi(x - \omega'_0 t, 0) .$$

- d) What is the “phase velocity” of the wave packet result in part (c) answer. This group “phase velocity” is the group velocity  $v_g$  for the wave packet. Show that the group velocity is the classical velocity one would expect classically for a particle of momentum  $p_0 = \hbar k_0$ .
2. Something that is never discussed in quantum texts (as far the instructor can tell) is the non-relativistic Doppler shift for matter waves. Perhaps this because one can always just work it for oneself.

- a) Given the de Broglie law

$$\lambda = \frac{h}{p}$$

and that one makes frame transformation to a frame with velocity  $v_0$  relative to the initial frame, find the transformation expressions for  $\lambda$ ,  $k$ , and momentum. The problem is all 1-dimensional. Use prime symbols to indicate quantities in the new frame.

- b) Now show that the group velocity transforms consistently: i.e.,

$$v'_g|_{k'_1} = v_g|_{k_1} - v_0$$

is obtained when one evaluates the group velocity in the primed frame. The subscript “1” denotes the central wavenumber of the wave packet in this case.

3. “Let’s play *Jeopardy!* For \$100, the answer is: This model of an atom is of historical and pedagogical interest, but it is of no use in modern practical calculations and from the modern standpoint is probably misleading rather than insight-giving.”

What is \_\_\_\_\_, Alex?

- a) Shrödinger’s model of the hydrogen atom  
 b) the Thomas-Fermi model of a many electron atom      c) Bohr’s model of the hydrogen atom  
 d) the liquid drop model of the atom  
 e) the model hydrogen atom of Leucippus and Democritos
4. The atomic theory was first proposed by the ancient Greeks Leucippus (5th century BCE) and Democritus (5th to 4th century BCE: he reputedly lived to be 100). The term atomos means uncut: e.g., the grass is atomos. The atomists started from a philosophical position that there had to be something to give stability to nature: obviously the macroscopic world was full of change: therefore what was imperishable or uncuttable—atoms—must be below perception. The modern quantum theory does indeed bear out some of their thinking. Microscopic particles can be created and destroyed, of course, but the members of a class are much more identical than macroscopic objects can ever be: fundamental particles like electrons and quarks are thought to be absolutely identical. Thus the forms particles can take are apparently eternal: a hydrogen atom today is the same in theory as one at any time in universal history.

The atomists tried to work out an atomic understanding of existence in general. For instance they constructed a cosmology using atoms that bears some resemblance to modern eternal inflationary cosmology in which there are infinitely many universes that are born out of primordial space-time foam and perhaps return to that—foam to foam. Unfortunately, the atomists got off on the wrong foot on the shape of the Earth: they were still flat Earthers when the round Earth theory was being established. Quite obviously to us, the atomists were badly non-experimental. Much of their thinking can be called rational myth. To a degree they were lucky in happening to be attracted to an essentially right idea.

The atomists were eventually stigmatized as atheists: they did not deny that gods exist, but didn’t leave anything for the gods to do. This may have been their downfall. The more orthodox

and popular philosophies of Plato, Aristotle, and the Stoics rejected atomism probably, among other things, for its seeming atheism. Christianity followed suit in this regard. The writings of the atomists only exist in fragments—and Democritus seems to have been as famous as Plato in his day. The Epicurean philosophers adopted atomism, but also suffered the stigmatization as atheists—and also hedonists who are, of course, the worst. But the atom idea lingered on through the centuries: Leucippus and Democritus, Epicurus, Lucretius (his surviving poem *De Rerum Natura* [*On Nature*] expounds atomism), Gassendi (17th century), Newton, Dalton: the chain is unbroken: it is not true that modern atomism has no historical or essential connection to ancient atomism.

A good account of ancient atomism can be found in David Furley's *The Greek Cosmologists*.

Now, without recurring to the top of this preamble, atomism was invented in:

- a) the early 19th century.                      b) the 17th century by Gassendi.            c) the 10th century CE.  
 d) the 5th century CE.                      e) the 5th century BCE.
5. Rutherford discovered the nucleus in 1911 by bombarding metal foils with alpha particles now known to be helium nuclei (atomic mass 4.0026). An alpha particle has positive charge  $2e$ . He expected the alpha particles to pass right through the foils with only small deviations. Most did, but some scattered off a very large angles. Using a classical particle picture of the alpha particles and the entities they were scattering off of he came to the conclusion that atoms contained most of their mass and positive charge inside a region with a size scale of  $\sim 10^{-15} \text{ m} = 1 \text{ fm}$ : this  $10^{-5}$  times smaller than the atomic size. (Note fm stands officially for femtometer, but physicists call this unit a fermi.) Rutherford concluded that there must be a dense little core to an atom: the nucleus.
- a) Why did the alpha particles scatter off the nucleus, but not off the electrons? **HINTS:** Think dense core and diffuse cloud. What is the force causing the scattering?
- b) If the alpha particles have kinetic energy 7.5 Mev, what is their de Broglie wavelength?
- c) The closest approach of the alpha particles to the nucleus was of order 30 fm. Would the wave nature of the alpha particles have had any effect? Note the wave-particle duality was not even suspected for massive particles in 1911.
6. In 1913, Niels Bohr presented his model of the hydrogen atom which was quickly generalized to the hydrogenic atom (i.e., the one-electron atom of any nuclear charge  $Z$ ). This model correctly gives the main hydrogenic atom energy levels and consists of a mixture of quantum mechanical and classical ideas. It is historically important for showing that quantization is somehow important in atomic structure and pedagogically it is of interest since it shows how simple theorizing can be done. But the model is, in fact, incorrect and from the modern perspective probably even misleading about the quantum mechanical nature of the atom. It is partially an accident of nature that it exists to be found. Only partially an accident since it does contain correct ingredients. And it is no accident that Bohr found it.

Bohr knew what he wanted: a model that would successfully predict the hydrogen atom spectrum which is a line spectrum showing emission at fixed frequencies. He knew from Einstein's photoelectric effect theory that electromagnetic radiation energy was quantized in amounts  $h\nu$  where  $h = 6.626 \times 10^{-27} \text{ erg s}$  was Planck's constant (which was introduced along with the quantization notion to explain black-body radiation in 1900) and  $\nu$  was frequency of the quantum of radiation. He recognized that Planck's constant had units of angular momentum. He knew from Rutherford's nuclear model of the atom that the positive charge of an atom was concentrated in region that was much smaller than the atom size and that almost all the mass of the atom was in the nucleus. He knew that there were negative electrons in atoms and they were much less massive than the nucleus. He knew the structure of atoms was stable somehow. By a judicious mixture of classical electromagnetism, classical dynamics, and quantum ideas he found his model. A more sophisticated mixture of these concepts would lead to modern quantum mechanics.

Let's see if we can follow the steps of the ideal Bohr—not the Bohr of history. **NOTE:** This a semi-classical question: Bohr, ideal or otherwise, knew nothing of the Schrödinger equation in 1913. Also note that this question uses Gaussian CGS units not MKS units.

- a) Bohr thought to build the electron system about the nucleus based on the electrostatic inverse square law with the electron system supported against collapse onto the nucleus by kinetic energy.

The nucleus was known to be much more massive than the electron, and so could be considered an immobile center of force. The electron—there is only one in a hydrogenic atom—was taken to be in orbit about the nucleus. Circular orbits seemed the simplest way to proceed. The electrostatic force law (in Gaussian cgs units) in scalar form for a circular orbit is

$$F = -\frac{Ze^2}{r^2},$$

where  $Ze$  is the nuclear charge,  $e$  is the electron charge, and  $r$  is the radial distance between nucleus and electron. What is the potential energy of the electron with the zero of potential energy for the electron at infinity as usual? **HINT:** If the result isn't obvious, you can get it using the work-potential energy formula:

$$V = -\int \vec{F} \cdot d\vec{r} + \text{constant}.$$

- b) Using the centripetal force law (which is really  $f = ma$  for uniform circular motion)

$$F = -\frac{mv^2}{r},$$

find an expression for the classical kinetic energy  $T$  of the electron in terms of  $Z$ ,  $e$ , and  $r$  alone.

- c) What is the total energy of the electron in the orbit?
- d) Classically an accelerating charge radiates. This seemed well established experimentally in Bohr's time. But an orbiting electron is accelerating, and so should lose energy continuously until it collapses into the nucleus: this catastrophe obviously doesn't happen. Electrons do not collapse into the nucleus. Also they radiate only at fixed frequencies which means fixed quantum energies by Einstein's photoelectric effect theory. So Bohr postulated that the electron could only be in certain orbits which he called stationary states and that the electron in a stationary state did not radiate. Only on transitions between stationary states was there an emission of radiation in a quantum or (to use an anachronism) a photon. To get the fixed energies of emission only certain energies were allowed for the stationary states. But the emitted photons didn't come out with equally spaced energies: ergo the orbits couldn't be equally spaced in energy. From the fact that Planck's constant  $h$  has units of angular momentum, Bohr hypothesized the orbits were quantized in equally spaced amounts of angular momentum. But  $h$  was not the spacing that worked. Probably after a bit of fooling around, Bohr found that  $h/(2\pi)$  or, as we now write it,  $\hbar$  was the spacing that gave the right answer. The allowed angular momenta were given by

$$L = n\hbar,$$

where  $n$  is any positive non-zero integer. The  $n$  was the first quantum number: we now call it the principal quantum number. It indeed determines the main spacing of the hydrogenic energy levels. Rewrite kinetic energy  $T$  in terms of  $n\hbar$  and solve for an expression for  $r$  in terms  $n$ ,  $\hbar$ ,  $Ze^2$  and  $m$  only. **HINT:** Recall the classical expression for angular momentum of particle in a circular orbit is  $L = mrv$ .

- e) Using the formula for  $r$  from the part (d) answer write an expression for the energy of a stationary state in terms of  $m$ ,  $c$ ,  $\alpha$ ,  $Z$ , and  $n$  only. The  $c$  is the speed of light and the  $\alpha$  is the fine structure constant: in Gaussian cgs units

$$\alpha = \frac{e^2}{\hbar c}.$$

(Real physicists use Gaussian cgs units). This formula for orbit energy turns out to be correct for the spacing of the main energy levels or shells as we would now call them. But a shell doesn't, in fact, have angular momentum  $n\hbar$ : it consists of has orbitals (as we now call them) with angular momenta in the range  $[0, n-1]$  in units of  $\hbar$  (e.g., Gr-139).

## Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

### 1 Geometrical Formulae

$$C_{\text{cir}} = 2\pi r \quad A_{\text{cir}} = \pi r^2 \quad A_{\text{sph}} = 4\pi r^2 \quad V_{\text{sph}} = \frac{4}{3}\pi r^3$$

### 2 Trigonometry

$$\frac{x}{r} = \cos \theta \quad \frac{y}{r} = \sin \theta \quad \frac{y}{x} = \tan \theta \quad \cos^2 \theta + \sin^2 \theta = 1$$

$$\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b) \quad \cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$$

$$\cos^2 \theta = \frac{1}{2}[1 + \cos(2\theta)] \quad \sin^2 \theta = \frac{1}{2}[1 - \cos(2\theta)] \quad \sin(2\theta) = 2 \sin(\theta) \cos(\theta)$$

$$\cos(a) \cos(b) = \frac{1}{2}[\cos(a - b) + \cos(a + b)] \quad \sin(a) \sin(b) = \frac{1}{2}[\cos(a - b) - \cos(a + b)]$$

$$\sin(a) \cos(b) = \frac{1}{2}[\sin(a - b) + \sin(a + b)]$$

### 3 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{[e^{h\nu/(kT)} - 1]} \quad B_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{[e^{hc/(kT\lambda)} - 1]}$$

$$B_\lambda d\lambda = B_\nu d\nu \quad \nu\lambda = c \quad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$k = 1.3806505(24) \times 10^{-23} \text{ J/K} \quad c = 2.99792458 \times 10^8 \text{ m/s}$$

$$h = 6.6260693(11) \times 10^{-34} \text{ J s} = 4.13566743(35) \times 10^{-15} \text{ eV s}$$

$$\hbar = \frac{h}{2\pi} = 1.05457168(18) \times 10^{-34} \text{ J s}$$

$$hc = 12398.419 \text{ eV \AA} \approx 10^4 \text{ eV \AA} \quad E = h\nu = \frac{hc}{\lambda} \quad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \quad \sigma = \frac{2\pi^5}{15} \frac{k^4}{c^2 h^3} = 5.670400(40) \times 10^{-8} \text{ W/m}^2/\text{K}^4$$

$$\lambda_{\max} T = \text{constant} = \frac{hc}{kx_{\max}} \approx \frac{1.4387751 \times 10^{-2}}{x_{\max}}$$

$$B_{\lambda, \text{Wien}} = \frac{2hc^2}{\lambda^5} e^{-hc/(kT\lambda)} \quad B_{\lambda, \text{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c} \nu = \frac{\omega}{c} \quad k_i = \frac{\pi}{L} n_i \quad \text{standing wave BCs} \quad k_i = \frac{2\pi}{L} n_i \quad \text{periodic BCs}$$

$$n(k) dk = \frac{k^2}{\pi^2} dk = \pi \left( \frac{2}{c} \right) \nu^2 d\nu = n(\nu) d\nu$$

$$\ln(z!) \approx \left( z + \frac{1}{2} \right) \ln(z) - z + \frac{1}{2} \ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \dots$$

$$\ln(N!) \approx N \ln(N) - N$$

$$\rho(E) dE = \frac{e^{-E/(kT)}}{kT} dE \quad P(n) = (1 - e^{-\alpha}) e^{-n\alpha} \quad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad f(x - vt) \quad f(kx - \omega t)$$

#### 4 Photons

$$KE = h\nu - w \quad \Delta\lambda = \lambda_{\text{scat}} - \lambda_{\text{inc}} = \lambda_C(1 - \cos\theta)$$

$$\lambda_C = \frac{h}{m_e c} = 2.426310238(16) \times 10^{-12} \text{ m} \quad e = 1.602176487(40) \times 10^{-19} \text{ C}$$

$$m_e = 9.1093826(16) \times 10^{-31} \text{ kg} = 0.510998918(44) \text{ MeV}$$

$$m_p = 1.67262171(29) \times 10^{-27} \text{ kg} = 938.272029(80) \text{ MeV}$$

$$\ell = \frac{1}{n\sigma} \quad \rho = \frac{e^{-s/\ell}}{\ell} \quad \langle s^m \rangle = \ell^m m!$$

#### 5 Matter Waves

$$\lambda = \frac{h}{p} \quad p = \hbar k \quad \Delta x \Delta p \geq \frac{\hbar}{2} \quad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Psi(x, t) = \int_{-\infty}^{\infty} \phi(k) \Psi_k(x, t) dk \quad \phi(k) = \int_{-\infty}^{\infty} \Psi(x, 0) \frac{e^{-ikx}}{\sqrt{2\pi}} dx$$

$$v_g = \left. \frac{d\omega}{dk} \right|_{k_0} = \frac{\hbar k_0}{m} = \frac{p_0}{m} = v_{\text{clas},0}$$

## 6 Non-Relativistic Quantum Mechanics

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \quad T = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \quad H\Psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

$$\rho = \Psi^* \Psi \quad \rho dx = \Psi^* \Psi dx$$

$$A\phi_i = a_i \phi_i \quad f(x) = \sum_i c_i \phi_i \quad \int_a^b \phi_i^* \phi_j dx = \delta_{ij} \quad c_j = \int_a^b \phi_j^* f(x) dx \quad [A, B] = AB - BA$$

$$P_i = |c_i|^2 \quad \langle A \rangle = \int_{-\infty}^{\infty} \Psi^* A \Psi dx = \sum_i |c_i|^2 a_i \quad H\psi = E\psi \quad \Psi(x, t) = \psi(x) e^{-i\omega t}$$

$$p_{\text{op}} \phi = \frac{\hbar}{i} \frac{\partial \phi}{\partial x} = p\phi \quad \phi = \frac{e^{ikx}}{\sqrt{2\pi}} \quad \frac{\partial^2 \psi}{\partial x^2} = \frac{2m}{\hbar^2} (V - E)\psi$$

$$|\Psi\rangle \quad \langle \Psi| \quad \langle x|\Psi\rangle = \Psi(x) \quad \langle \vec{r}|\Psi\rangle = \Psi(\vec{r}) \quad \langle k|\Psi\rangle = \Psi(k) \quad \langle \Psi_i|\Psi_j\rangle = \langle \Psi_j|\Psi_i\rangle^*$$

$$\langle \phi_i|\Psi\rangle = c_i \quad 1_{\text{op}} = \sum_i |\phi_i\rangle \langle \phi_i| \quad |\Psi\rangle = \sum_i |\phi_i\rangle \langle \phi_i|\Psi\rangle = \sum_i c_i |\phi_i\rangle$$

$$1_{\text{op}} = \int_{-\infty}^{\infty} dx |x\rangle \langle x| \quad \langle \Psi_i|\Psi_j\rangle = \int_{-\infty}^{\infty} dx \langle \Psi_i|x\rangle \langle x|\Psi_j\rangle \quad A_{ij} = \langle \phi_i|A|\phi_j\rangle$$

$$Pf(x) = f(-x) \quad P \frac{df(x)}{dx} = \frac{df(-x)}{d(-x)} = -\frac{df(-x)}{dx} \quad Pf_{e/o}(x) = \pm f_{e/o}(x) \quad P \frac{df_{e/o}(x)}{dx} = \mp \frac{df_{e/o}(x)}{dx}$$

## 7 Special Relativity

$$c = 2.99792458 \times 10^8 \text{ m/s} \approx 2.998 \times 10^8 \text{ m/s} \approx 3 \times 10^8 \text{ m/s} \approx 1 \text{ ly/yr} \approx 1 \text{ ft/ns}$$

$$\beta = \frac{v}{c} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \gamma(\beta \ll 1) = 1 + \frac{1}{2}\beta^2 \quad \tau = ct$$

Galilean Transformations

$$\begin{aligned} x' &= x - \beta\tau \\ y' &= y \\ z' &= z \\ \tau' &= \tau \end{aligned}$$

$$\beta'_{\text{obj}} = \beta_{\text{obj}} - \beta$$

Lorentz Transformations

$$\begin{aligned} x' &= \gamma(x - \beta\tau) \\ y' &= y \\ z' &= z \\ \tau' &= \gamma(\tau - \beta x) \end{aligned}$$

$$\beta'_{\text{obj}} = \frac{\beta_{\text{obj}} - \beta}{1 - \beta\beta_{\text{obj}}}$$

$$\ell = \ell_{\text{proper}}\sqrt{1-\beta^2} \quad \Delta\tau_{\text{proper}} = \Delta\tau\sqrt{1-\beta^2}$$

$$m = \gamma m_0 \quad p = mv = \gamma m_0 c \beta \quad E_0 = m_0 c^2 \quad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \quad E = \sqrt{(pc)^2 + (m_0 c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0 c^2)^2} - m_0 c^2 = (\gamma - 1)m_0 c^2$$

$$f = f_{\text{proper}}\sqrt{\frac{1-\beta}{1+\beta}} \quad \text{for source and detector separating}$$

$$f(\beta \ll 1) = f_{\text{proper}}\left(1 - \beta + \frac{1}{2}\beta^2\right)$$

$$f_{\text{trans}} = f_{\text{proper}}\sqrt{1-\beta^2} \quad f_{\text{trans}}(\beta \ll 1) = f_{\text{proper}}\left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1}\tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1}x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\text{intersection}}}{\gamma} = x'_{\text{scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}} \quad \tau' = \frac{\tau_{\text{intersection}}}{\gamma} = \tau'_{\text{scale}}\sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

$$\theta_{\text{Mink}} = \tan^{-1}(\beta)$$