Electromagnetic Induction

1) Electromagnetic Induction

Is ubiquitous in Nature
and techonology
— but not obvious
— it was first discovered
~1830 in England by Faraday
and independently in the US
by Joseph Henry.
Faraday gets most of the credit
since published first.

What is it qualitatively?

A changing magnetic field
(a B-field) creates
an EMF (electromotive force),

2)

It's ineluctable
— it just everywhere.

If you move relative to a B-field, there's an induced EMF in your frame — just moving relative to Earth's field.

— If change a B-field in your frame of rest by moving a magnet or changing a current there's an induced EMF

Electromagnetic radiation (i.e., light) depends on induction.

In technology, induction is the effect that allows electric generaton that turn mechanical energy into electric generators and transformers that allow step ups/downs in potential. Transformers are everywhere — you ordinarily don't notice since they are shielded for reasons of safety.

2) Faraday's Law of Induction

There are actually two Faraday's laws of induction

Maxwell Faraday version (an exact classical law)

Motional EMF — exact

— but in short form they have the same formula:

derived result in many important cases

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Induced emf around the closed contour that links the magnetic flux Φ

— It's caused by the changing magnetic flux

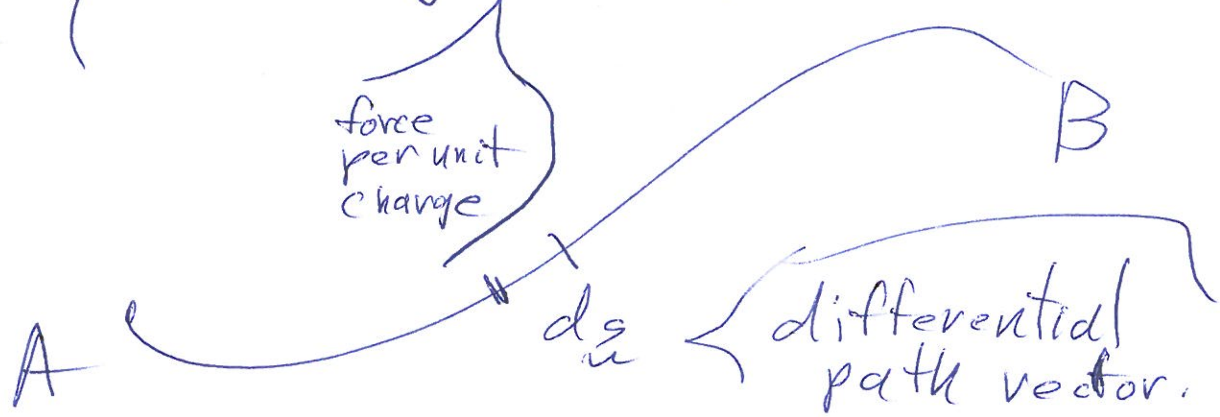
minus is part of the convention

Φ is the magnetic flux linked by a closed contour in space.

3) What is EMF anyway?

It's the force per unit charge integrated along some path at one instant in time (Griffiths)

$$\underbrace{\mathcal{E}}_{\text{emf}} = \int \underbrace{\underset{\sim}{f}}_{\text{force per unit charge}} \cdot d\underset{\sim}{s}$$



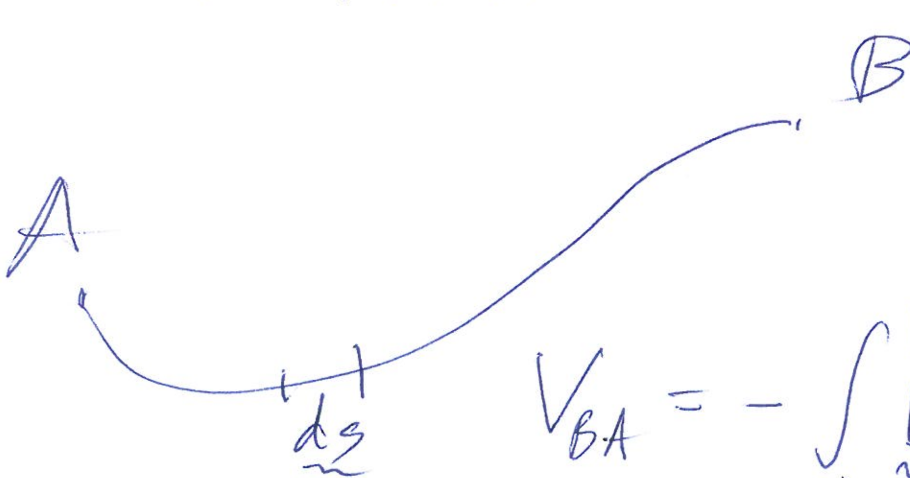
So electromotive force (EMF) is NOT a force (as the term is defined in physics) but the name is traditional, and so we are stuck with it.

EMF is often confused with electrical potential. ~~and~~

— They are often equal in value and that's the main reason why.

— Even when they are NOT equal, EMF plays a role similar to potential.

For example
Say you go from A to B in an E-field.



$$V_{BA} = -\int_A^B \underset{\sim}{E} \cdot d\underset{\sim}{s}$$

Now you actually moved a charge q along that path with <u>no</u> change in speed from A to B, then the <u>work-kinetic energy</u> theorem says

in general the speed can change along the path but $v_B = v_A$

$$0 = \Delta KE = W_{net}$$

$$= W_{E\text{-field}} + W_{other}$$

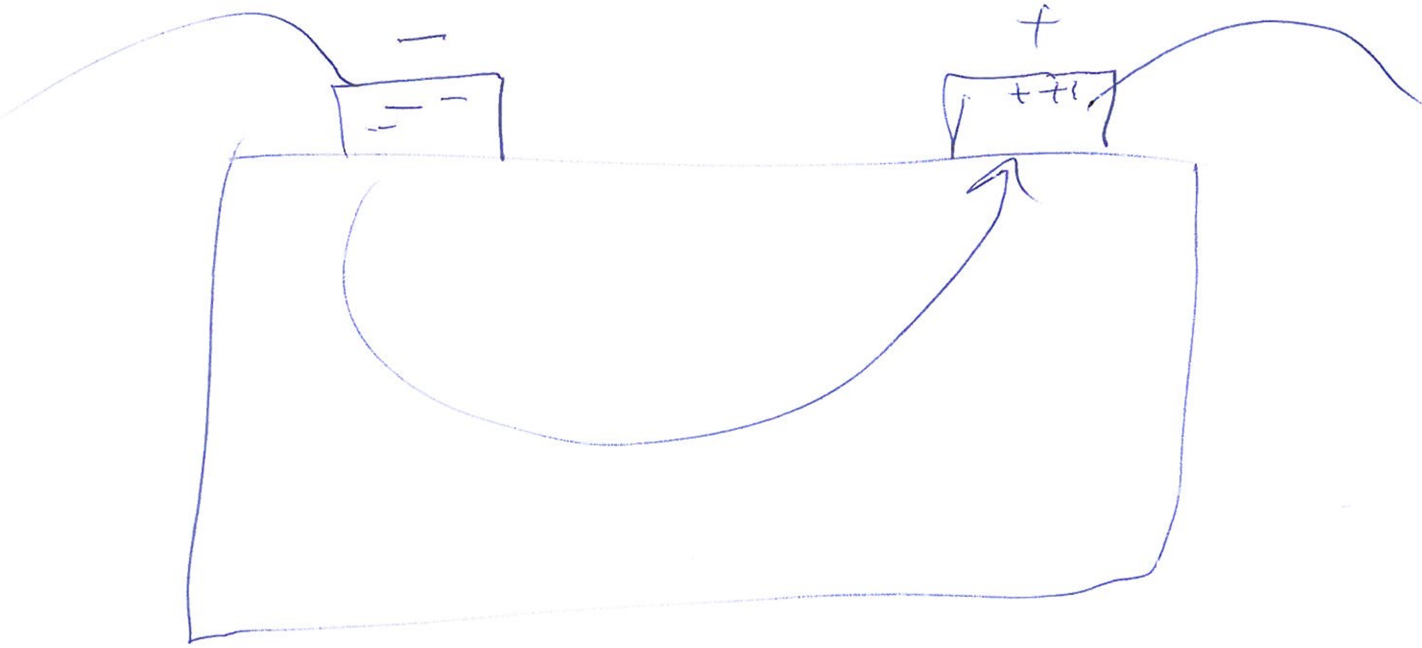
$$= \underbrace{q\int_A^B \underset{\sim}{E}\cdot d\underset{\sim}{s}}_{-V_{BA}} + \underbrace{q\int_A^B \underset{\sim}{f}_{other}\cdot d\underset{\sim}{s}}_{\mathcal{E}}$$

$$\therefore \quad \mathcal{E} = V_{BA} = -\int_A^B \underset{\sim}{E}\cdot d\underset{\sim}{s}$$

f_{other} is an EMF force per unit charge, but it's not <u>the electric field</u>

It's something else.

This situation happens in batteries



$$V_{battery} = V_+ - V_-$$

Some force inside the battery ~~[illegible] carries charge~~ pushes charge from the ~~positive to the negative~~ negative to the positive terminal against the E-field set up by the charge distribution mostly on the terminals.

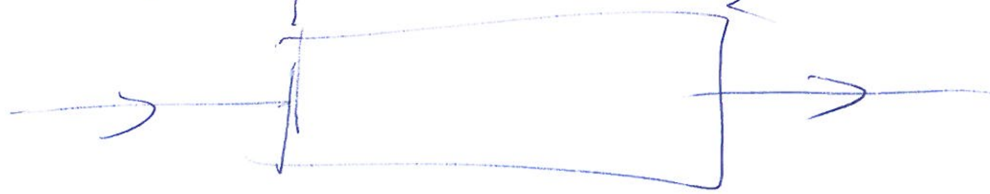
In this case the EMF force is also an electric field force, but a microscopic one ~~can~~ at the atomic level.

$$V_+ - V_- = \mathcal{E}$$

often the case.

~~Similarly~~ since there is no acceleration in steady flow

Similarly in resistors



$$V_{Resistor} = V_1 - V_2$$

The electric field pushes charge from 1 to 2 down a potential drop

But the charge does **NOT** accelerate and so there must be an EMF force

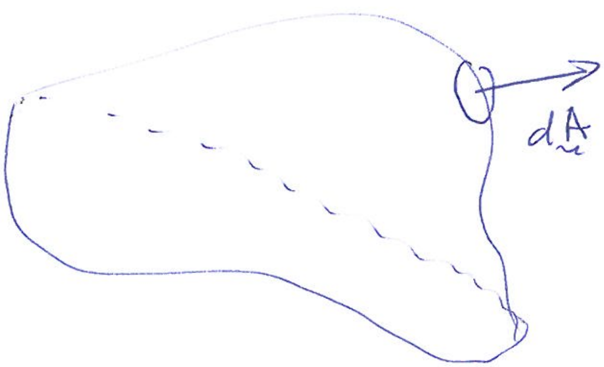
giving $\mathcal{E} = V_{12}$

- Again the EMF force is ~~as~~ actually an E-field force at the microscopic level acting against flowing charge.

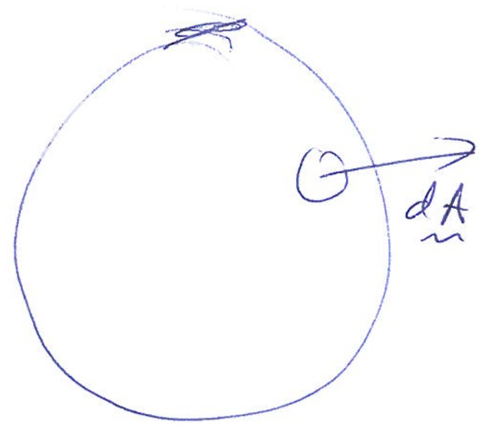
The EMF force can be the magnetic force and other forces too in principle — but ~~often usall~~ usually it's some ~~version~~ manifestation of the electromagnetic force.

4) Magnetic Flux Φ

Consider any surface in space. There may be a physical material there or just an imagined surface



Open surface



Closed surface

$d\underset{\sim}{A}$ is a differential surface area vector

It has area dA and points normal to the surface

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The differential area is so small there is no ambiguity about the normal direction

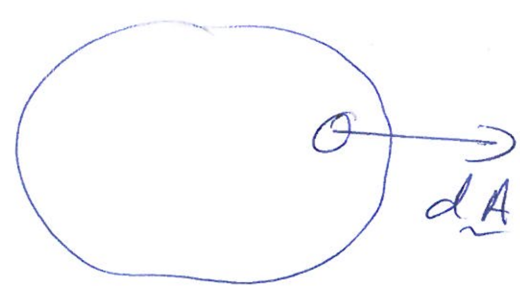
But

Which way does it point "in" or "out".

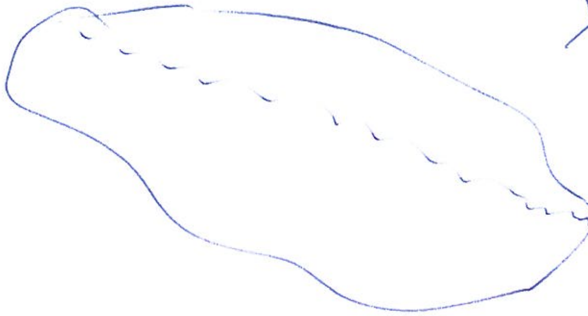
In any case, you must set a convention and stick to it.

If the surface is closed like a sphere,

The differential area vector points outward by universal convention.



If a surface is open, you must choose sides (and never switch sides — unless it's convenient to do so.



open surface

The magnetic flux thru differential area $d\mathbf{A}$ is

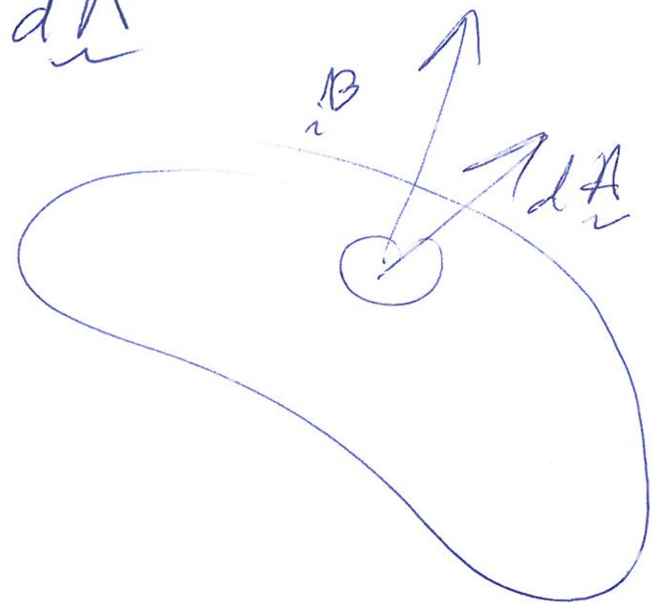
$$d\Phi = \mathbf{B} \cdot d\mathbf{A}$$

called a <u>flux</u> even though neither $\mathbf{B}$ nor $d\mathbf{A}$ need be changing. Just a jargon term.

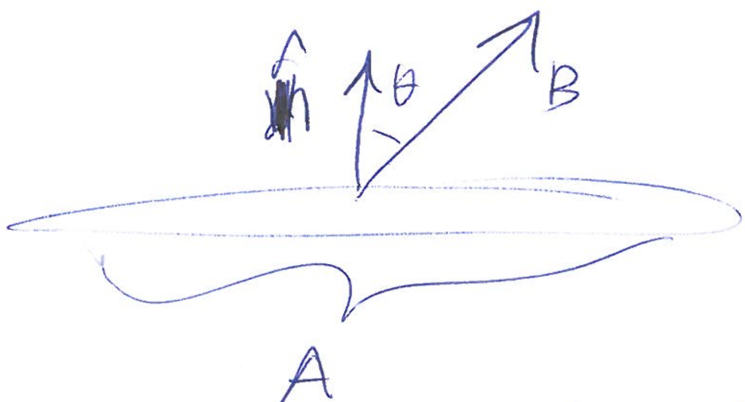
Why this definition? Well it turned out to be useful in ~~des~~ setting up the formalism of physical law.

For a finite surface

$$\Phi = \int_{\text{surface}} \underset{\sim}{B} \cdot d\underset{\sim}{A}$$



example



A flat bit of surface of area A and a constant $\underset{\sim}{B}$-field at angle θ to the normal vector $\hat{n}$

$$\Phi = \underset{\sim}{B} \cdot A\hat{n} = BA\cos\theta$$

For a <u>closed surface</u>

$$\Phi = \oint \underset{\sim}{B} \cdot d\underset{\sim}{A} = 0 \quad \left\{ \text{Gauss's Law for magnetism} \right.$$

This is analogous to Gauss's law (for electricity)

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{q_{enclosed}}{\varepsilon_0}.$$

But there is NO magnetic charge (and people have wondered why not), and

so

$$\oint \underset{\sim}{B} \cdot d\underset{\sim}{A} = 0$$

Both Gauss's law and Gauss's law for magnetism are members of Maxwell's 4 equations (like the Maxwell-Faraday law of induction

The 4th equation is Ampère's law with Maxwell's displacement current)

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4) Faraday's Law
Maxwell Faraday Version

— This is an exact classical law of physics and one of the 4 Maxwell equations that determine all of classical electromagnetism.

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{s} = -\frac{d}{dt}\int \underset{\sim}{B} \cdot d\underset{\sim}{A}$$

around a contour fixed in the frame of the observer.

~~Contour~~ Surface linked by contour

The EMF force here is caused by an induced electric field

No change causes this $\underset{\sim}{E}$-field.

It's an <u>induced $\underset{\sim}{E}$-field</u>.

<u>Note</u> $\oint \underset{\sim}{E} \cdot d\underset{\sim}{s} = 0$

for an $\underset{\sim}{E}$-field caused by charge

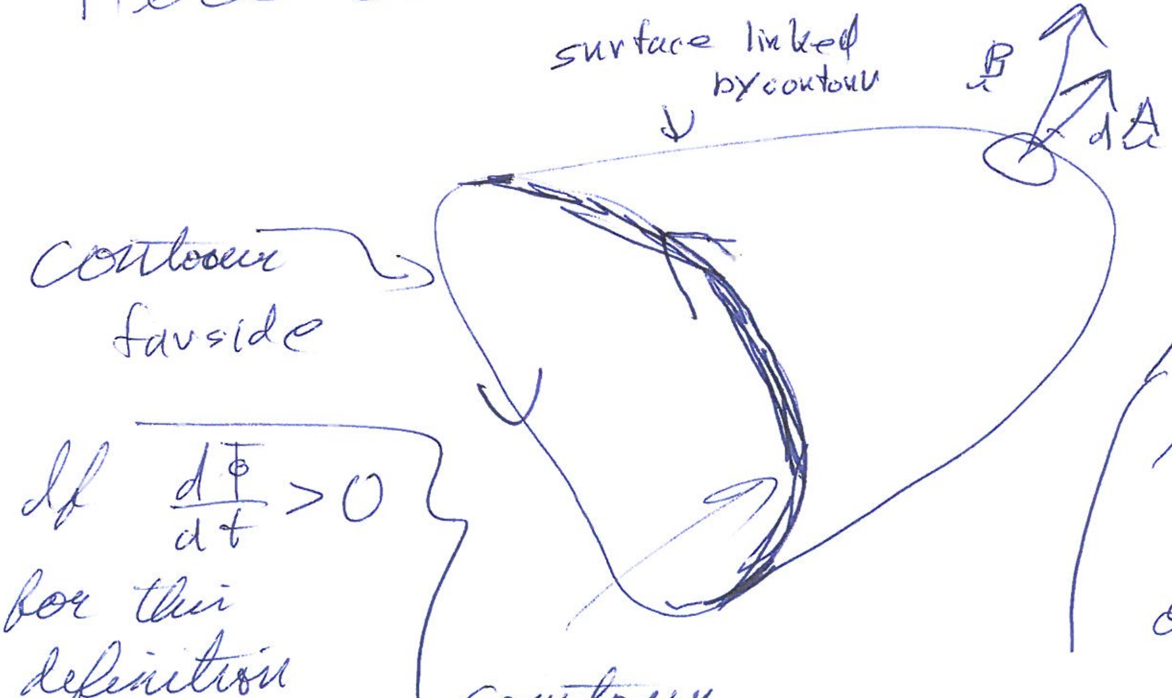
<u>and</u> for such an $\underset{\sim}{E}$-field a <u>potential</u> can be defined throughout space.

For an <u>induced $\underset{\sim}{E}$-field</u> no ~~polen~~ potential can be defined if you go all around the contour, since $\oint \underset{\sim}{E} \cdot d\underset{\sim}{s} \neq 0$

<u>But</u> if con if you ~~gou~~ go part way around. —— and this is often done as with electric generators — and call that that potential.

What direction is the $\overset{\text{induced}}{}$ EMF?

Here conventions tell all



say the $d\underset{\sim}{A}$ vector points out this way

If $\frac{d\Phi}{dt} > 0$ for this definition of $d\underset{\sim}{A}$, the induced $\underset{\sim}{E}$-field points as shown

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If you look at the diagram, the induced E-field causes an EMF that would drive a current in the direction shown ⟶ if there were a conductor along the contour.

The induced E-field and EMF are present in any case.

It's actually tricky to remember the conventions, but Lenz's law makes it relatively easy to get the direction.

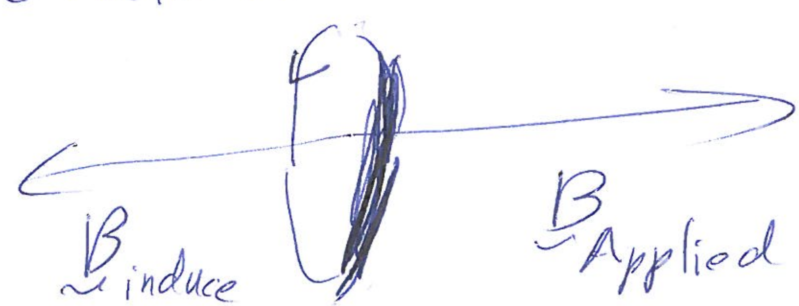
Lenz's law is NOT a real law — the name is conventional — it's a result derived from the conventions of Faraday's law.

Lenz's Law

If a conduct~~ing~~ wire were along the contour, the induced current would create an induced B-field that opposed the changes ~~the~~ in the applied B-field,

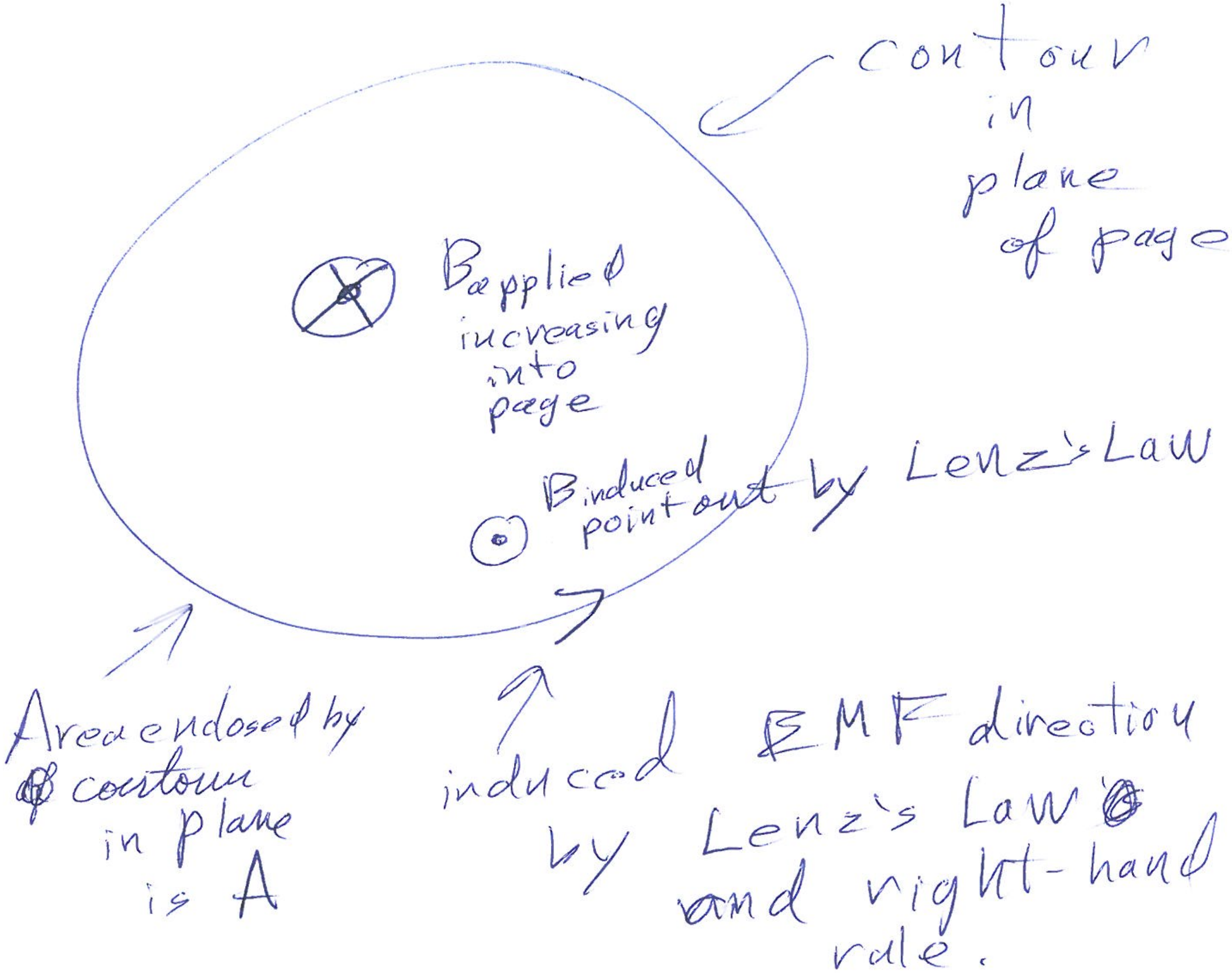
Lenz's law works whether a ~~current is~~ conductor is present or not.

The induced B-field is related to the EMF by a right-hand rule



right thumb along $\underset{\sim}{B}_{induced}$ and fingers curl in the direction of the induced B-field and EMF

Example



$$\Phi = \underset{\sim}{B} \cdot A\hat{n}$$

$$= BA \text{ in this case}$$

$$\mathcal{E} = -\frac{d}{dt}(BA)$$

$$\mathcal{E} = -\frac{dB}{dt}A$$

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~~Ell~~ =

If $\frac{dB}{dt} > 0$,

$\mathcal{E}$ is counterclockwise

If $\frac{dB}{dt} < 0$, $\mathcal{E}$ is clockwise.

Say $A = 1 m^2$ MKS units

and $\frac{dB}{dt} = 10^{-4}\ T/s$

$$\mathcal{E} = -10^{-4}\ \frac{Tm^2}{s}$$

$$= 10^{-4}\ V \text{ counterclockwise}$$

$d\underset{\sim}{F} = I\, d\underset{\sim}{s} \times \underset{\sim}{B}$

$N = C/s \cdot m \cdot T$

$V = \frac{N \cdot m}{C}$

$= \frac{C/s \cdot m \cdot T \cdot m}{C}$

$= Tm^2/s$

The units work out correctly

No Potential is definable if you round and round the contour.

But if you go around just once, one could if you like write $V = \mathcal{E}$.

If the contour was a wire of resistance R

$$I = V/R = \mathcal{E}/R$$

Say $R = 10^{-3}\,\Omega$

$$I = 10^{-4}\,V / 10^{-3}\,\Omega$$

$$= 0.1\,A.$$

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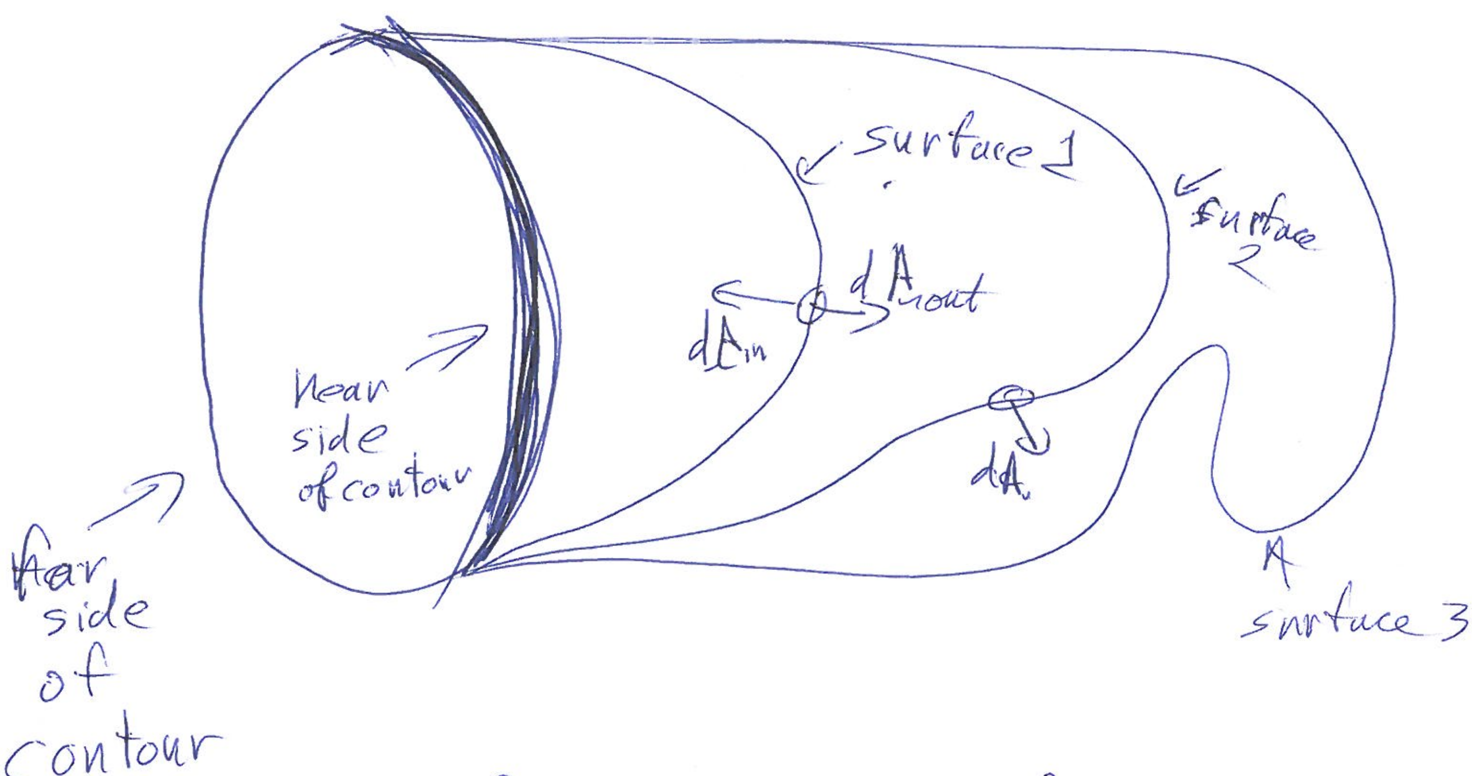
But, But what <u>surface</u> is used calculate the flux for a given contour?



$$-\Phi_1 + \Phi_2 = -\int_1 \underset{\sim}{B} \cdot d\underset{\sim}{A}_{out} + \int_2 \underset{\sim}{B} \cdot dA$$

$$= \int_1 \underset{\sim}{B} \cdot d\underset{\sim}{A}_{in} + \int_2 \underset{\sim}{B} \cdot d\underset{\sim}{A}$$

$$= \oint_{\text{closed surface } 1+2} \underset{\sim}{B} \cdot dA = 0$$ by Gauss's law for magnetism

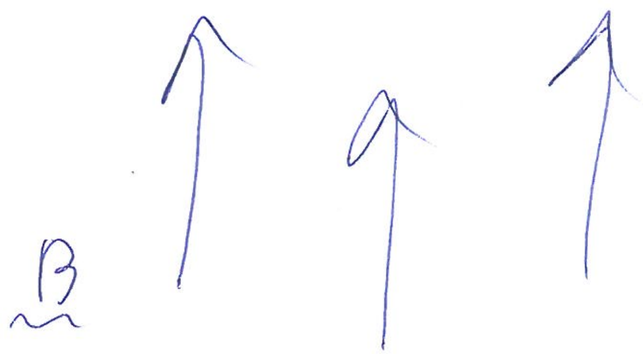
So $\Phi_1 = \Phi_2$.

Since surfaces 1 and 2 are general, all surfaces linked by the contour can be used and you will get the same result.

<u>But</u> some surfaces are much ~~easy~~ easier to do the calculation for.

5) Faraday's Law Motional EMF Version

Say you had a $\underset{\sim}{B}$-field that was constant with time.



Thru any contour at rest in the frame of the B-field

$$\mathcal{E} = -\frac{d\Phi}{dt} = 0$$ by Maxwell-Faraday law

since $\underset{\sim}{B}$ is constant with respect to time.

But what about a moving contour?

There will be an EMF around that contour.

The EMF force is the magnetic force.

Called the motional EMF

$$d\mathcal{E} = \underbrace{\underset{\sim}{v}_{tot} \times \underset{\sim}{B}}_{\text{magnetic force per unit charge}} \cdot \underbrace{d\underset{\sim}{s}}_{\text{differential bit of contour at one instant in time}}$$

$$\underset{\sim}{v}_{tot} = \underset{\sim}{v}_{contour} + \underset{\sim}{v}_{\text{charge relative to contour}}.$$

If you go around a contour

$$\mathcal{E} = \oint \underset{\sim}{v}_{tot} \times \underset{\sim}{B} \cdot d\underset{\sim}{s} \, .$$

Why go all the way around the contour?

Well you are usually thinking of a <u>contour</u> that is actually has a conducting wire and one wants to know the total EMF driving a current.

a loop or current loop

~~If there is a~~

There is a cute vector identity that leads to a simplification in the current carrying wire case

$$(\underset{\sim}{v}_{tot} \times \underset{\sim}{B}) \cdot d\underset{\sim}{s}$$

$$= d\underset{\sim}{s} \cdot (\underset{\sim}{v}_{tot} \times \underset{\sim}{B})$$

$$= \underset{\sim}{B} \cdot (d\underset{\sim}{s} \times \underset{\sim}{v}_{tot})$$

Proof beyond our scope Jackson P. -2 But not very ~~actually~~ hard actually

$\angle\, d\underset{\sim}{s}$

At an instant in time $d\underset{\sim}{s}$ is parallel to $\underset{\sim}{v}$ relatative to ~~current~~ contour

$$\therefore \quad d\underset{\sim}{s} \times \underset{\sim}{v}_{tot} = d\underset{\sim}{s} \times (\underset{\sim}{v}_{contour} + \underset{\sim}{v}_{rel})$$

$$= d\underset{\sim}{s} \times \underset{\sim}{v}_{cont} + \underbrace{d\underset{\sim}{s} \times \underset{\sim}{v}_{rel}}$$

0 because they are parallel.

$$= \underset{\sim}{B} \cdot (d\underset{\sim}{s} \times \underset{\sim}{v}_{con})$$

$$= d\underset{\sim}{s} \cdot (\underset{\sim}{v}_{con} \times \underset{\sim}{B})$$

$$= (\underset{\sim}{v}_{con} \times \underset{\sim}{B}) \cdot d\underset{\sim}{s}$$

So the current velocity cancels out.

$$\mathcal{E} = \oint \underbrace{\underline{v}_{con}}_{\text{we can suppress the subscript if we know what we mean.}} \times \underline{B} \cdot d\underline{\ell}$$

You could actually use this formula to find the EMF around the moving ~~loop~~ current loop. $\Big\}$ motional emf

~~But there is an ec~~

But there is an easier way.
The motional emf Faraday's law

$$\underset{\sim}{\mathcal{E}} = -\frac{d\Phi}{dt}$$

emf around a closed contour that links magnetic flux Φ

where here the magnetic flux is changing because the linking contour is changing with time and the B-field is constant in time.

The motional EMF (ME) Faraday's law is NOT actually a law. (Unlike the Maxwell-Faraday's law which is an exact classical result)

It is a derived result which is often exactly correct, often ~~app~~ correct to good approximation,

but can be wrong
sometimes.

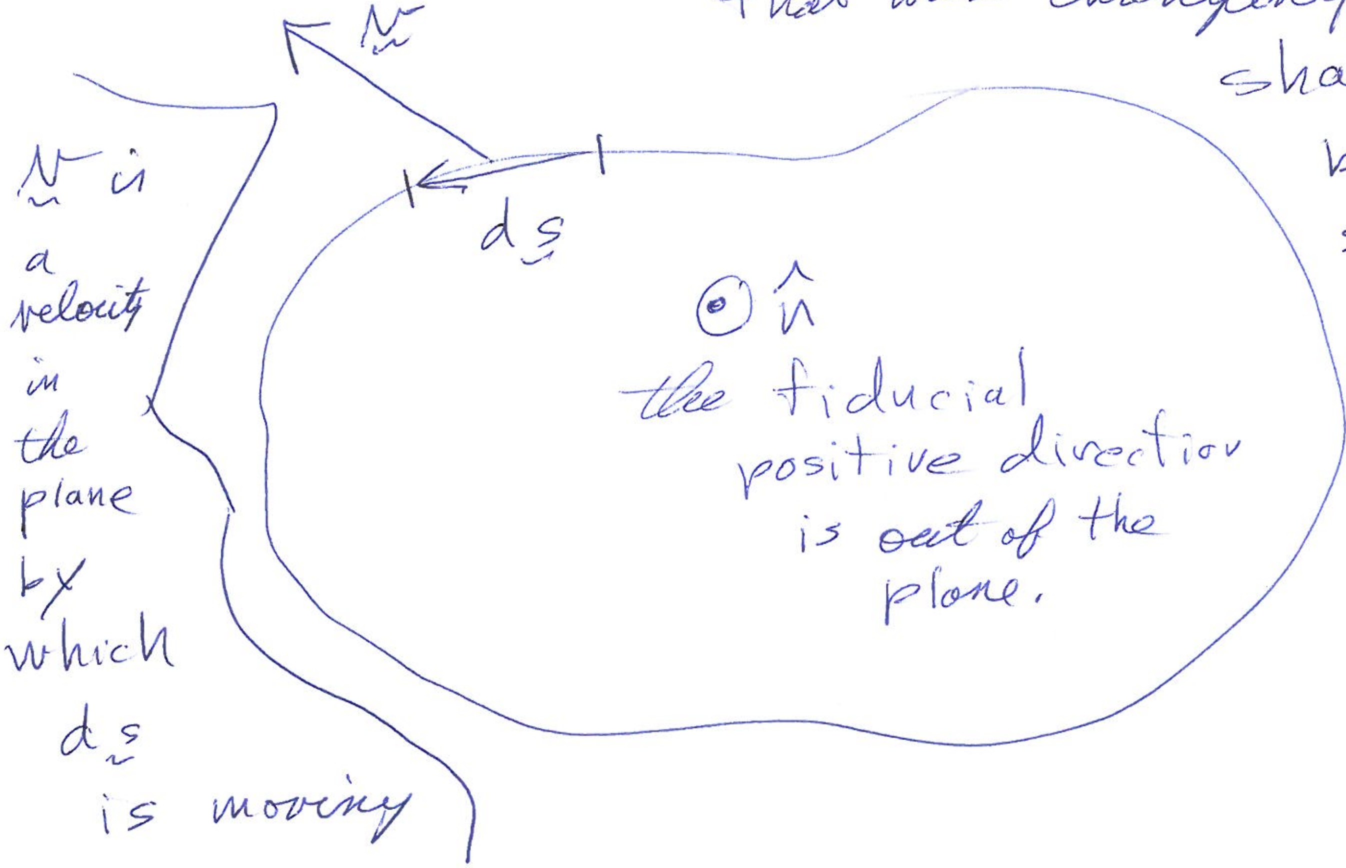
In fact, the cases where
~~A~~ the ME Faraday's law
are wrong seem to be
rather rare in cases
of interest → I don't
know any — but then
I've NOT thought about
it much.

One applies the motional EMF
Faraday's law like
the Maxwell-Faraday law
(Mutatis Mutandis).
Lenz's law works for

the ME Faraday's law too.

Proof of Motional EMF Faraday's law in a special case

Say we had a planar contour that was changing shape, but staying planar.

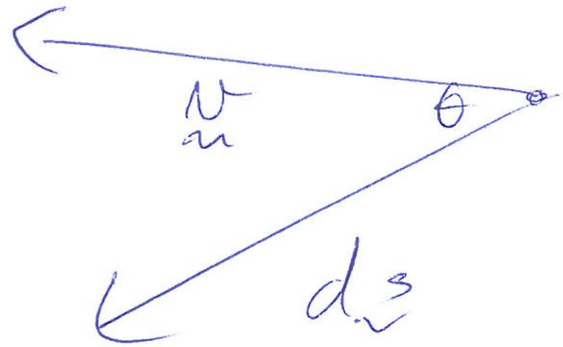


$$\underbrace{d\mathcal{E} = \mathbf{v} \times \mathbf{B} \cdot d\mathbf{s}}$$

We assume a conducting wire case, and so the relative charge velocity can be neglected.

$$d\mathcal{E} = \mathbf{v} \times \mathbf{B} \cdot d\mathbf{s}$$

$$\left.\begin{aligned} &= d\mathbf{s} \cdot (\mathbf{v} \times \mathbf{B}) \\ &= \mathbf{B} \cdot (d\mathbf{s} \times \mathbf{v}) \end{aligned}\right\} \text{Jackson P. -2}$$



$$d\mathbf{s} \times \mathbf{v} = ds\, \underbrace{v \sin\theta}_{v_\perp} (-\hat{n})$$

$v_\perp$ the ~~perpendic~~ perpendicular component

$v_\perp > 0$ for $\theta > 0$

$v_\perp < 0$ for $\theta < 0$

~~If~~ $\mathbf{B}$ points up, the motional emf is clockwise. This is what the minus sign means.

$$\Delta\mathcal{E} \simeq -\mathbf{B} \cdot \hat{n}\, \underbrace{\Delta s\, v_\perp}_{\frac{\Delta A}{\Delta t}}$$

Note if $ds \doteq \Delta s$ stretches in Δt that is a higher order term

$$\Delta S(t + \Delta t) = \Delta S(t) + \left(\frac{\Delta S}{\Delta t}\right)_{\text{stretching}} \Delta t$$

and this cancels out when $\Delta t \to 0$

As one notes stretching

~~without~~ with $v_\perp = 0$
doesn't ~~increase a~~
change area

$$\cancel{\Delta}\mathcal{E} \cong -\frac{B\cdot\hat{n}\,\Delta A}{\Delta t}$$

$$\cancel{\Delta}\mathcal{E} \cong -\frac{\sum_i B_i\cdot\hat{n}\,\Delta A_i}{\Delta t}$$

Summing up all the contributions around the contour

$$\mathcal{E} \cong -\frac{\left(\sum_j B_j\cdot\hat{n}\,\Delta A_j(t+\Delta t) - \sum_j B_j\cdot\hat{n}\,\Delta A_j(t)\right)}{\Delta t}$$

$$\cong -\frac{\Delta\left[\int B\cdot dA\right]}{\Delta t}$$

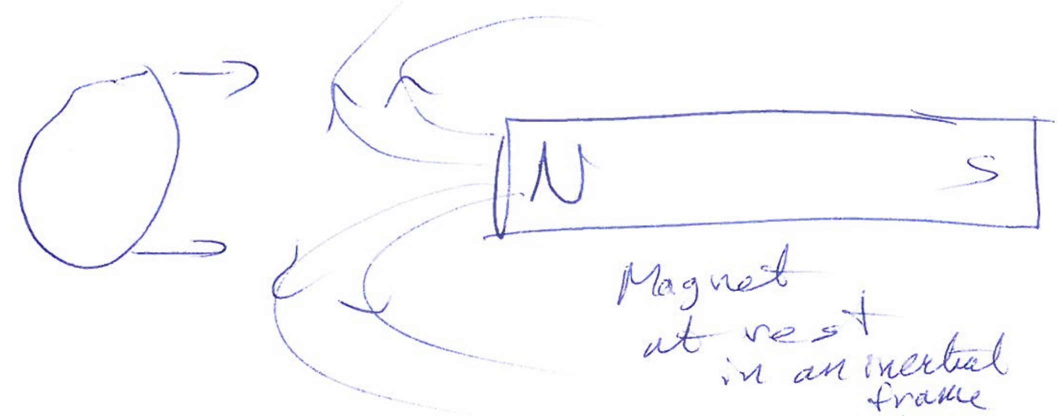
Now adding up flux over the whole linked area

$$\mathcal{E} = \lim_{\Delta t\to 0} -\frac{\Delta[\int B\cdot dA]}{\Delta t} = -\frac{d\Phi}{dt}$$

I ~~think~~ this proof is exactly correct.

Examples of Motional EMF ~~Farad~~

1) case a



Moving rigid loop ~~with~~ with constant velocity and orientation

The Motional EMF Faraday's law probably works accurately (textbooks imply as much)

But one can calculate ~~it from~~ the motional emf from the ~~the~~ ~~B-f~~ magnetic force law in any case.

What if we switch to the inertial frame in which

– the loop is at rest
and the magnet is moving

<u>Case 6</u>

at rest

Well the EMF should
be the same.
– Frame-independent

Can we use the ~~Maxwell-Faraday's~~
Faraday's law in this case
to get an exact result.

Yes & No.

The MF Faraday's will be exact
for the changing B-field.

but the electromagnetic field

Of the magnets transforms in going from ~~it~~ ~~it's own rest frame~~ the magnet's own rest frame to the rest frame ~~fram~~ of the loop

$$\underset{\sim}{E}' = \underset{\sim}{E}'(\underset{\sim}{E}, \underset{\sim}{B})$$

$$\underset{\sim}{B}' = \underset{\sim}{B}'(\underset{\sim}{E}, \underset{\sim}{B})$$

Jackson 552

in loop frame

in magnet frame

Even if $\underset{\sim}{E} = 0$

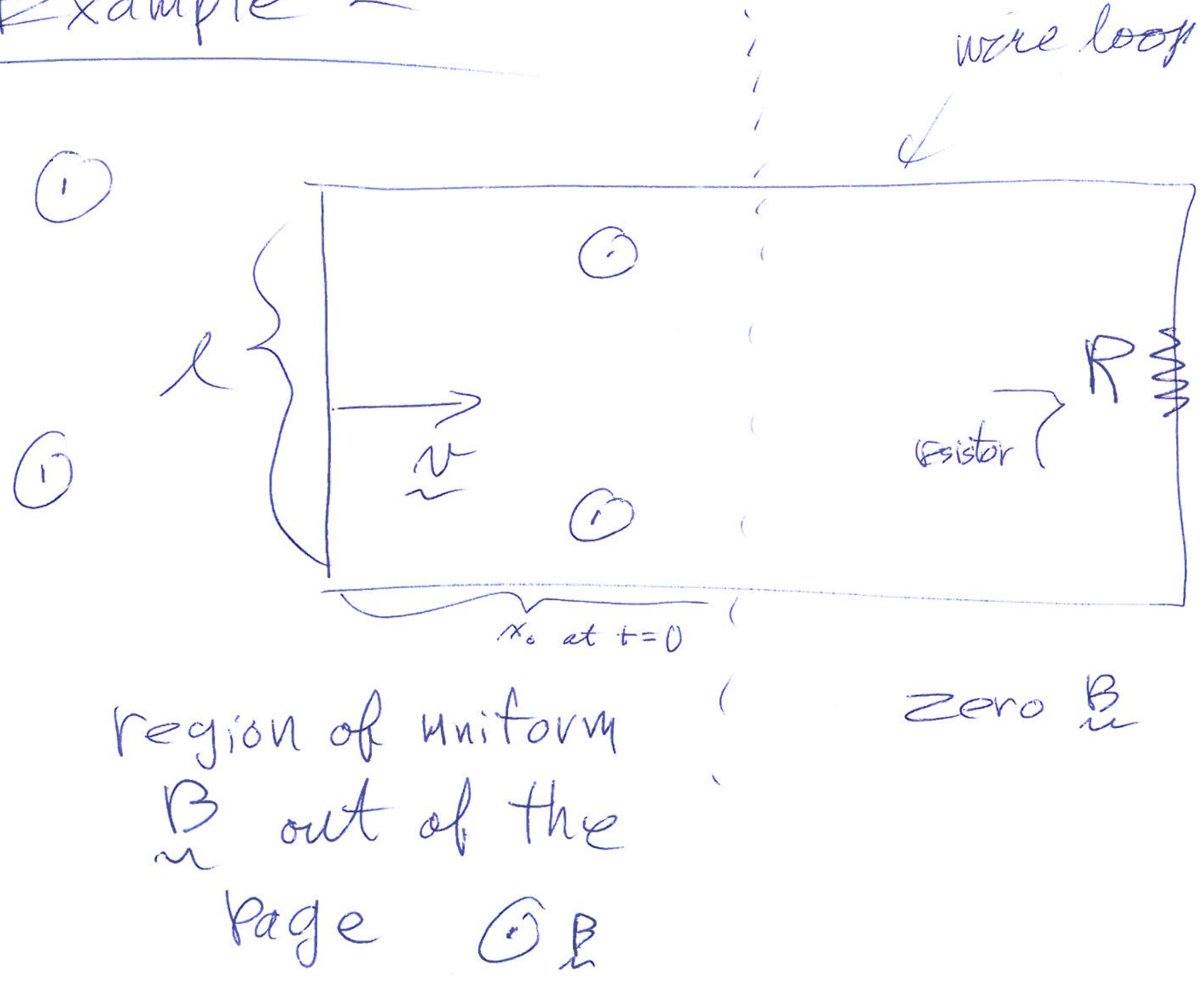
$\underset{\sim}{E}'$ is not zero in general.

So it's tricky

But if the Motional EMF — Faraday's

works for case a, it ~~shou~~ will work for case b. (with the untransformed $\underset{\sim}{B}$ used, not $\underset{\sim}{B}'$)

Example 2



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The loop is pushed to right at velocity $\underset{\sim}{v}$

Using the Motional EMF Faraday's law

Does $B_{induced}$ have an effect on the EMF? Yes, but for single loops it's usually small enough to ignore. It's a complex feedback effect. But for coils it is important

$$\mathcal{E} = -\frac{d(BA)}{dt}$$ where $A = l(x_0 - vt)$

$$= -Bl(-v)$$

$$= Blv$$ since positive the EMF is counterclockwise just as Lenz's law tells use.

Using Kirchhoff's voltage law (which is really about EMFs not ~~potentials~~ except where they are the same) and Ohm's law

$$\mathcal{E} = IR$$

$$I = \frac{\mathcal{E}}{R} = \frac{Blv}{R}$$

$B_{induced} \sim \frac{\mu_0}{2\pi l} I_{induced}$

$\sim \frac{\mu_0}{2\pi} \frac{v}{R} B$

$\sim 2\times10^{-7} \frac{v}{R} B$

So $B_{ind} \ll B$ for reasonable v & R.

The power output in the resistor is

$$P_{output} = I V_{drop}$$
$$= I \mathcal{E}$$
$$= \frac{\mathcal{E}^2}{R} = \frac{(Blv)^2}{R}$$

So we've got energy for nothing?

No. It takes power to push the loop.

Recall

$$\underline{F} = I\, d\underline{l} \times \underline{B}$$

$$\underline{F}_{top} = I x B \hat{y}$$

$$\underline{F}_{bottom} = I x B(-\hat{y})$$

$$\underline{F}_{side} = I l B(-\hat{x})$$

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The top & bottom forces cancel each other.

<u>But</u> the side magnetic force doesn't.

The pusher must cancel

is

$$\underset{\sim}{F}_{push} = -\underset{\sim}{F}_{side}$$

$$= I\ell B\hat{x}$$

$$W_{push} = I\ell B\Delta x$$

$$P_{push} = I\ell B v$$

$$= IB\ell v$$

$$= \frac{(B\ell v)^2}{R} = P_{output}.$$

So the pusher supplies the

that gets emitted as waste heat in the resistor.

So no energy has been created.

But energy has been transformed

Mechanical energy → "electrical energy" (loosely speaking)

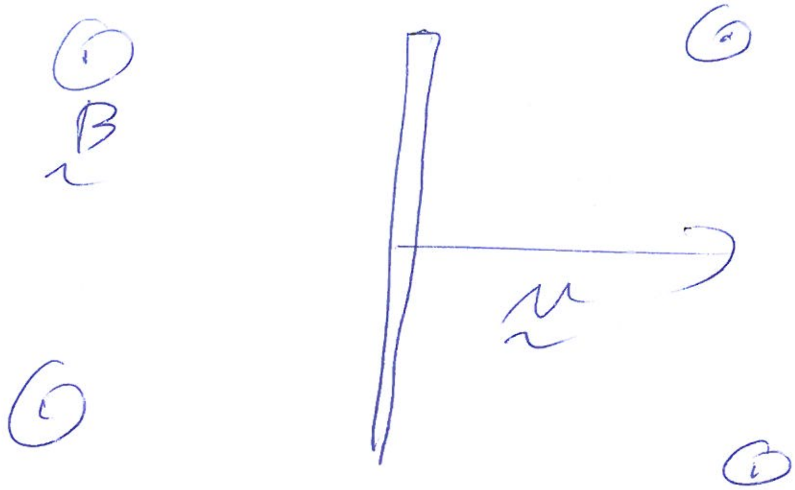
→ ~~Waste~~ heat

~~Note~~

In a crude sense the loop is an electric generator.

If we pulled it in and out we'd create an Alternating Current (AC current).

What if we just pushed a conducting wire segment thru a B field?



Well getting it started would take some <u>work</u>, but after that it would move at constant velocity without a pushing force

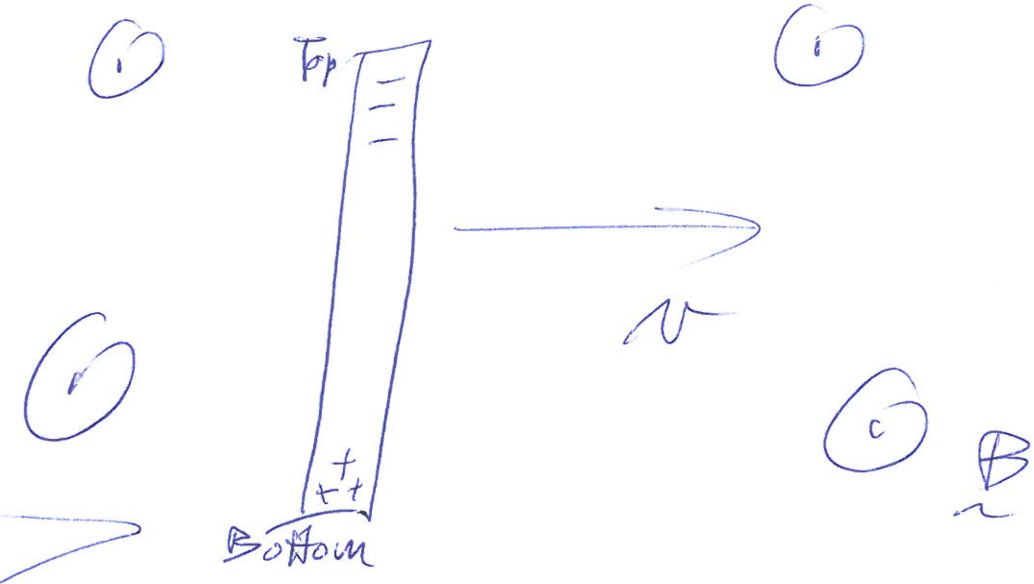
$$\mathcal{E} = \int \underset{\sim}{v} \times \underset{\sim}{B} \cdot ds$$

$= B\ell v$ downward.

But no current would flow

after the constant velocity was established.

What happens is that the wire becomes electrically polarized



charge pushed here by the magnetic force

$$V_{Top \to bottom} = - \int \mathbf{E} \cdot d\mathbf{s}$$

At every point

$$\mathbf{v} \times \mathbf{B} - \mathbf{E} = 0$$

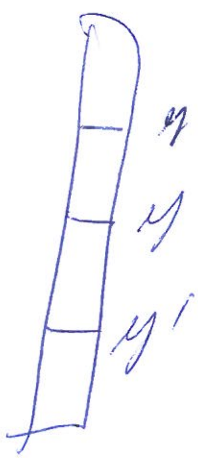
to keep charge static

or $vB = E$

equal in magnitude forces per unit charge, but opposite in direction.

The ~~⊕~~ charge arranges itself to cancel the magnetic force.

$$\underset{\text{constant}}{vB} = E = \int_0^y \frac{k\rho(y')(y-y')}{(y-y')^3}\,dy$$



An integral equation for $\rho(y)$ the charge density.

~~$\frac{dE}{dy} = 0$~~ Too tough for us to solve

maybe not solvable in strict one-dimensionality,

since as $y' \to y$, $\frac{\rho(y')}{(y-y')^2} \to \infty$, unless $\rho(y) \to 0$

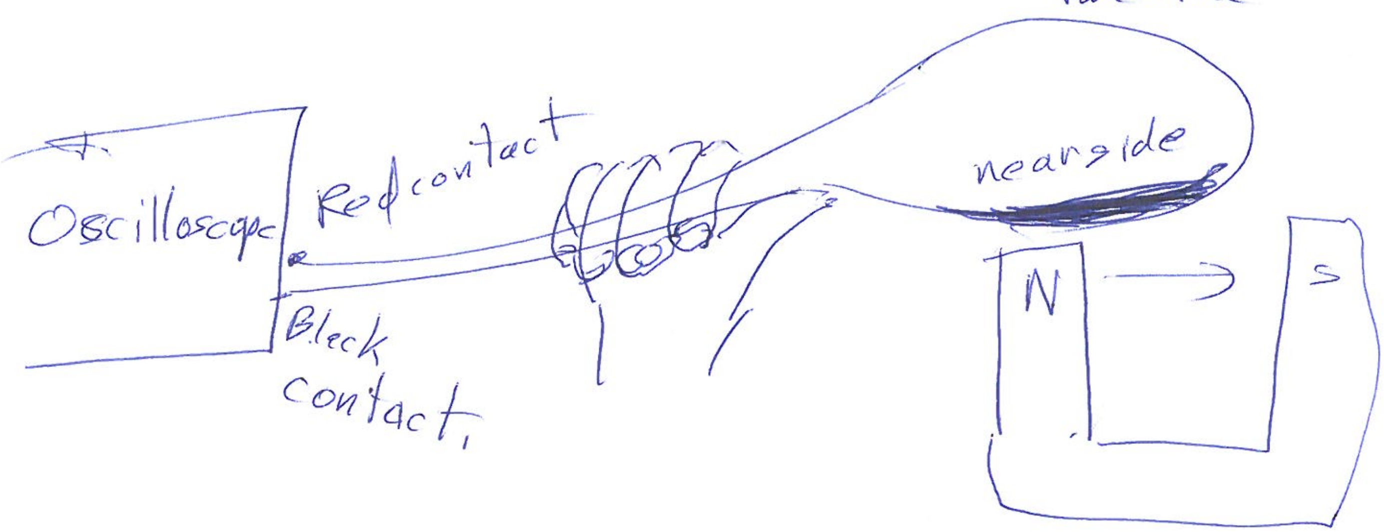
which is clearly not a solution.

Probably narrow rings of charge integrated up would be about right and would work. — a hard problem.

6) Practicalities

Task 1

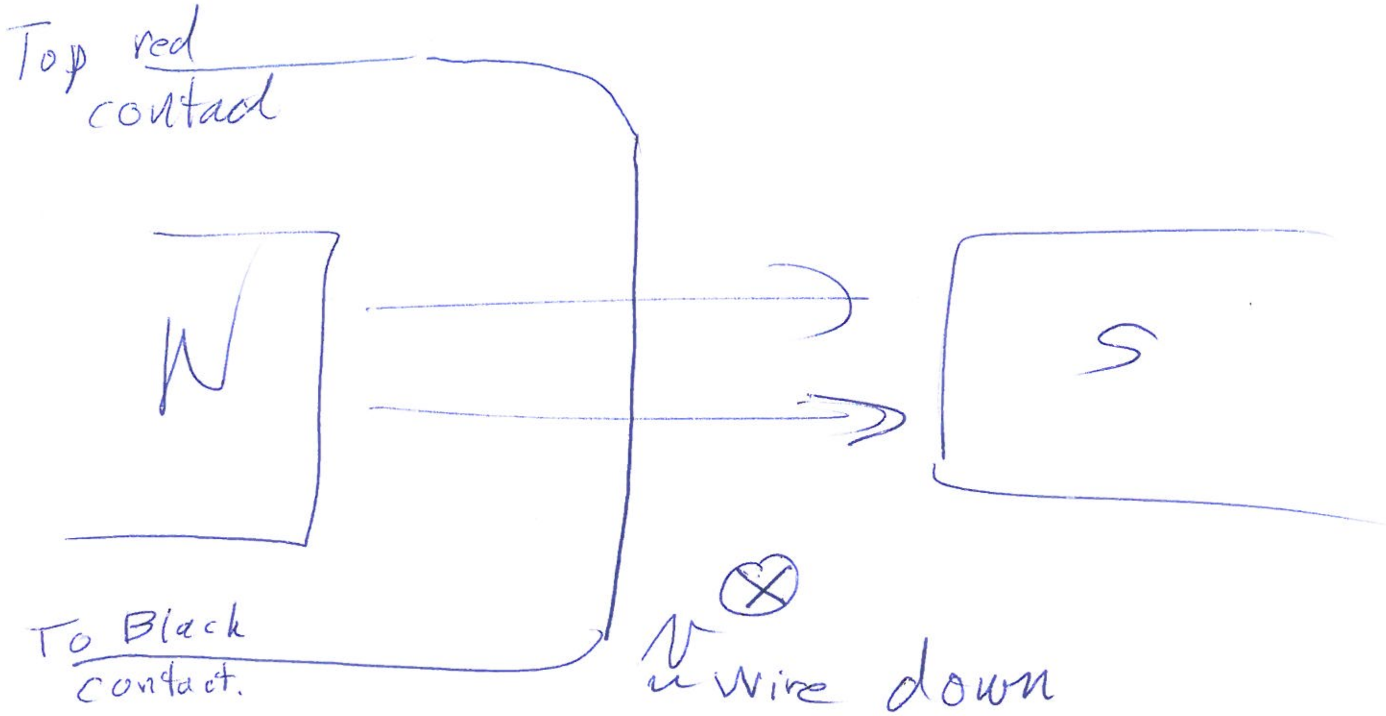
long, stiffer wire works best



+ve charge at red contact causes an upward deflection of oscilloscope trace

+ve charge at black contact causes a downward of trace

Top view



$$\underset{\sim}{F} = q\,\underset{\sim}{v} \times \underset{\sim}{B}$$

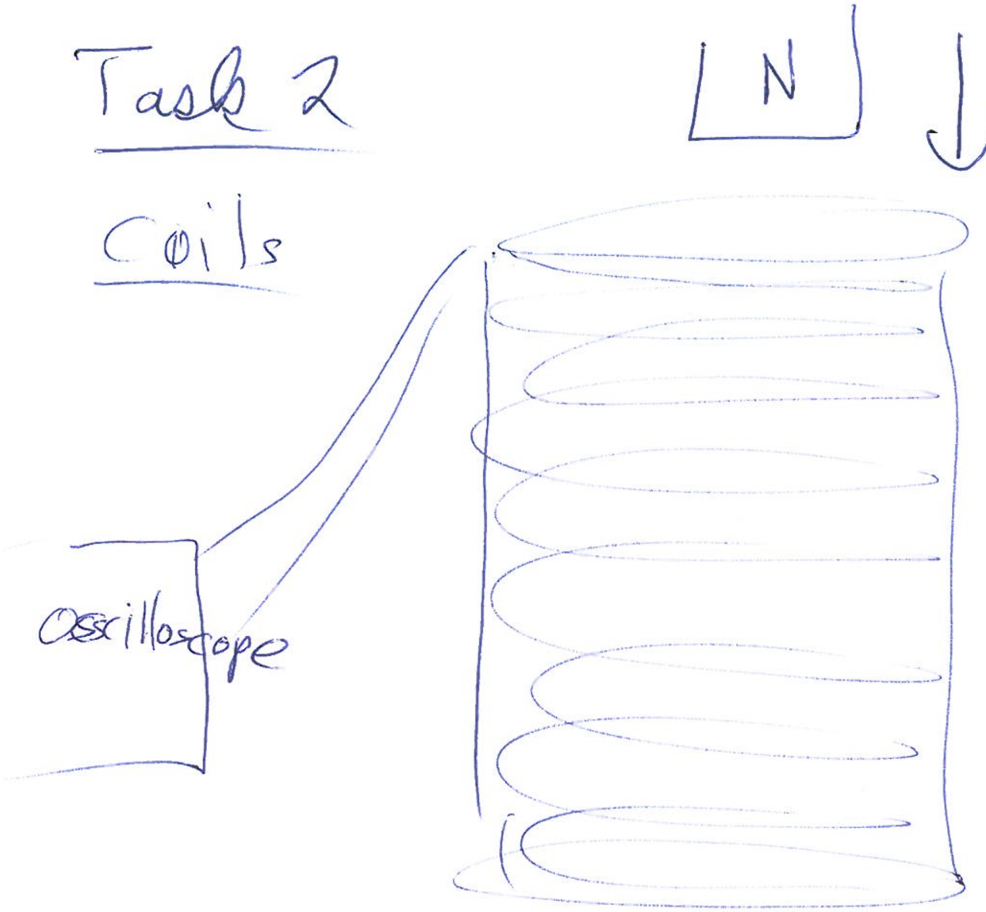
~~= low~~

= magnetic force points { upward on page? / downward?

Which way does trace deflect?

Task 2

Coils



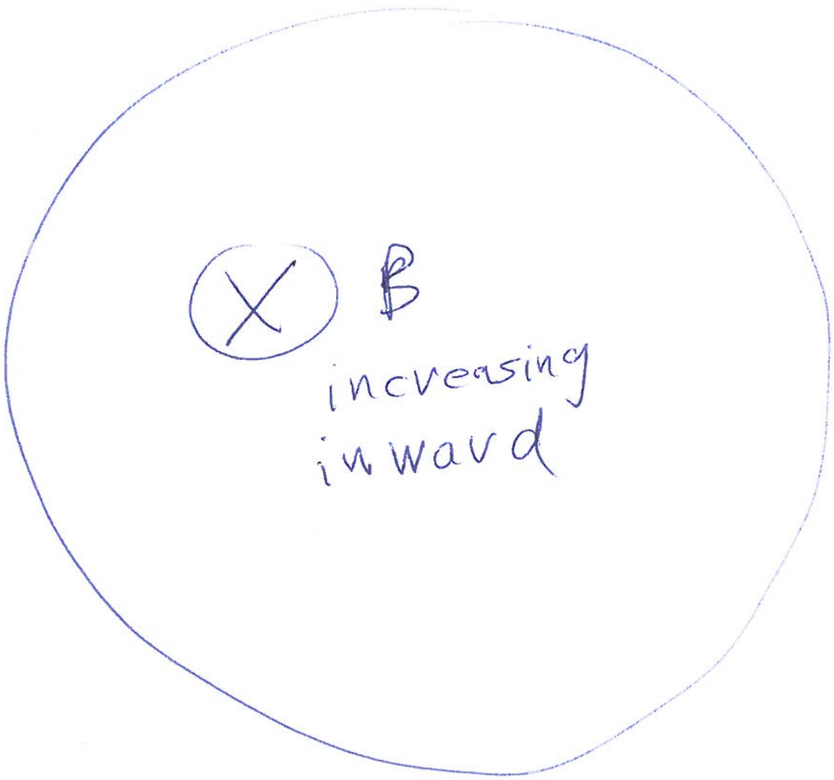
causes magnetic flux thru the turns to increase

Which way does the current flow?

- clockwise or counterclockwise

Does it flow up or down?

Top view



Which way does the induced B-field point?

Which way does current flow?

Does it flow up or down the coil.