

Modern Physics 305: 2nd Exam
2008 April 23 Wednesday

NAME:

Instructions: There are 10 multiple-choice questions each worth 2 marks for a total of 20 marks altogether. Choose the **BEST** answer, completion, etc., and darken fully the appropriate circle on the table provided below. Read all responses carefully. **NOTE** long detailed preambles and responses won't depend on hidden keywords: keywords in such preambles and responses are bold-faced capitalized.

There are **THREE** full answer questions each worth 10 marks for a total of 30 marks altogether. Answer them all on the paper provided. It is important that you **SHOW (SHOW, SHOW, SHOW)** how you got the answer.

This is a **CLOSED-BOOK** exam. **NO** cheat sheets allowed. An equation sheet is provided. Calculators are permitted for calculations. Cell phones **MUST** be turned off. The test is out of 50 marks altogether.

This a 50-minute test. Remember your name (and write it down on the exam too).

Answer Table for the Multiple-Choice Questions

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	6.	O	O	O	O	O
2.	O	O	O	O	O	7.	O	O	O	O	O
3.	O	O	O	O	O	8.	O	O	O	O	O
4.	O	O	O	O	O	9.	O	O	O	O	O
5.	O	O	O	O	O	10.	O	O	O	O	O

001 qmult 00500 1 4 3 easy deducto-memory: Bohr atom

1. “Let’s play *Jeopardy!* For \$100, the answer is: This model of an atom is of historical and pedagogical interest, but it is of no use in modern practical calculations and from the modern standpoint is probably misleading rather than insight-giving.”

What is _____, Alex?

- a) Schrödinger’s model of the hydrogen atom
 b) the Thomas-Fermi model of a many electron atom
 c) Bohr’s model of the hydrogen atom d) the liquid drop model of the atom
 e) the model hydrogen atom of Leucippos and Democritos

SUGGESTED ANSWER: (c) That the Bohr model exists is sort of an accident of nature. As far as I can see, it’s interest nowadays is only historical and pedagogical. Its picture of the atom seems to me to be somewhat misleading even.

Wrong answers:

- a) Schrödinger’s model of the hydrogen atom is the analytic solution of his equation for that atom: this model the basis of all modern atomic physics. We never call it the Schrödinger model though: all Schrödinger’s glory is subsumed in his equation itself.
 b) The Thomas-Fermi atom is still very useful.
 d) The liquid drop model of the nucleus is still useful for insight.
 e) The answer applies to the atoms of the ancient Greeks too, but not to this particular question. Leucippos and Democritos never heard of hydrogen.

Redaction: Jeffery, 2001jan01

002 qmult 00600 1 4 5 easy deducto-memory: uncertainty principle

2. “Let’s play *Jeopardy!* For \$100, the answer is: $\Delta x \Delta p \geq \hbar/2$ or $\sigma_x \sigma_p \geq \hbar/2$.

What is _____, Alex?

- a) an equality b) a standard deviation
 c) the Heisenberg **CERTAINTY** principle d) the Cosmological principle
 e) the Heisenberg **UNCERTAINTY** principle

SUGGESTED ANSWER: (e)

Wrong answers:

- c) Don’t be fooled by that missing prefix.

Redaction: Jeffery, 2001jan01

002 qmult 00720 1 1 1 easy memory: stationary state

3. A system in a stationary state will:

- a) not evolve in time. b) evolve in time.
 c) both evolve and not evolve in time. d) occasionally evolve in time.
 e) violate the Heisenberg uncertainty principle.

003 qmult 00500 1 4 2 easy deducto-memory: tunneling

7. “Let’s play *Jeopardy!* For \$100, the answer is: This effect occurs because wave functions can extend (in an exponentially decreasing way albeit) into the classically forbidden region: i.e., the region where a classical particle would have negative kinetic energy.”

What is _____, Alex?

- a) stimulated radiative emission b) quantum mechanical tunneling
c) quantization d) symmetrization e) normalization

SUGGESTED ANSWER: (b)

Wrong answers:

- d) Symmetrization is another fundamental property of quantum systems—but beyond our scope.

Redaction: Jeffery, 2001jan01

011 qmult 00410 1 4 4 easy deducto-memory: spherical harmonics 2

Extra keywords: mathematical physics

8. “Let’s play *Jeopardy!* For \$100, the answer is: They form a basis or complete set for the 2-dimensional space of the surface a sphere which is usually described by the angular coordinates of spherical polar coordinates.”

What are the _____, Alex?

- a) Hermite polynomials b) Laguerre polynomials
c) associated Laguerre polynomials d) spherical harmonics
e) Chebyshev polynomials

SUGGESTED ANSWER: (d)

Wrong answers:

- a) These turn up as factors in the solution of the 1-dimensional simple harmonic oscillator problem.
c) These turn up as factors in the solution of the radial part part of the Coulomb potential 2-body problem
e) These have uses in numerical computation.

Redaction: Jeffery, 2008jan01

011 qmult 00500 1 4 2 easy deducto-memory: spdf designations

9. Conventionally, the spherical harmonic eigenstates for angular momentum quantum numbers

$$\ell = 0, 1, 2, 3, 4, \dots$$

are designated by:

- a) a, b, c, d, e , etc.
b) s, p, d, f , and then alphabetically following f : i.e., g, h , etc.
c) x, y, z, xx, yy, zz, xxx , etc.

- d) A, C, B, D, E, etc.
- e) \$@%&*!!

SUGGESTED ANSWER: (b)

Wrong Answers:

- a) This is the way it should be, not the way it is.
- e) Only in Tasmanian devilish.

Redaction: Jeffery, 2001jan01

012 qmult 00190 1 1 2 easy memory: hydrogen wave functions

10. The hydrogenic atom eigenstate wave functions contain a factor that causes them to:
- a) increase exponentially with radius.
 - b) decrease exponentially with radius.
 - c) increase logarithmically with radius.
 - d) increase quadratically with radius.
 - e) increase linearly with wavelength.

SUGGESTED ANSWER: (b) The wave function must decrease rapidly with radius in order for it to be normalizable.

Wrong answers:

- a) Exactly wrong.

Redaction: Jeffery, 2001jan01

012 qfull 02100 1 3 0 easy math: positronium solution

11. Positronium is an exotic atom consisting of an electron and its antiparticle the positron. It was predicted to exist in 1934 (or even earlier) shortly after the positron was discovered in 1932. Positronium was experimentally discovered in 1951. Positronium cannot exist long because the electron and positron will mutually annihilate usually producing two γ -rays although more γ -rays are possible since there are multiple modes of annihilation. Positronium frequently forms in excited states and decays by radiative transitions to the ground state unless it annihilates first by some mode. Positronium transition spectra and annihilation γ -ray spectra provide a fine test of quantum mechanics and quantum electrodynamics. Neglecting annihilation effects, spin effects, and relativistic effects, positronium to first order is Schrödinger-solution hydrogenic atom. We just consider this simplified positronium in this problem.
- a) What are the positronium total mass and reduced mass?
 - b) What is the formula for the energy of the energy levels of positronium?
 - c) How does the emitted/absorbed photon of a positronium line transition (i.e., transition between energy levels) compare to the corresponding line transition photon of the Schrödinger-solution **HYDROGEN** atom? By “corresponding”, we mean that the photons result from transitions that have the same initial and final principal quantum numbers.

SUGGESTED ANSWER:

a) The total and reduced masses are, respectively,

$$m = 2m_e \approx 1.82 \times 10^{-30} \text{ kg} \quad \text{and} \quad m = \frac{m_e^2}{2m_e} = \frac{m_e}{2} = 4.6 \times 10^{-31} \text{ kg} .$$

Note we are neglecting the small change in mass due to binding energy since that is a relativistic effect.

b) The general Schrödinger-solution hydrogenic atom energies are given by the formula

$$E_n = -\frac{1}{2}m_e c^2 \alpha^2 \frac{Z^2 m_{\text{reduced}}}{n^2 m_e} = -E_{\text{Ryd}} \frac{Z^2 m_{\text{reduced}}}{n^2 m_e} \approx -13.606 \times \frac{Z^2 m_{\text{reduced}}}{n^2 m_e} \text{ eV} .$$

Specializing this formula for positronium (which has $Z = 1$) gives

$$E_n = -\frac{1}{4}m_e c^2 \alpha^2 \frac{1}{n^2} = -\frac{E_{\text{Ryd}}}{2} \frac{1}{n^2} \approx -6.803 \times \frac{1}{n^2} \text{ eV} .$$

c) The positronium photon has just a bit more than half the frequency and the energy and just a bit less than twice the wavelength of the corresponding Schrödinger-solution hydrogen atom photon. This is just because all the energy-level energies of positronium have just a bit more than half those of the Schrödinger-solution hydrogen atom. The slight difference from having exactly “half” in the above statements is that the reduced mass of positronium is $m_e/2$ while the reduced mass of the hydrogen atom is slightly less than m_e because the proton mass is not infinite. The reduced mass appears linearly in the energy level formula, and so all the positronium energy levels are just a bit more than half the hydrogen atom energy levels.

Redaction: Jeffery, 2008jan01

003 qfull 00100 2 3 0 moderate math: infinite square well in 1-d

12. You are given the time-independent Schrödinger equation

$$H\psi(x) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x) = E\psi(x)$$

and the infinite square well potential

$$V(x) = \begin{cases} 0, & x \in [0, a]; \\ \infty & \text{otherwise.} \end{cases}$$

- a) What must the wave function be outside of the well (i.e., outside of the region $[0, a]$) in order to satisfy the Schrödinger equation? Why?
- b) What boundary conditions must the wave function satisfy? Why must it satisfy these boundary conditions?

- c) Reduce Schrödinger's equation inside the well to an equation of the same form as the **CLASSICAL** simple harmonic oscillator differential equation with all the constants combined into a factor of $-k^2$, where k is newly defined constant. What is k 's definition?
- d) Solve for the general solution for a **SINGLE** k value, but don't impose boundary conditions or normalization yet. A solution by inspection is adequate. Why can't we allow solutions with $E \leq 0$? Think carefully: it's not because k is imaginary when $E < 0$.
- e) Use the boundary conditions to eliminate most of the solutions with $E > 0$ and to impose quantization on the allowed set of distinct solutions (i.e., on the allowed k values). Give the general wave function with the boundary conditions imposed and give the quantization rule for k in terms of a dimensionless quantum number n . Note that the multiplication of a wave function by an arbitrary global phase factor $e^{i\phi}$ (where ϕ is arbitrary) does not create a physically distinct wave function (i.e., does not create a new wave function as recognized by nature.) (Note the orthogonality relation used in expanding general functions in eigenfunctions also does not distinguish eigenfunctions that differ by global phase factors either: i.e., it gives the expansion coefficients only for distinct eigenfunctions. So the idea of distinct eigenfunctions arises in pure mathematics as well as in physics.)
- f) Normalize the solutions.
- g) Determine the general formula for the eigenenergies in terms of the quantum number n .

SUGGESTED ANSWER:

- a) Outside the well any wave function is zero in order to satisfy the Schrödinger equation. This is because if the potential goes to infinity over a finite region, the only reasonable way to satisfy the Schrödinger equation is with a zero wave function in that region.
- b) For a finite potential the wave function and its 1st derivative must be continuous: the 1st derivative is allowed to have kinks. If the potential becomes infinite at a point, then the first derivative is allowed to have finite discontinuities and the wave function is allowed to have kinks at that point. In our case, all the well walls require by themselves is that the wave function be continuous there and thus be zero there. It is known (but exactly how is seldom gone into) that in this case no condition is imposed on the continuity of the 1st derivative of the wave function and no condition is needed. I append a note discussing the continuity of the wave function and its 1st derivative below: it's prolix.
- c) Inside the well one has

$$H\psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = E\psi .$$

Defining

$$k = \frac{\sqrt{2mE}}{\hbar} ,$$

we obtain

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 \psi .$$

This last equation has the same form as the classical simple harmonic oscillator differential equation.

d) By inspection and lots of experience, the general solution for $E > 0$ is

$$\psi(x) = A \sin(kx) + B \cos(kx) ,$$

where A and B are constants. This solution, of course, only applies inside the well. Outside of the well $\psi = 0$ everywhere.

We cannot allow $E \leq 0$ as we show in the following. If we did allow $E \leq 0$, we would have the differential equation

$$\frac{\partial^2 \psi}{\partial x^2} = \kappa^2 \psi ,$$

where

$$\kappa = \frac{\sqrt{2m|E|}}{\hbar} .$$

Note by our definition, k would be imaginary in this case, but that has no consequence since the eigenvalues in our Hermitian operator equation for $E \leq 0$ are still real.

For $E < 0$, the general solution is

$$\psi = Ae^{\kappa x} + Be^{-\kappa x} ,$$

where A and B are constants. Neither of the terms of this solution are ever zero (unless $A = B = 0$) and since one term is strictly increasing and the other strictly decreasing, only one zero can be created by linear combination. The linear combination that gives the one zero at any x satisfies the ratio

$$\frac{A}{B} = -e^{-2\kappa x} .$$

Because there is only one zero at most, the $E < 0$ solution cannot satisfy the boundary conditions and must be ruled out. For $E = 0$, the general solution is

$$\psi = Ax + B ,$$

where A and B are constants. This solution can only be zero at one point (unless $A = B = 0$), and thus cannot satisfy the boundary conditions and must be ruled out. If $A = B = 0$ for $E \leq 0$, the boundary conditions are satisfied, but the solutions cannot be normalized, and so must be ruled out. So all cases of $E \leq 0$ give physically invalid solutions.

Note there is a general proof that $E > V_{\min}$, except that $E = V_{\min}$ is allowed for a constant wave function solution to a system with periodic boundary conditions: see the solution to the problem suggested by Griffiths's problem Gr-24:2.2. For the infinite square well, the boundary conditions are not periodic and $V_{\min} = 0$. Thus we find that solutions must have $E > 0$ by the general proof.

- e) To satisfy the boundary conditions (ψ continuous, but no continuity constraint on $\partial\psi/\partial x$ because of the infinite potential), we must have $\psi(0) = \psi(ka) = 0$. Thus, $B = 0$ (i.e., no cosine solutions are allowed) and

$$k = \frac{n\pi}{a},$$

where n must be an integer. The fact that n must be an integer gives the quantization of allowed states: the boundary conditions have imposed this quantization, in fact. The number n is the dimensionless quantum number.

The $n = 0$ case gives a zero eigenfunction which cannot be normalized and the negative n values because of the oddness of the sine function do not give physically distinct solutions from their positive counterparts (i.e., the $-n$ values). Recall wave functions that differ by a global phase factor (i.e., $e^{i\phi}$ where ϕ is any number) are not physically distinct: nature does not recognize them as different states. There are actually infinitely many mathematical states for each physically distinct state.

Finally, we find that n runs over all positive integers only: $n = 1, 2, 3, \dots$. The allowed solutions are

$$\psi_n(x) = A \sin\left(\frac{n\pi}{a}x\right).$$

A few other remarks can be made. We can see that k is in fact a wavenumber since the solution is periodic for every $\Delta x = 2\pi/k$. The wavelength λ is in fact that Δx :

$$\lambda = \frac{2\pi}{k} = \frac{2a}{n}.$$

Consequently, we find

$$n\frac{\lambda}{2} = a$$

which implies that the n th wave function will have n antinodes and $n + 1$ nodes. Two of the nodes are on the boundaries, of course.

- f) For normalization we require

$$\begin{aligned} 1 &= A^2 \int_0^a \sin^2(kx) dx = A^2 \frac{1}{k} \int_0^{ka} \sin^2(y) dy \\ &= A^2 \frac{1}{2k} \int_0^{ka} [1 - \cos(2y)] dy = A^2 \frac{1}{2k} \left[y - \frac{\sin(2y)}{2} \right] \Big|_0^{ka=n\pi} \\ &= A^2 \frac{1}{2k} (ka) = A^2 \frac{a}{2}, \end{aligned}$$

and thus

$$A = \sqrt{\frac{2}{a}},$$

where we have chosen A to be pure real. Thus the normalized general solution is

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right).$$

g) The energy of the n th eigenstate is given by

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m} \left(\frac{\pi}{a}\right)^2 n^2.$$

Thus the energies are quantized with n being the quantum number. The quantization is imposed by the boundary conditions and the requirement of normalizability. All bound quantum states are in fact quantized. But we won't prove that here.

NOTE: Herein we consider the continuity properties of the wave function and its 1st derivative at some length. This note has never been perfected. Once we go to the infinite potential case, it's just maundering on and has to be all cleaned up when I get a chance.

First note that the time independent Schrödinger equation leads directly to the following integral

$$\frac{\partial\psi}{\partial x}\Big|_{a+\epsilon} - \frac{\partial\psi}{\partial x}\Big|_{a-\epsilon} = \int_{a-\epsilon}^{a+\epsilon} dx \frac{\partial^2\psi}{\partial x^2} = \int_{a-\epsilon}^{a+\epsilon} dx \frac{2m}{\hbar^2} [V(x) - E] \psi(x),$$

where a is any point and ϵ is a small displacement parameter. It is a given that E is finite or zero. If the potential and wave function are finite or zero everywhere, we find

$$\lim_{\epsilon \rightarrow 0} \left[\frac{\partial\psi}{\partial x}\Big|_{a+\epsilon} - \frac{\partial\psi}{\partial x}\Big|_{a-\epsilon} \right] = 0.$$

This result follows even V or $\psi(x)$ is discontinuous at a : the integral is still of a finite integrand over a zero area in the limit that $\epsilon \rightarrow 0$.

Thus the 1st derivative of the wave function must be continuous at points of finite potential and wave function: kinks in the 1st derivative are allowed in principle. But if the 1st derivative is continuous, then the wave function itself must be continuous and kink-free. A kink in the wave function causes a discontinuity in the 1st derivative and a discontinuity in the wave function causes an infinite discontinuity in the 1st derivative.

To sum up, if V is finite or zero, then the wave function must be continuous and kink-free. The 1st derivative is allowed have kinks. Note V can be discontinuous.

Now what if potential goes to positive infinity? First let us consider the case where there is an infinite potential over a finite region. We set one boundary of the infinite wall at $x = 0$ for convenience.

To find the solution let us first allow the potential to be a finite constant V for $x < 0$. For $x > 0$, we set the potential to 0. I think we can always consider zones close enough to the wall that the potential on either side can be considered as constants. We assume $0 < E < V$: I don't think this is unduly restrictive: recall $E > V_{\text{minimum}}$, except for periodic boundary condition cases (Gr-24). The solutions close the wall are

$$\psi(x) = Ae^{\kappa x} \quad \text{and} \quad \psi(x) = B \sin(kx) + C \cos(kx) ,$$

where

$$\kappa = \pm \sqrt{\frac{2m}{\hbar^2}(V - E)} \quad \text{and} \quad k = \pm \sqrt{\frac{2m}{\hbar^2}E} .$$

Only the positive solution for κ is allowed by the normalizability condition. We cannot specify A , B , and C exactly without defining the whole potential and finding an expression for the whole wave function. We don't want to do that since we are trying to see if can get a general understanding.

Since we are first considering a finite wall, we require continuous wave function and its 1st derivative. Thus at $x = 0$ we demand

$$A = C \quad \text{and} \quad A\kappa = Bk .$$

Now if we let V become large, $\psi(x < 0)$ must become small and thus A and C must become small. But nothing demands that Bk become small since κ is growing large as V grows large. If we let $V \rightarrow \infty$, then A and C go to zero, but Bk can stay non-zero since $A\kappa$ can stay non-zero and finite. In this way wave function stays continuous at $x = 0$ and in a limiting sense so does its 1st derivative even though in direct sense there is discontinuity in the 1st derivative.

But does nature take the limit such that $A\kappa$ stays finite non-zero? Well nature certainly doesn't let $A\kappa$ go to infinity since that would make Bk go to infinity which seems implausible. If $A\kappa$ goes to zero, then either B or k goes to zero and then the wave function and its 1st derivative are both zero at $x = 0$. If one requires the wave function and its 1st derivative to be zero at the wall, then there are no solutions to the infinite square well problem. We know in nature that systems approaching the infinite square well do have a spectrum of solutions, and so conclude that does take the limit such that Bk stays finite non-zero.

But one doesn't really like to appeal to observation. Isn't there some general mathematical argument? Since for any specific system, there is a mathematical solution for A and if it always gives $A\kappa = Bk$ for when $V \rightarrow \infty$ (as we seem to think nature demands), then there must be some general mathematical proof that $A\kappa = Bk$ for when $V \rightarrow \infty$. But I can't see what it could be?

Maybe I'm being over-idealizing. There are no infinite potentials nor even any finite, sharp wall potentials. Maybe nearly sharp finite wall potential approximated as sharp infinite wall potential just allows one to use only the continuity condition on the wave function because one is admitting

at the outset one isn't treating the wave function in the neighborhood of the nearly sharp wall at all correctly. Anyway Gr-60 treats the finite square well, but that doesn't elucidate the general sharp wall case for me.

The following may be gibberish: I've no patience right now to figure out if I was talking sense when I wrote it.

So far so good: now on to the pathological cases. Can the wave function be infinite? Not over a finite region for that would give an infinite probability of finding the particle in that region: a probability greater than 1 is not allowed. Can we allow a Dirac delta region of the wave function? In the limiting sense of a very highly peaked wave function region I don't see why not, but I confess I don't know how to treat a Dirac delta function magnitude squared (which is probability density). The 1st derivative could be discontinuous across the Dirac delta function region, but only by a finite amount. Thus the wave function will be continuous if kinked across the Dirac delta region (not counting where it shoots to "infinity"). So in a sense the wave function must be continuous even if it shoots at a point to a Dirac delta infinity.

What if the potential shoots to a Dirac delta infinity at point a ? Well the 1st derivative can have a finite discontinuity, but the wave function must stay continuous. But the wave function doesn't have to go to zero at a . What if the potential shoots to more than a Dirac delta infinity at a point. Well this is a physically impossible case. No potential is really ever infinite and a Dirac delta infinity is just a way of compress a very high potential that acts over a small region into a neat mathematical form.

We leave further thought on this *sine die*.

We have considered an infinite wall, but what about an infinite drop potential. Well below the drop the drop looks like an infinite wall. So this case reduces to the last. The particle is always on the drop side.

We could go on considering pathological cases all night, but enough already. Infinite potentials are an idealization anyway. Still it's necessary to know how to treat them in the correct limiting way.

Redaction: Jeffery, 2001jan01

001 qfull 01100 2 5 0 moderate thinking: Bohr atom

13. In 1913, Niels Bohr presented his model of the hydrogen atom which was quickly generalized to the hydrogenic atom (i.e., the one-electron atom of any nuclear charge Z). This model correctly gives the main hydrogenic atom energy levels and consists of a mixture of quantum mechanical and classical ideas. It is historically important for showing that quantization is somehow important in atomic structure and pedagogically it is of interest since it shows how simple theorizing can be done. But the model is, in fact, incorrect and from the modern perspective probably even misleading about the quantum mechanical nature of the atom. It is partially an accident of nature that it exists to be found. Only partially an accident since it does contain correct ingredients.

And it is no accident that Bohr found it. Bohr knew what he wanted: a model that would successfully predict the hydrogen atom spectrum which is a line spectrum showing emission at fixed frequencies. He knew from Einstein's photoelectric effect

theory that electromagnetic radiation energy was quantized in amounts $h\nu$ where $h = 6.62606896(33) \times 10^{-27}$ erg s was Planck's constant (which was introduced along with the quantization notion to explain black-body radiation in 1900) and ν was frequency of the quantum of radiation. He recognized that Planck's constant had units of angular momentum. He knew from Rutherford's nuclear model of the atom that the positive charge of an atom was concentrated in region that was much smaller than the atom size and that almost all the mass of the atom was in the nucleus. He knew that there were negative electrons in atoms and they were much less massive than the nucleus. He knew the structure of atoms was stable somehow. By a judicious mixture of classical electromagnetism, classical dynamics, and quantum ideas he found his model. A more sophisticated mixture of these concepts would lead to modern quantum mechanics.

Let's see if we can follow the steps of the ideal Bohr—not the Bohr of history.

NOTE: This a semi-classical question: Bohr, ideal or otherwise, knew nothing of the Schrödinger equation in 1913. Also note that this question uses Gaussian CGS units not MKS units. The most relevant distinction is that electric charge

$$e_{\text{CGS}} = \frac{e_{\text{MKS}}}{\sqrt{4\pi\epsilon_0}}$$

which implies the fine structure constant in CGS is

$$\alpha = \frac{e^2}{\hbar c} .$$

Astronomy is all Gaussian CGS by the way.

- a) Bohr thought to build the electron system about the nucleus based on the electrostatic inverse square law with the electron system supported against collapse onto the nucleus by angular momentum. The nucleus was known to be much tinnier than the electron system which gives the atom its volume. The nucleus could thus be a considered an immobile point center of force at the origin of the relative nucleus-electron coordinate system frame. This frame is non-inertial, but classically can be given an inertial-frame treatment if the electron is given a reduced mass given by

$$m = \frac{m_e m_{\text{nucleus}}}{m_e + m_{\text{nucleus}}} \approx m_e \left(1 - \frac{m_e}{m_{\text{nucleus}}} \right) ,$$

where m_e the electron mass and m_{nucleus} is the nucleus mass. The approximation is valid for $m_e/m_{\text{nucleus}} \ll 1$ which is true of hydrogen and most hydrogenic systems, but not, for example, for positronium (a bound electron and positron). The electron—there is only one in a hydrogenic atom—was taken to be in orbit about the nucleus. Circular orbits seemed the simplest way to proceed. The electrostatic force law (in Gaussian cgs units) in scalar form for a circular orbit is

$$\vec{F} = -\frac{Ze^2}{r^2} \hat{r} ,$$

where Ze is the nuclear charge, e is the electron charge, and r is the radial distance to the electron, and \hat{r} is a unit vector in the radial direction.

What is the potential energy of the electron with the zero of potential energy for the electron at infinity as usual? **HINT:** If the result isn't obvious, you can get it using the work-potential energy formula:

$$V = - \int \vec{F} \cdot d\vec{r} + \text{constant} .$$

- b) Using the centripetal force law (which is really $F = ma$ for uniform circular motion)

$$\vec{F} = -\frac{mv^2}{r} \hat{r} ,$$

find an expression for the classical kinetic energy T of the electron in terms of Z , e , and r alone.

- c) What is the total energy of the electron in the orbit?
- d) Classically an accelerating charge radiates. This seemed well established experimentally in Bohr's time. But an orbiting electron is accelerating, and so should lose energy continuously until it collapses into the nucleus: this catastrophe obviously doesn't happen. Electrons do not collapse into the nucleus. Also they radiate only at fixed frequencies which means fixed quantum energies by Einstein's photoelectric effect theory. So Bohr postulated that the electron could only be in certain orbits which he called stationary states and that the electron in a stationary state did not radiate. Only on transitions between stationary states (sometimes called quantum jumps or leaps) was there an emission of radiation in a quantum of radiation or (to use an anachronism) a photon. To get the fixed energies of emission only certain energies were allowed for the stationary states. But the emitted photons didn't come out with equally spaced energies: ergo the orbits couldn't be equally spaced in energy. From the fact that Planck's constant h has units of angular momentum, Bohr hypothesized the orbits were quantized in equally spaced amounts of angular momentum. But h was not the spacing that worked. Probably after a bit of fooling around, Bohr found that $h/(2\pi)$ or, as we now write it, \hbar was the spacing that gave the right answer. The allowed angular momenta were given by

$$L = n\hbar ,$$

where n is any positive non-zero integer. The n is now called the principal quantum number, but its meanings in the Bohr model and in modern quantum mechanics are somewhat different. The principal quantum number n determines the main spacing of the hydrogenic energy levels.

Rewrite kinetic energy T in terms of $n\hbar$ and solve for an expression for r in terms n , \hbar , Ze^2 , and m only. **HINT:** Recall the classical expression for angular momentum of particle in a circular orbit is $L = mrv$.

- e) Using the formula for r from the part (d) answer write an expression for the energy of a stationary state in terms of m , c , α , Z , and n only. The c is the speed of light and the α is the fine structure constant: recall that in Gaussian cgs units

$$\alpha = \frac{e^2}{\hbar c} .$$

This formula for orbit energy turns out to be correct for the spacing of the main energy levels. But these energy levels doesn't, in fact, have angular momentum $n\hbar$: it consists of has orbitals (as we now call them) with angular momenta in the range $[0, n - 1]$ in units of \hbar (e.g., Gr-139).

SUGGESTED ANSWER:

a) Behold:

$$V = - \int_{\infty}^r \vec{F} \cdot d\vec{r}' = - \int_{\infty}^r F(-dr') = \int_{\infty}^r F dr' = \int_{\infty}^r \frac{Ze^2}{r'} dr' = -\frac{Ze^2}{r'} \Big|_{\infty}^r = -\frac{Ze^2}{r} ,$$

where we have used the work-potential energy formula.

b) The electrostatic force is the force of the centripetal force law. Thus

$$-\frac{Ze^2}{r^2} \hat{r} = -\frac{mv^2}{r} \hat{r}$$

which immediately yields

$$T = \frac{1}{2}mv^2 = \frac{1}{2} \frac{Ze^2}{r} .$$

c) The total energy is

$$E = T + V = -\frac{1}{2} \frac{Ze^2}{r} .$$

d) Well

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m \left(\frac{L}{mr} \right)^2 = \frac{L^2}{2mr^2} = \frac{n^2\hbar^2}{2mr^2} = \frac{1}{2} \frac{Ze^2}{r} ,$$

and thus

$$r = \frac{n^2\hbar^2}{mZe^2} = \frac{\hbar}{m_e e^2} \frac{m_e n^2}{Z} .$$

Note the m_e in the last version of the expression is to allow us to define the fiducial radius for the hydrogenic atom: i.e., the Bohr radius. The fiducial radius for a hydrogenic atom is, in fact, the Bohr radius given by

$$a_0 = \frac{\hbar^2}{me^2} = \frac{\hbar}{m c \alpha} = \frac{\lambda_C}{2\pi\alpha} = 0.52917720859(36) \times 10^{-8} \text{ cm} \approx 0.529 \text{ \AA} ,$$

where

$$\alpha = \frac{e^2}{\hbar c}$$

is the fine structure constant in Gaussian cgs units and

$$\lambda = \frac{h}{mc}$$

is the Compton wavelength.

e) Behold

$$E = -\frac{1}{2} \frac{Ze^2}{r} = -\frac{1}{2} mc^2 \alpha^2 \frac{Z^2}{n^2} .$$

To go a bit further note

$$E = -\frac{1}{2} mc^2 \alpha^2 \frac{Z^2}{n^2} = -\frac{1}{2} m_e c^2 \alpha^2 \frac{m}{m_e} \frac{Z^2}{n^2} = -E_{\text{ryd}} \frac{m}{m_e} \frac{Z^2}{n^2} \approx -13.606 \text{ eV} \times \frac{m}{m_e} \frac{Z^2}{n^2} .$$

where E_{ryd} is the Rydberg energy. I like the Rydberg expression

$$E_{\text{ryd}} = \frac{1}{2} m_e c^2 \alpha^2 \approx 13.606 \text{ eV}$$

since it clearly shows the E_{ryd} is an energy because of the $m_e c^2$ factor: the other factors are dimensionless. So this is the form I've committed to memory. It's much better than those clusters of obscure constants one often sees.

Note: The Bohr atom gets the main energy levels correct and their dependence on n . But, in fact, the main energy levels correspond to n different angular momentum values.

We should also note that historically Bohr didn't postulate that angular momentum quantization, but derived it from other postulates which were rather crude (PN-195–197). Modern presentations of the Bohr atom prefer to regard angular momentum quantization as a postulate and avoid those crude postulates.

Redaction: Jeffery, 2001jan01

Equation Sheet for Modern Physics

These equation sheets are intended for students writing tests or reviewing material. Therefore they are neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

1 Constants

$$\begin{aligned}
 c &= 2.99792458 \times 10^8 \text{ m/s} \approx 2.998 \times 10^8 \text{ m/s} \approx 3 \times 10^8 \text{ m/s} \approx 1 \text{ yr/yr} \approx 1 \text{ ft/ns} \\
 e &= 1.602176487(40) \times 10^{-19} \text{ C} \\
 E_{\text{Rydberg}} &= 13.60569193(34) \text{ eV} \\
 g_e &= 2.0023193043622 \quad (\text{electron } g\text{-factor}) \\
 h &= 6.62606896(33) \times 10^{-34} \text{ J s} = 4.13566733(10) \times 10^{-15} \text{ eV s} \\
 hc &= 12398.419 \text{ eV \AA} \approx 10^4 \text{ eV \AA} \\
 \hbar &= 1.054571628(53) \times 10^{-34} \text{ J s} = 6.58211899(16) \times 10^{-16} \text{ eV s} \\
 k &= 1.3806504(24) \times 10^{-23} \text{ J/K} = 0.8617343(15) \times 10^{-4} \text{ eV/K} \approx 10^{-4} \text{ eV/K} \\
 m_e &= 9.10938215(45) \times 10^{-31} \text{ kg} = 0.510998910(13) \text{ MeV} \\
 m_p &= 1.672621637(83) \times 10^{-27} \text{ kg} = 938.272013(23), \text{ MeV} \\
 \alpha &= e^2/(4\pi\epsilon_0\hbar c) = 7.2973525376(50) \times 10^{-3} = 1/137.035999679(94) \approx 1/137 \\
 \lambda_C &= h/(m_e c) = 2.4263102175(33) \times 10^{-12} \text{ m} = 0.0024263102175(33) \text{ \AA} \\
 \mu_B &= 5.7883817555(79) \times 10^{-5} \text{ eV/T}
 \end{aligned}$$

2 Geometrical Formulae

$$C_{\text{cir}} = 2\pi r \quad A_{\text{cir}} = \pi r^2 \quad A_{\text{sph}} = 4\pi r^2 \quad V_{\text{sph}} = \frac{4}{3}\pi r^3$$

3 Trigonometry

$$\frac{x}{r} = \cos \theta \quad \frac{y}{r} = \sin \theta \quad \frac{y}{x} = \tan \theta \quad \cos^2 \theta + \sin^2 \theta = 1$$

$$\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b) \quad \cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)$$

$$\cos^2 \theta = \frac{1}{2}[1 + \cos(2\theta)] \quad \sin^2 \theta = \frac{1}{2}[1 - \cos(2\theta)] \quad \sin(2\theta) = 2 \sin(\theta) \cos(\theta)$$

$$\cos(a) \cos(b) = \frac{1}{2} [\cos(a - b) + \cos(a + b)] \quad \sin(a) \sin(b) = \frac{1}{2} [\cos(a - b) - \cos(a + b)]$$

$$\sin(a) \cos(b) = \frac{1}{2} [\sin(a - b) + \sin(a + b)]$$

4 Blackbody Radiation

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{[e^{h\nu/(kT)} - 1]} \quad B_\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{[e^{hc/(kT\lambda)} - 1]}$$

$$B_\lambda d\lambda = B_\nu d\nu \quad \nu\lambda = c \quad \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$$

$$E = h\nu = \frac{hc}{\lambda} \quad p = \frac{h}{\lambda}$$

$$F = \sigma T^4 \quad \sigma = \frac{2\pi^5}{15} \frac{k^4}{c^2 h^3} = 5.670400(40) \times 10^{-8} \text{ W/m}^2/\text{K}^4$$

$$\lambda_{\max} T = \text{constant} = \frac{hc}{kx_{\max}} \approx \frac{1.4387751 \times 10^{-2}}{x_{\max}}$$

$$B_{\lambda, \text{Wien}} = \frac{2hc^2}{\lambda^5} e^{-hc/(kT\lambda)} \quad B_{\lambda, \text{Rayleigh-Jeans}} = \frac{2ckT}{\lambda^4}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c} \nu = \frac{\omega}{c} \quad k_i = \frac{\pi}{L} n_i \quad \text{standing wave BCs} \quad k_i = \frac{2\pi}{L} n_i \quad \text{periodic BCs}$$

$$n(k) dk = \frac{k^2}{\pi^2} dk = \pi \left(\frac{2}{c} \right) \nu^2 d\nu = n(\nu) d\nu$$

$$\ln(z!) \approx \left(z + \frac{1}{2}\right) \ln(z) - z + \frac{1}{2} \ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \dots$$

$$\ln(N!) \approx N \ln(N) - N$$

$$\rho(E) dE = \frac{e^{-E/(kT)}}{kT} dE \quad P(n) = (1 - e^{-\alpha}) e^{-n\alpha} \quad \alpha = \frac{h\nu}{kT}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad f(x - vt) \quad f(kx - \omega t)$$

5 Photons

$$KE = h\nu - w \quad \Delta\lambda = \lambda_{\text{scat}} - \lambda_{\text{inc}} = \lambda_C(1 - \cos\theta)$$

$$\ell = \frac{1}{n\sigma} \quad \rho = \frac{e^{-s/\ell}}{\ell} \quad \langle s^m \rangle = \ell^m m!$$

6 Matter Waves

$$\lambda = \frac{h}{p} \quad p = \hbar k \quad \Delta x \Delta p \geq \frac{\hbar}{2} \quad \Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Psi(x, t) = \int_{-\infty}^{\infty} \phi(k) \Psi_k(x, t) dk \quad \phi(k) = \int_{-\infty}^{\infty} \Psi(x, 0) \frac{e^{-ikx}}{\sqrt{2\pi}} dx$$

$$v_g = \left. \frac{d\omega}{dk} \right|_{k_0} = \frac{\hbar k_0}{m} = \frac{p_0}{m} = v_{\text{clas},0}$$

7 Non-Relativistic Quantum Mechanics

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \quad T = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \quad H\Psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

$$\rho = \Psi^* \Psi \quad \rho dx = \Psi^* \Psi dx$$

$$A\phi_i = a_i\phi_i \quad f(x) = \sum_i c_i\phi_i \quad \int_a^b \phi_i^* \phi_j dx = \delta_{ij} \quad c_j = \int_a^b \phi_j^* f(x) dx$$

$$[A, B] = AB - BA$$

$$P_i = |c_i|^2 \quad \langle A \rangle = \int_{-\infty}^{\infty} \Psi^* A \Psi dx = \sum_i |c_i|^2 a_i \quad H\psi = E\psi \quad \Psi(x, t) = \psi(x)e^{-i\omega t}$$

$$p_{\text{op}}\phi = \frac{\hbar}{i} \frac{\partial \phi}{\partial x} = p\phi \quad \phi = \frac{e^{ikx}}{\sqrt{2\pi}} \quad \frac{\partial^2 \psi}{\partial x^2} = \frac{2m}{\hbar^2} (V - E)\psi$$

$$|\Psi\rangle \quad \langle \Psi| \quad \langle x|\Psi\rangle = \Psi(x) \quad \langle \vec{r}|\Psi\rangle = \Psi(\vec{r}) \quad \langle k|\Psi\rangle = \Psi(k) \quad \langle \Psi_i|\Psi_j\rangle = \langle \Psi_j|\Psi_i\rangle^*$$

$$\langle \phi_i|\Psi\rangle = c_i \quad 1_{\text{op}} = \sum_i |\phi_i\rangle\langle \phi_i| \quad |\Psi\rangle = \sum_i |\phi_i\rangle\langle \phi_i|\Psi\rangle = \sum_i c_i|\phi_i\rangle$$

$$1_{\text{op}} = \int_{-\infty}^{\infty} dx |x\rangle\langle x| \quad \langle \Psi_i|\Psi_j\rangle = \int_{-\infty}^{\infty} dx \langle \Psi_i|x\rangle\langle x|\Psi_j\rangle \quad A_{ij} = \langle \phi_i|A|\phi_j\rangle$$

$$Pf(x) = f(-x) \quad P \frac{df(x)}{dx} = \frac{df(-x)}{d(-x)} = -\frac{df(-x)}{dx} \quad Pf_{e/o}(x) = \pm f_{e/o}(x)$$

$$P \frac{df_{e/o}(x)}{dx} = \mp \frac{df_{e/o}(x)}{dx}$$

$$Y_{0,0} = \frac{1}{\sqrt{4\pi}} \quad Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2} \cos(\theta) \quad Y_{1,\pm 1} = \mp \left(\frac{3}{8\pi}\right)^{1/2} \sin(\theta) e^{\pm i\phi}$$

$$L^2 Y_{\ell m} = \ell(\ell+1) \hbar^2 Y_{\ell m} \quad L_z Y_{\ell m} = m \hbar Y_{\ell m} \quad |m| \leq \ell \quad m = -\ell, -\ell+1, \dots, \ell-1, \ell$$

0	1	2	3	4	5	6	...
<i>s</i>	<i>p</i>	<i>d</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>	...

9 Hydrogenic Atom

$$\psi_{n\ell m} = R_{n\ell}(r) Y_{\ell m}(\theta, \phi) \quad \ell \leq n-1 \quad \ell = 0, 1, 2, \dots, n-1$$

$$a_z = \frac{a_0}{Z} \left(\frac{m_e}{m_{\text{reduced}}} \right) \quad a_0 = \frac{\hbar}{m_e c \alpha} = \frac{\lambda_C}{2\pi \alpha} \quad m_{\text{reduced}} = \frac{m_1 m_2}{m_1 + m_2}$$

$$R_{10} = 2a_z^{-3/2} e^{-r/a_z} \quad R_{20} = \frac{1}{\sqrt{2}} a_z^{-3/2} \left(1 - \frac{1}{2} \frac{r}{a_z} \right) e^{-r/(2a_z)}$$

$$R_{21} = \frac{1}{\sqrt{24}} a_z^{-3/2} \frac{r}{a_z} e^{-r/(2a_z)}$$

$$R_{n\ell} = - \left\{ \left(\frac{2}{na_z} \right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3} \right\}^{1/2} e^{-\rho/2} \rho^\ell L_{n+\ell}^{2\ell+1}(\rho) \quad \rho = \frac{2r}{nr_z}$$

$$L_q(x) = e^x \left(\frac{d}{dx} \right)^q (e^{-x} x^q) \quad \text{Rodrigues's formula for the Laguerre polynomials}$$

$$L_q^j(x) = \left(\frac{d}{dx} \right)^j L_q(x) \quad \text{Associated Laguerre polynomials}$$

$$\langle r \rangle_{n\ell m} = \frac{a_z}{2} [3n^2 - \ell(\ell+1)]$$

Nodes = $(n - 1) - \ell$ not counting zero or infinity

$$E_n = -\frac{1}{2}m_e c^2 \alpha^2 \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} = -E_{\text{Ryd}} \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} \approx -13.606 \times \frac{Z^2}{n^2} \frac{m_{\text{reduced}}}{m_e} \text{ eV}$$

10 Spin, Magnetic Dipole Moment, Spin-Orbit Interaction

$$S_{\text{op}}^2 = \frac{3}{4}\hbar \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad s = \frac{1}{2} \quad s(s+1) = \frac{3}{4} \quad S = \sqrt{s(s+1)}\hbar = \frac{\sqrt{3}}{2}\hbar$$

$$S_{z,\text{op}} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad m_s = \pm \frac{1}{2} \quad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mu_b = \frac{e\hbar}{2m_e} = 9.27400915(26) \times 10^{-24} \text{ J/T} = 5.7883817555(79) \times 10^{-5} \text{ eV/T}$$

$$\mu_{\text{nuclear}} = \frac{e\hbar}{2m_p} = 5.05078324(13) \times 10^{-27} \text{ J/T} = 3.1524512326(45) \times 10^{-8} \text{ eV/T}$$

$$\vec{\mu}_\ell = -g_\ell \mu_b \frac{\vec{L}}{\hbar} \quad \mu_\ell = g_\ell \mu_b \ell(\ell+1) \quad \mu_{\ell,z} = -g_\ell \mu_b \frac{L_z}{\hbar} \quad \mu_{\ell,z} = -g_\ell \mu_b m_\ell$$

$$\vec{\tau} = \vec{\mu} \times \vec{B} \quad PE = -\vec{\mu} \cdot \vec{B} \quad \vec{F} = \Delta(\vec{\mu} \cdot \vec{B}) \quad F_z = \sum_j \mu_j \frac{\partial B_j}{\partial z} \quad \vec{\omega} = \frac{g_\ell \mu_b}{\hbar} \vec{B}$$

$$\vec{J} = \vec{L} + \vec{S} \quad J = \sqrt{j(j+1)}\hbar \quad j = |\ell - s|, |\ell - s + 1|, \dots, \ell + s \quad \text{triangle rule}$$

$$J_z = m_j \hbar \quad m_j = -j, -j + 1, \dots, j - 1, j$$

$$E(n, \ell, \pm 1/2, j) = -\frac{E_{\text{Ryd}}}{n^2} \frac{m}{m_e} \left[1 + \frac{\alpha^2}{n^2} \left(\frac{n}{j + 1/2} - \frac{3}{4} \right) \right]$$

11 Special Relativity

$$c = 2.99792458 \times 10^8 \text{ m/s} \approx 2.998 \times 10^8 \text{ m/s} \approx 3 \times 10^8 \text{ m/s} \approx 1 \text{ ly/yr} \approx 1 \text{ ft/ns}$$

$$\beta = \frac{v}{c} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad \gamma(\beta \ll 1) = 1 + \frac{1}{2}\beta^2 \quad \tau = ct$$

Galilean Transformations

$$\begin{aligned} x' &= x - \beta\tau \\ y' &= y \\ z' &= z \\ \tau' &= \tau \end{aligned}$$

$$\beta'_{\text{obj}} = \beta_{\text{obj}} - \beta$$

Lorentz Transformations

$$\begin{aligned} x' &= \gamma(x - \beta\tau) \\ y' &= y \\ z' &= z \\ \tau' &= \gamma(\tau - \beta x) \end{aligned}$$

$$\beta'_{\text{obj}} = \frac{\beta_{\text{obj}} - \beta}{1 - \beta\beta_{\text{obj}}}$$

$$\ell = \ell_{\text{proper}} \sqrt{1 - \beta^2} \quad \Delta\tau_{\text{proper}} = \Delta\tau \sqrt{1 - \beta^2}$$

$$m = \gamma m_0 \quad p = mv = \gamma m_0 c \beta \quad E_0 = m_0 c^2 \quad E = \gamma E_0 = \gamma m_0 c^2 = mc^2$$

$$E = mc^2 \quad E = \sqrt{(pc)^2 + (m_0 c^2)^2}$$

$$KE = E - E_0 = \sqrt{(pc)^2 + (m_0 c^2)^2} - m_0 c^2 = (\gamma - 1)m_0 c^2$$

$$f = f_{\text{proper}} \sqrt{\frac{1 - \beta}{1 + \beta}} \quad \text{for source and detector separating}$$

$$f(\beta \ll 1) = f_{\text{proper}} \left(1 - \beta + \frac{1}{2}\beta^2 \right)$$

$$f_{\text{trans}} = f_{\text{proper}} \sqrt{1 - \beta^2} \quad f_{\text{trans}}(\beta \ll 1) = f_{\text{proper}} \left(1 - \frac{1}{2}\beta^2\right)$$

$$\tau = \beta x + \gamma^{-1} \tau' \quad \text{for lines of constant } \tau'$$

$$\tau = \frac{x - \gamma^{-1} x'}{\beta} \quad \text{for lines of constant } x'$$

$$x' = \frac{x_{\text{intersection}}}{\gamma} = x'_{x \text{ scale}} \sqrt{\frac{1 - \beta^2}{1 + \beta^2}} \quad \tau' = \frac{\tau_{\text{intersection}}}{\gamma} = \tau'_{\tau \text{ scale}} \sqrt{\frac{1 - \beta^2}{1 + \beta^2}}$$

$$\theta_{\text{Mink}} = \tan^{-1}(\beta)$$