

ELECTRIC CHARGE, ELECTRIC FIELD, ELECTRIC FORCE

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ABSTRACT

Lecture notes on what the title says.

Subject headings: keywords — keywords

1. INTRODUCTION

2. ELECTRIC FIELD LINES

2.1. Two Equal Point Charges

3. THE ELECTRIC FIELD IN THE FAR-FIELD LIMIT

In this section, we consider the electric field for a localizable distribution of charge in the far-field limit.

In this context “localizable” means that the distribution can be enclosed in a finite closed surface.

We will do the derivations for discrete sets of charge only for mental clarity. One can also take the continuum limit and turn sums into integrals if needed.

Consider a set of point charges q_i located at points \vec{r}_i . The origin is close to the center of distribution somehow defined. It doesn't have to be very close and we leave how close unspecified, but it is relatively close to the center when we specify that that we are in the far-field limit. The far-field limit is when the size scale of the distribution and the distance of the center to the origin are small compared to the distance to the location where we evaluate the electric field.

In many theoretically interesting and practically important cases, it is useful to put the origin at the center of the distribution somehow defined. The center of mass position is often a useful place.

The general electric field at position \vec{r} for the distribution is

$$\vec{E}(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|^3} (\vec{r} - \vec{r}_i) . \quad (1)$$

We now write

$$\vec{r} - \vec{r}_i = r \left(\hat{r} - \frac{\vec{r}_i}{r} \right) , \quad (2)$$

where r is the magnitude of \vec{r} as usual and \hat{r} is the unit vector in the direction of \vec{r} .

Next we use the law of cosines to write

$$\frac{1}{|\vec{r} - \vec{r}_i|^3} = \frac{1}{(r^2 + r_i^2 - 2rr_i \cos \theta_i)^{3/2}} = \frac{1}{r^3 [1 - 2(r_i/r) \cos \theta_i + (r_i/r)^2]^{3/2}} , \quad (3)$$

where θ_i is the angle between \vec{r} and \vec{r}_i .

Substituting equations (2) and (3) into equation (1), we obtain

$$\vec{E}(\vec{r}) = \sum_i \frac{kq_i}{r^2} \frac{(\hat{r} - \vec{r}_i/r)}{[1 - 2(r_i/r) \cos \theta_i + (r_i/r)^2]^{3/2}} . \quad (4)$$

In the far-field limit, $r_i/r \ll 1$ for all \vec{r}_i by our definition of the far-field limit.

We will explore the far-field limit by Taylor's series expanding equation (4) to first order in small r_i/r . The higher orders terms will be dropped. In general, of course, they can be

worked out to any order one likes and those higher order terms have uses, but they are beyond our scope.

In the far-field limit, the higher the order of the term, the smaller the contribution of the term to the electric field. Each term is of order r_i/r smaller than the previous term. As one goes farther and farther into the far-field, eventually each term in decreasing order becomes negligible.

Expanding equation (4) in a Taylor's series, we get

$$\begin{aligned}\vec{E}(\vec{r}) &= \sum_i \frac{kq_i}{r^2} \left(\hat{r} - \frac{\vec{r}_i}{r} \right) \left(1 + 3\frac{r_i}{r} \cos \theta_i + \dots \right) \\ &= \sum_i \frac{kq_i}{r^2} \left(\hat{r} - \frac{\vec{r}_i}{r} + 3\frac{r_i}{r} \cos \theta_i \hat{r} + \dots \right),\end{aligned}\tag{5}$$

where we have only shown the terms explicitly to 1st order in r_i/r .

Remember the Taylor's series in general will not converge if the r_i/r values get too large and in this case the Taylor's series does not give the electric field value.

The zeroth order term is the monopole term. The 1st order term is the dipole term. The undisplayed 2nd order term is the quadrupole term. The undisplayed 3rd order term is the octupole term. Higher order terms don't have very common names, but the 6th order term has been called the hexacontatetrapole (<http://physics.unl.edu/~tgay/content/multipoles.html>). The expansion itself is called a multipole expansion and the coefficients of the powers of r_i/r are components of what are called the multipole moments—which tensors (e.g. Jackson 1975, p. 138). But don't worry about tensors in general now—a zeroth order tensor is a scalar and 1st order tensor is a vector and that is all we need. The Coulomb constant k is not included as a factor in the multipole moments.

If we truncate to the zeroth order term, we get

$$\vec{E}(\vec{r}) = \sum_i \frac{kq_i}{r^2} \hat{r} = \frac{kq}{r^2} \hat{r},\tag{6}$$

where

$$q = \sum_i q_i \tag{7}$$

is the net charge or the monopole moment of the distribution. (The monopole moment is a scalar or rank zero tensor.) If one is far enough from the distribution, only the monopole term is significant. The above development shows explicitly how any localizable charge distribution begins to look like a point charge at the origin if one gets sufficiently far away from the origin.

Note that the monopole moment is actually origin independent, and thus is an intrinsic property of the distribution.

We now define the dipole moment \vec{p} by the formula

$$\vec{p} = \sum_i \vec{r}_i q_i \tag{8}$$

(e.g., Jackson 1975, p. 137), where we note the dipole moment is a vector, has dimensions of length times charge, and in MKS units has units of m C. (The dipole moment is a vector, as just noted, and so is a rank 1 tensor.) The traditional symbol for the dipole moment \vec{p} is the same as for momentum. Context must decide which is meant. We can't have a new unique symbol for every variable in physics. We'd run out of symbols or have to invent new ones which would be horrible. We have to recycle symbols.

The dipole moment is generally dependent on the origin. But if the monopole moment is zero, then it becomes origin independent and thus an intrinsic property of the distribution. The proof is simple. Say \vec{p} is the dipole moment for a distribution with $q = \sum_i q_i = 0$ for an origin at $\vec{r} = 0$. Now let \vec{p}_0 be the dipole moment relative to an origin at \vec{r}_0 . We find that

$$\vec{p}_0 = \sum_i (\vec{r}_i - \vec{r}_0) q_i = \sum_i \vec{r}_i q_i - \vec{r}_0 \sum_i q_i = \vec{p} - 0 = \vec{p} \tag{9}$$

Since \vec{r}_0 is general, the dipole moment for any origin is \vec{p} .

With the definitions of the monopole and dipole moments, we find to 1st order that the electric field is

$$\vec{E}(\vec{r}) = \frac{kq}{r^2} \hat{r} + k \left[\frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} \right] \quad (10)$$

(e.g., Jackson 1975, p. 138), where we have used the fact

$$\vec{p} \cdot \hat{r} = \left(\sum_i \vec{r}_i q_i \right) \cdot \hat{r} = \sum_i r_i q_i \cos \theta_i . \quad (11)$$

Equation (10) is a very useful expression for finding the electric field in the far-field limit for general localizable charge distributions.

If the charge distribution is overall neutral (i.e., the monopole moment $q = 0$) and the dipole moment is non-zero (i.e., $\vec{p} \neq 0$), then the charge distribution is usually called a dipole or an electric dipole (to distinguish it from magnetic dipoles which we get to in a later lecture). The 1st order far-field dipole electric field (or dipole electric field for brevity) is

$$\vec{E}(\vec{r}) = k \left[\frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} \right] . \quad (12)$$

It has to be emphasized that equation (12) is only the 1st order far-field dipole electric field. If you get close enough to the charge distribution, the higher order terms will become important and eventually the Taylor's series expansion will fail in general.

Equation (12) is actually very useful since microscopic dipoles are everywhere in nature. Many molecules (and some atoms??) have permanent dipole moments and all molecules and atoms can have induced dipole moments when an external field polarizes them. Induced macroscopic dipoles occur all the time too.

Equation (12) shows explicitly that the dipole electric field falls off as $1/r^3$ which is significantly faster than the monopole electric field fall of $1/r^2$. So the forces dipoles tend to be shorter range than monopole electric fields. For example, consider a monopole of charge q and a dipole of charges q and $-q$. In the far-field limit at radial coordinate r from the

origin centered in the charge distribution, the dipole field will be smaller on average than the monopole field by a factor of order $|\vec{p}/q|/r$ and this factor is much less than 1 since we are in the far-field limit. We can see this from equation (12) for a dipole electric field in the far-field limit:

$$\vec{E}(\vec{r}) = k \left[\frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} \right] \cdot \sim \frac{kq}{r^2} \frac{|\vec{p}/q|}{r} . \quad (13)$$

We can now look at some special cases of the far-field electric fields of charge distributions.

3.1. Two Equal Point Charges

Say we have had two point charges of charge q separated by a distance $2a$. The midpoint between the charges is the origin. The displacement vectors of the charges are $-\vec{a}$ and \vec{a} .

The general electric field is

$$\vec{E}(\vec{r}) = \frac{kq}{|\vec{r} + \vec{a}|^3}(\vec{r} + \vec{a}) + \frac{kq}{|\vec{r} - \vec{a}|^3}(\vec{r} - \vec{a}) . \quad (14)$$

The monopole moment is, of course, just $2q$.

The dipole moment is

$$\vec{p} = \sum_i \vec{r}_i q_i = -\vec{a}q + \vec{a}q = 0 . \quad (15)$$

This dipole moment is origin-dependent, of course. But putting the origin at the midpoint makes the dipole moment zero which is actually simplifying. It causes us to have no dipole term in the Taylor's series expansion.

Thus to 1st order the far-field electric field is just the monopole field

$$\vec{E}(\vec{r}) = \frac{k(2q)}{r^2} \hat{r} . \quad (16)$$

We have already consider the electric field structure of this charge distribution in § 2.1.

3.2. The Simple Dipole

The simple dipole consists of two point charges of charge $-q$ and q . Let's say that they are separated by a distance $2a$ and put origin at the midpoint. The displacement vectors of the charges are $-\vec{a}$ and \vec{a} .

The general electric field is

$$\vec{E}(\vec{r}) = \frac{k(-q)}{|\vec{r} + \vec{a}|^3}(\vec{r} + \vec{a}) + \frac{kq}{|\vec{r} - \vec{a}|^3}(\vec{r} - \vec{a}) . \quad (17)$$

The monopole moment is, of course, zero.

The dipole moment is

$$\vec{p} = \sum_i \vec{r}_i q_i = -\vec{a}(-q) + \vec{a}q = 2q\vec{a} . \quad (18)$$

The dipole moment is origin independent since the monopole moment is zero.

The dipole electric field (i.e, the 1st order far-field dipole electric field) is

$$\vec{E}(\vec{r}) = k \left[\frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} \right] . \quad (19)$$

which, of course, looks just like the general formula the dipole electric field.

What is the structure of the simple dipole electric field?

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3.3. An Alternate Formula for the Dipole Electric Field: Optional

Just for completeness, there is another way to write the formula for the dipole electric field (i.e, the 1st order far-field dipole electric field). First note that

$$\vec{E}(\vec{r}) = k \left[\frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} \right] = k \left[\frac{3p \cos \theta \hat{r} - \vec{p}}{r^3} \right] , \quad (20)$$

where θ is the angle between the dipole moment and \hat{r} (the unit vector pointing to the location where the field is being evaluated).

If one defines a set spherical polar coordinates with the dipole moment defining the positive z direction, then

$$\vec{p} = p \cos \theta \hat{r} + p \cos \left(\theta + \frac{\pi}{2} \right) \hat{\theta} = p \cos \theta \hat{r} - p \sin \theta \hat{\theta} , \quad (21)$$

where \hat{r} and $\hat{\theta}$ are the unit vectors of the the radial and polar angle components of the spherical polar coordinates.

Now First note that

$$\begin{aligned} \vec{E}(\vec{r}) &= k \left[\frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{r^3} \right] = k \left[\frac{3p \cos \theta \hat{r} - p \cos \theta \hat{r} + p \sin \theta \hat{\theta}}{r^3} \right] \\ &= k \left[\frac{2p \cos \theta \hat{r} + p \sin \theta \hat{\theta}}{r^3} \right] , \end{aligned} \quad (22)$$

where the last expression is the alternate formula we sought.

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Fig. 1.— Schematic structure of a simple dipole electric field shown in cross section.

REFERENCES

- Arfken, G. 1970, *Mathematical Methods for Physicists* (New York: Academic Press)
- Barger, V. D., & Olson, M. G. 1987, *Classical Electricity and Magnetism* (Boston: Allyn and Bacon, Inc.)
- Enge, H. A. 1966, *Introduction to Nuclear Physics* (Reading, Massachusetts: Addison-Wesley Publishing Company)
- French, A. P. 1971, *Newtonian Mechanics: The M.I.T. Introductory Physics Series* (New York: W.W. Norton & Company)
- Goldstein, H., Poole, C., & Safko, J. 2002, *Classical Mechanics*, 3rd Edition (San Francisco: Addison-Wesley)
- Griffiths, D. J. 1999, *Introduction to Electrodynamics* (Upper Saddle River, New Jersey: Prentice Hall)
- Jackson, J. D. 1975 *Classical Electrodynamics* (New York: John Wiley & Sons)
- Neugebauer, O. 1969, *The Exact Sciences in Antiquity* (New York: Dover) (Ne)
- Ohanian, H. C. 1988, *Classical Electrodynamics* (Boston: Allyn and Bacon, Inc.)
- Serway, R. A. & Jewett, J. W., Jr. 2008, *Physics for Scientists and Engineers*, 7th Edition (Belmont, California: Thomson)
- Tipler, P. A., & Mosca, G. 2008, *Physics for Scientists and Engineers*, 6th Edition (New York: W.H. Freeman and Company)
- Weber, H. J., & Arfken, G. B. 2004, *Essential Mathematical Methods for Physicists* (Amsterdam: Elsevier Academic Press)

Wolfson, R. & Pasachoff, J. M. 1990, *Physics: Extended with Modern Physics* (London: Scott, Foresman/Little, Brown Higher Education)