Ch 32 Inductance

Well we've actually already in Ch. 31 been dealing with inductance.

Faraday's law (and the universal flux rule) tell us changing magnetic fields create induced emfs, which if ~~there~~ is a conductor around create induced currents and induced B-fields.

In this chapter we focus on changing currents causing changing B-fields causing induction.

§ 32.1 Self Inductance & Inductance

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

$$\therefore \mathcal{E} \propto \frac{dB}{dt} \propto \frac{dI}{dt}$$

So area or angle of the flux surface isn't fixed

for certain setup where the path of the emf is fixed and a current gives rises to the B

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which is pretty much always true in circuit setup

(not in electric motors or generators or free propagating electromagnetic waves)

So in circuit setups

$$\mathcal{E} \propto \frac{dI}{dt}$$

and we can write

$$\mathcal{E} = -L\frac{dI}{dt}$$

For cases where the induced emf is in the ~~system that~~ circuit that gave rise to it.

where L is called ~~the~~ an <u>inductance</u> or <u>self inductance</u>

L is determined by the geometry of the system or device

A device designed to have an inductance is an *inductor* (and any magnetic materials around, but we won't go into that much)

L is independent of I and $\mathcal{E}$ ideally.

— but circuits will have ineluctable ~~accidental~~ self-inductances like it or not — usually those are pretty small.

The minus sign in

$$\mathcal{E} = -L\frac{dI}{dt}$$

is conventional and tells us the induced emf opposes ~~the~~ changes in current. — One can use the conventions to find the ~~sense~~ sense of the emf, but Lenz's law is easier

~~We did an example of an~~

opposes changes in current. Maybe not for circuit cases

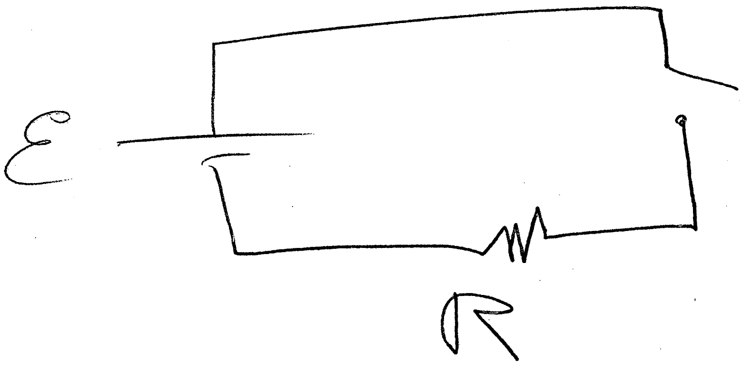
32-6

Units of L

$$[L] = \left[\frac{\mathcal{E}}{\frac{dI}{dt}}\right] = \frac{V}{A/s} = \frac{J\,s^2}{C^2} = \frac{J}{A^2} \equiv \underbrace{1\,H}$$

a henry for Joseph Henry who independently discovered Faraday's law about the same time as Faraday.

Ex: Ineluctable Inductance



~~We've~~ When the circuit is closed we've ignored the ~~uiie~~ phase of current from 0 to steady state

$$I = \mathcal{E}/R$$

— for circuits without inductors, that phase is pretty ~~fast~~ short, but it isn't zero time:

a) because it does ~~take~~ a finite time for the static guide charge & electric fields to establish a steady state.

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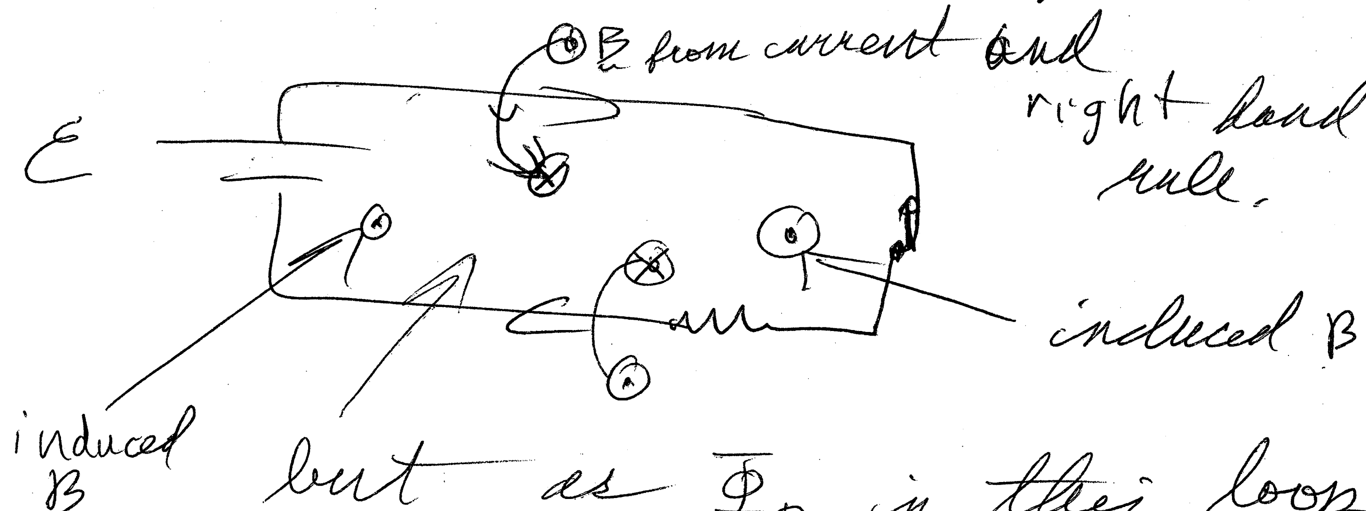
b) Which is actually a subset of (a), self-inductance.

— as current flow



but as Φ_B in this loop grows

an induced $\mathcal{E}$ induced is created by Faraday's law around the loop with the sense of

~~b~~ opposing $\underline{B}$'s grows by created an induced $\underline{B}$

32-8

$\mathcal{E}_{induced}$
~~opposed~~ $\mathcal{E}$.

~~So really the or~~

Say $L_{circuit}$ is the ineluctable inductance, then ignoring other delay effects (which may or may not be valid since L is likely pretty tiny in this case) Kirchhoff's voltage law gives

$$\mathcal{E} - \underbrace{L_{cir}\frac{dI}{dt}}_{\mathcal{E}_{induced}} - IR = 0$$

going around the loop.

For such circuits L is

32-10) usually negligibly small depending on how accurate you are being and we usually assume $L_{cir} = 0$

But an actual inductor device with a finite L will have an effect as we'll see.

Ex Calculation of L for an ideal Solenoid

— we did this example before and so this is a repetition

an ideal solenoid — one that can ~~treated~~ is infinite in length even if it isn't.

32-12

$\therefore\ \mathcal{E} = -\frac{d\Phi_B}{dt}$

net emf

$= N\mathcal{E}_{turn}$

$= -\cancel{\frac{\mu_0 \pi r^2 N^2}{l}} \frac{dI}{dt}$

$\mu_0 n^2 V$

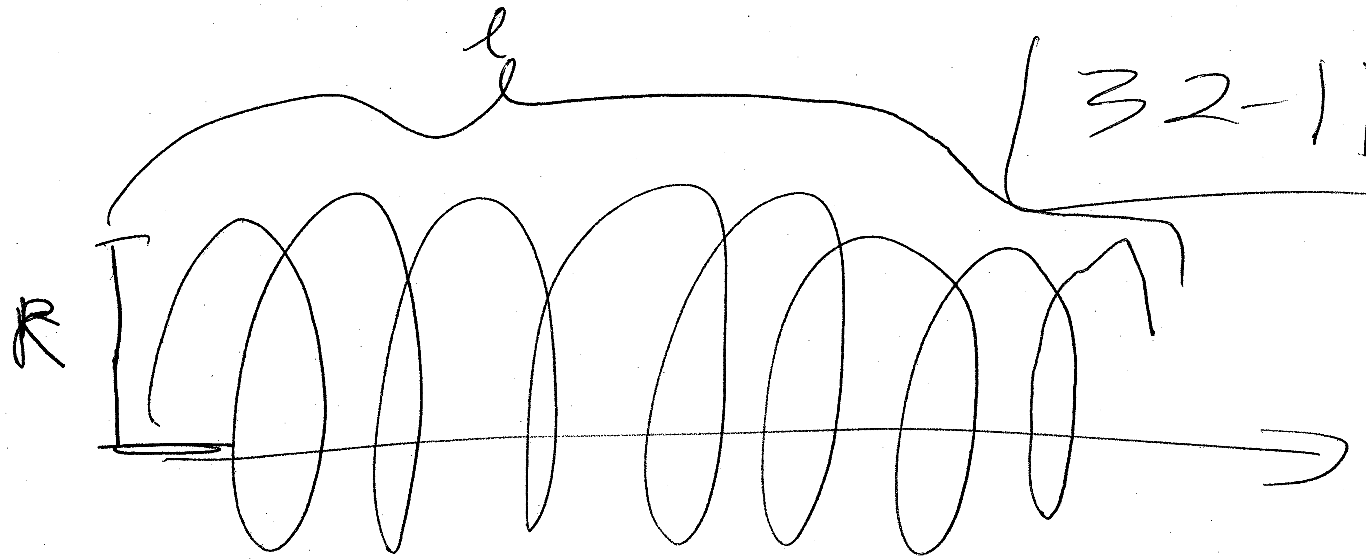
$L = \mu_0 n^2 V_{volume}$

$\therefore\ \cancel{L = \frac{\mu_0 \pi R^2 N^2}{l}}$ (SJ-899 agree)

So the inductance depends on the geometry.

— filling the solenoid with a magnetic material would enhance L by a factor of μ_r (the relative permeability (Wiki: inductance)) (e.g., soft iron)

actually this depends on applied magnetic flux usually and simplicity is lost,



$$B = \begin{cases} \mu_0 n I & \text{inside} \\ 0 & \text{outside} \end{cases}$$ from Ampere's law.

So inside uniform in direction and magnitude

n is number of turns per unit length $= N/l$ where N is the number of turns

$$\Phi_{B\,\text{one turn}} = \pi R^2 B = \pi R^2 \mu_0 n I$$

$$\cancel{= \mu_0 \pi R^2 \frac{N}{l} I}$$

$$\Phi_B = N \Phi_{B\,\text{one turn}} \cancel{= \mu_0 \pi R^2 \frac{N^2}{l} I}$$

total flux linked by solenoid

$$= n l \pi R^2 \mu_0 n I$$

$$= \mu_0 n^2 I V_{\text{volume}}$$

(Volume $= l \pi R^2$)

but we won't worry about such complications.

Let's evaluate a fiducial value for L

$$\mu_0 = 4\pi \times 10^{-7} \text{ recall}$$

$$R = 1\,\text{cm} = .01\,\text{m}$$

$$\ell = 1\,\text{m}$$

$$N = 1000, \quad n = 10^3$$

$$V = \pi R^2 \ell \cong 3 \cdot 10^{-4} \cdot 1 = 3 \cdot 10^{-4}$$

$$L \cong \overbrace{4\pi \times 10^{-7} \cdot 3 \cdot 10^{-4} \cdot 10^{6}}^{= \mu_0 V_{solen} n^2}$$

$$= 40 \cdot 10^{-5}$$

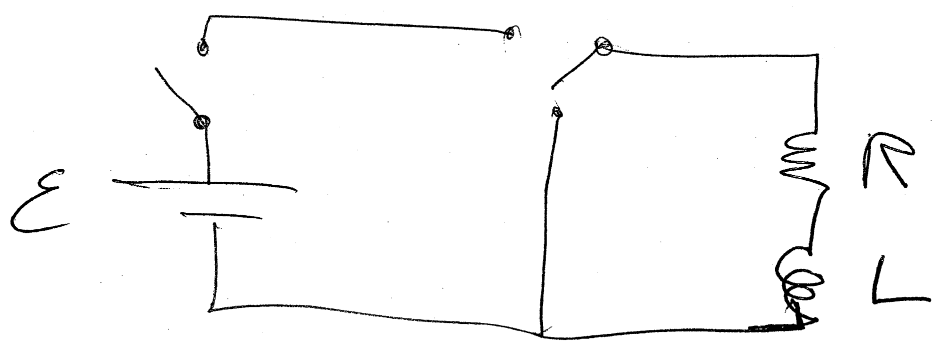
$$= 4 \times 10^{-4}\,\text{H}$$

$$= .4\,\text{mH}$$

Say $\frac{dI}{dt} = 1\,\text{A/s}$, $\mathcal{E}_{ind} = .4 \times 10^{-3}\,\text{V}$
$= .4\,\text{mV}$

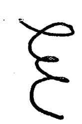
§ 32.2 RL circuits

RL means they have a resistor and an inductor.

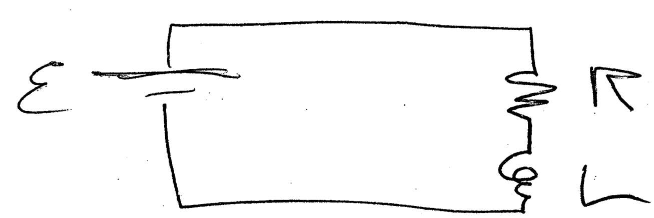


— we assume the wires are ideal — no resistance, no inductance.

inductor symbol



If we close to get



, then

Kirchhoff's voltage law gives

$$\mathcal{E} - \underbrace{L\frac{dI}{dt}} - IR = 0$$

Count as a drop in direction of current flow (source into a rise if $\frac{dI}{dt} < 0$)

induced emf opposes ~~the applied~~ current the applied emf tries to establish.

This is a differential equation (DE) not an algebraic one.

— Actually a linear, 1st order, homogeneous DE.

It can be solved — but the techniques are beyond our scope.

32-16

So we'll just write ~~doon~~ down the solution and confirm it ~~[illegible]~~ below.

$$I = I_\infty \left(1 - e^{-t/\tau}\right)$$

where $\tau = \frac{L}{R}$

is the e-folding time or time constant

When a steady state is reached $\frac{dI}{dt} = 0$

and ~~so~~ $\mathcal{E} - I_\infty R = 0$

then and $I_\infty = \frac{\mathcal{E}}{R}$

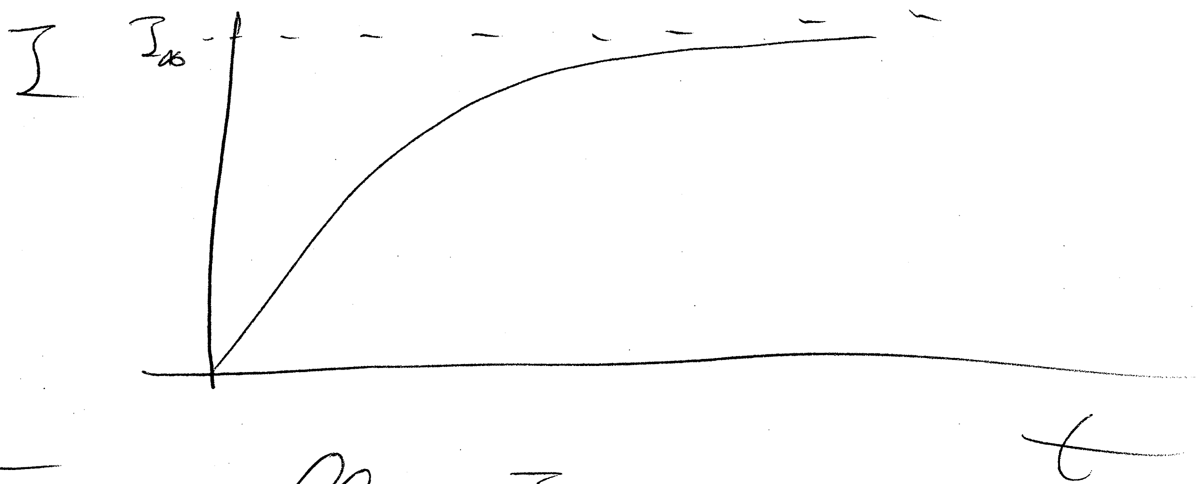
$$\frac{dI}{dt} = I_\infty \left(\frac{e^{-t/\tau}}{\tau}\right)$$

$$\mathcal{E} - L \frac{\mathcal{E}}{R} \frac{e^{-t/\tau}}{\tau} - R \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right) = 0$$

$$\therefore \quad 1 - e^{-t/\tau} - \left(1 - e^{-t/\tau}\right) = 0$$

and so we've confirmed the solution.

~~Formally~~



Formally I never reaches I_∞.

But in any real circuit there is uncertainty in current measurements ΔI_{err} and current fluctuations ΔI_{fl} (due all kinds of low level things — thermal fluctuations, stray E-fields (although they may not penetrate the conductors much.)

32-18)

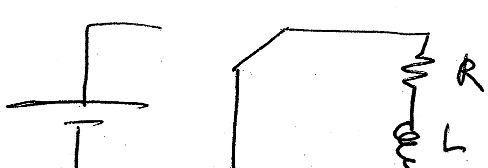
Once $|I - I_\infty| < \Delta I_{err}$ or ΔI_{fl}

then practically I has reached I_∞.

This may happen within a few e-folding times depending on how precise one is being.

The inductor slows the rise of the current to I_∞.

The other case of the circuit is when on suddenly



Now $\underbrace{I_0}_{\text{initial current of the new phase.}} = I_\infty$

In this case Kirchoff's voltage law gives

$$\underbrace{-IR}_{\text{a potential drop}} - \underbrace{L\frac{dI}{dt}}_{\frac{dI}{dt}<0} = 0$$

a ~~voltage~~ potential drop

Now a potential rise that tries to maintain the current. — the induced emf opposes changes.

We know the final state $I_\infty = 0$

when steady state is reached I_∞ is constant, but then

$I_\infty R = 0$ for the Voltage law

$\therefore I_\infty = 0.$

32-20

We can guess

$$I = I_0 e^{-t/\tau}$$

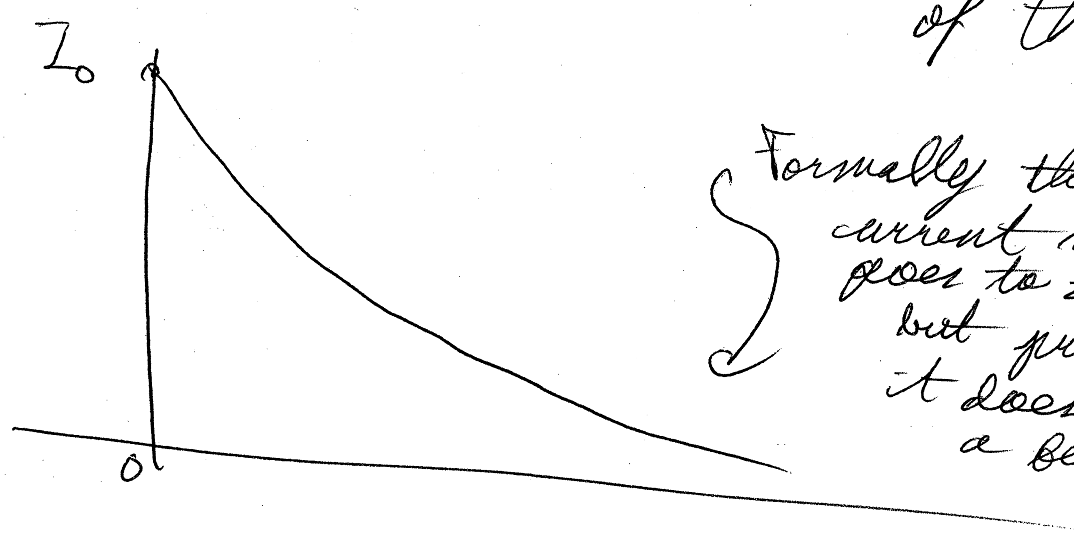
$$\therefore \; = I_0 e^{-t/\tau} R - L\left(-\frac{1}{\tau}\right) I_0 e^{-t/\tau} = 0$$

$$= R + R = 0$$

and so

$$I = I_0 e^{-t/\tau} \text{ is}$$

the solution of the DE



Formally the current never goes to zero, but practically it does after a few e-folding times depending on what one's error & fluctuations are

§ 32.3 Energy in a Magnetic Field

The power transferred in an inductor is

$$P = V_{\text{drop across}}\, I$$

at any instant.

Now $V_{\text{drop across}} = \mathcal{E}_{\text{induced}}$

from The electrostatic force pushing the current along

From the induced electric field preventing acceleration of current.

32-22

$$\mathcal{E} = L\frac{dI}{dt}$$

$$P = L\frac{dI}{dt}I$$ Assuming $I > 0$.

if I is increasing the power is going out of the circuit somewhere

if I is decreasing power is coming from somewhere into the circuit

Energy released $= \int L\frac{dI}{dt} I\, dt$

$= L\int_0^I I'\,dI' = \frac{1}{2}LI^2$

$$U = E_{stored} = \frac{1}{2} L I^2$$

if you create a current starting from $I = 0$ and raising it to I.

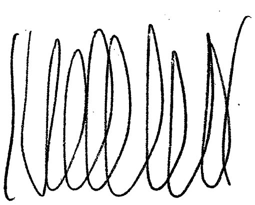
We don't call this stored energy potential energy — the term doesn't seem to be a good fit. Isn't a fit.

But after our experience with electric fields, it should come as NO surprise that the energy is stored in the magnetic field created by the current.

32-24

Just as with E-field case, we can't prove this — beyond our scope, but we can elucidate it.

Let's approximate an inductor as an ideal solenoid.



$$B = \begin{cases} \mu_0 n I & \text{inside} \\ 0 & \text{outside} \end{cases}$$

recall ~~inside~~

n is the number of turns per unit length.

Now recall $L_{\text{ideal solenoid}} = \mu_0 \frac{N^2 A}{\ell}$ (see ST-899)

$L = \mu_0 n^2 V_{sol}$

$\therefore$ $U = \frac{1}{2} L I^2$

$$= \frac{1}{2} \mu_0 \cancel{N^2 A}\, n^2 V_{vol} \frac{B^2}{\mu_0^2 n^2}$$

$$= \cancel{\frac{N^2 A}{\ell^2}} \cancel{\ell} \frac{B^2}{2\mu_0}$$

$$= \cancel{\ell A}\, V_{vol} \frac{B^2}{2\mu_0}$$

~~volume~~ of solenoid.

$\therefore$ one might guess

$$u = \frac{U}{V} = \frac{B^2}{2\mu_0}$$

(And we are right. This is the general result — but we can't prove it.)

is the energy density of magnetic field. (for vacuum) which air nearly is magnetically speaking,

Recall $u = \frac{1}{2} \varepsilon_0 E^2$

is the energy density of the electric field.

32-26)

As we'll see in Ch 34

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

~~and~~

so $$\frac{u_B}{u_E} = \frac{\frac{B^2}{2\mu_0}}{\frac{1}{2}\varepsilon_0 E^2}$$

$$= \frac{1}{(\mu_0 \varepsilon_0)} \frac{B^2}{E^2}$$

$$= c^2 \frac{B^2}{E^2}$$

which ~~may mea~~ probably means something important, but it eludes me at the moment.

Ex for Fiducial case

$u_E = \frac{1}{2}\varepsilon_0 E^2$ $\qquad$ $u_B = \frac{B^2}{2\mu_0}$

Say $E = 1\ V/m$ $\qquad$ $B = 1\ T$

$\varepsilon_0 = 8.854 \times 10^{-12}$ $\qquad$ $\mu_0 = 4\pi \times 10^{-7}$

$u_E \simeq 4.4 \times 10^{-12}\ J/m^3$ $\qquad$ $u_B = \frac{1}{8\pi} \times 10^{7}$

$\simeq 4 \times 10^{5}\ J/m^3$

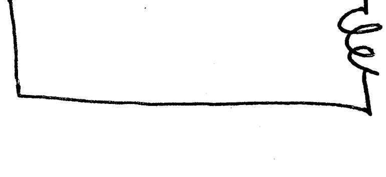
which numbers are
actually consistent with
what we'd ordinarily think

1 V/m is not a very large
E-field

but 1 T is a pretty big
B-field.

Ex 32.3

What happens to the
energy of an inductor
in the fiducial decaying
current case



at $t = 0$
$I = I_0$

We know that
$I = I_0 e^{-t/\tau}$

32-28

where $\tau = \frac{L}{R}$

$$U = \frac{1}{2} L I^2$$

$$= \frac{1}{2} L I_0^2 e^{-2t/\tau}$$

~~is the~~ show declining energy. $P_{out} = \frac{dU}{dt} = -R I_0^2 e^{-2t/\tau}$

~~lost~~ less than zero and so a loss

d the resistor ~~the~~ power is

$$P = VI = \cancel{\frac{V^2}{R}} = I^2 R$$

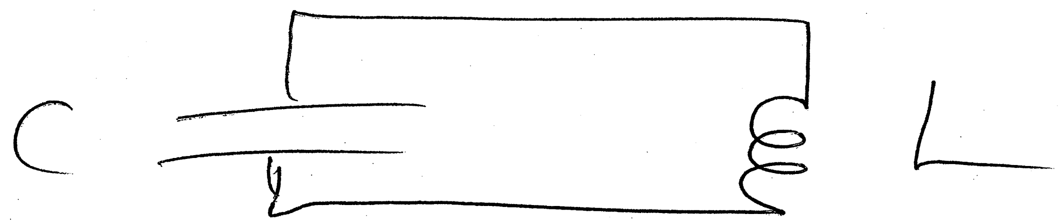
which is going into waste heat (which is something we ~~thermodynamics~~).

$$\therefore P_{out} = R I_0 e^{-2t/\tau}$$

output of resistor to waste heat.

So the ~~power~~ energy ~~flow~~ stored in the magnetic field is converted to waste heat in the resistor.

§32.5 Oscillation in an LC circuit.



a capacitor inductor circuit

— So LC circuit.

— As we see it here, it's a bit isolated and abstract

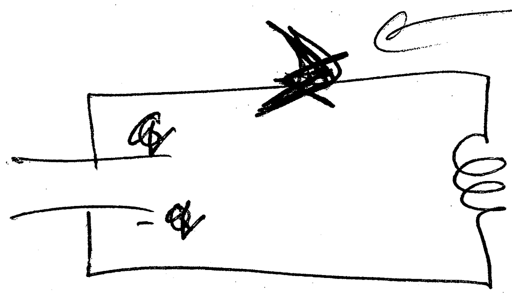
— but LC circuits are included in ~~O~~ electronic oscillators to create sinusoidal electrical signals

and for filters
and tuners,
but it goes beyond
our scope to go into
the practicalities (right now. We do a bit in Ch. 33)
— we just look at
the essence.

Apply Kirchoff voltage law
around the loop.

initial guess at a current direction

$V = \frac{q}{C}$



$$Cq - \underbrace{L\frac{dI}{dt}}_{} = 0$$

rise as we drawn it

a drop for our chosen direction assuming $\frac{dI}{dt} > 0$

This assumption does work out.

Now $I = -\frac{dq}{dt}$

— the current we count as positive comes at the expense of the charge on the capacitor $\therefore$

$$\therefore \quad \frac{q}{C} + L\frac{d^2q}{dt^2} = 0$$

$$\therefore \quad \frac{d^2q}{dt^2} = -\frac{1}{LC}\,q$$

$$[L] = \frac{V}{A/s}$$

$$[C] = \frac{C}{V}$$

$$\left[\frac{1}{\sqrt{LC}}\right] = \sqrt{\frac{(A/s)\,V}{V\,C}}$$

$$= \frac{1}{\sqrt{s^2}}$$

$$= \frac{1}{s}$$

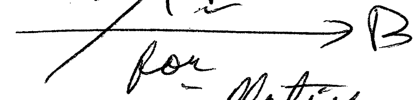
define $\omega = \sqrt{\frac{1}{LC}}$

$$\therefore \quad \frac{d^2q}{dt^2} = -\omega^2 q$$

which is the simple harmonic oscillator differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

also $\frac{d^2\theta}{dt^2} = -\omega_0^2\,\theta$



for oscillating ~~mag~~ electric dipole

32-32

~~The systems~~ systems are physically different, but they have the same mathematical description

— decades of experience tells me the general ~~solution~~ solution is

$$q = A\cos\omega t + B\sin\omega t$$

$$\frac{d^2q}{dt^2} = -A\omega^2 \cancel{\cos}\,\omega t - B\omega^2\sin\omega t$$

$$= -\omega^2 q$$

and ~~that~~ confirm ~~the~~ solution.

~~Say~~ A and B are constants set by <u>initial conditions</u>

Nothing in the physics of the

32-34

So increasing either L or C increases the period and decreases the oscillation frequency.

Example

Say at time zero
$$q = q_0$$
and we've just ~~hooked~~ connected the capacitor to a zero current inductor.

$$q = q_0 \cos(\omega t) + B \sin \omega t$$

The time zero condition gives $\cancel{A} = q$

$$I = -\frac{dq}{dt} = + q_0 \omega \sin \omega t - B\omega \cos \omega t$$

system tells us what they are.

— This is often the case in physical system — well always except maybe the universe as a whole — physical tells us something, but to know everything we need the initial conditions or the boundary conditions.

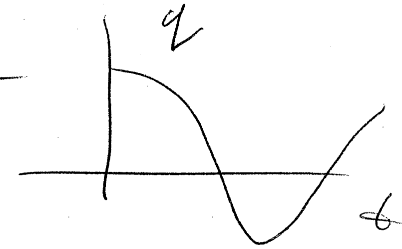
~~Example~~ ~~Cat~~ The sinusoidal function are periodic. Every time t increases by Δt such that $\omega \Delta t = 2\pi$, the q value repeats.

$$\therefore \quad P = \frac{2\pi}{\omega} = 2\pi\sqrt{LC} = 2\pi\sqrt{LC}$$ is the period.

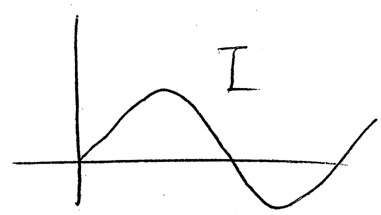
Since $I = 0$ at time zero

$$B = 0$$

$$\therefore \quad q = q_0 \cos \omega t$$



$$I = q_0 \omega \sin \omega t$$



$$\left.\begin{array}{l} U_{cap} = \dfrac{q^2}{2C} = \dfrac{q_0^2}{2C}\cos^2 \omega t \\ U_{ind} = \dfrac{1}{2} L I^2 \end{array}\right\} \text{ recall}$$

$$= \frac{1}{2} L q_0^2 \omega^2 \sin^2 \omega t = \frac{1}{2}\frac{1}{C} q_0^2 \sin^2 \omega t$$

$$U_{cap} + U_{ind} = \frac{1}{2}\frac{q_0^2}{C}\left(\cos^2 \omega t + \sin^2 \omega t\right)$$

$$= \frac{1}{2}\frac{q_0^2}{C}$$

— a constant.

— the energy oscillates back and forth between capacitor and inductor energy but is overall conserved.

There is no way in the system for energy to come from or go to.

— In LRC circuit, energy can be dissipated in the resistor and ~~the~~ initial energy decays.

Example periods

$$P = \begin{cases} 2\pi\sqrt{LC} \\ \underbrace{2\pi \text{ s}}_{6\text{s}} \text{ for } L = 1H,\ C = 1F \end{cases}$$

L32-7

$$2\pi\underbrace{\sqrt{10^{-3}\cdot 10^{-6}}}\quad \left\{\begin{array}{l} L = 1\,mH \\ F = 1\mu F \end{array}\right.$$

$$\cong 20 \times 10^{-5}$$

$$= 2 \times 10^{-4}\,s$$

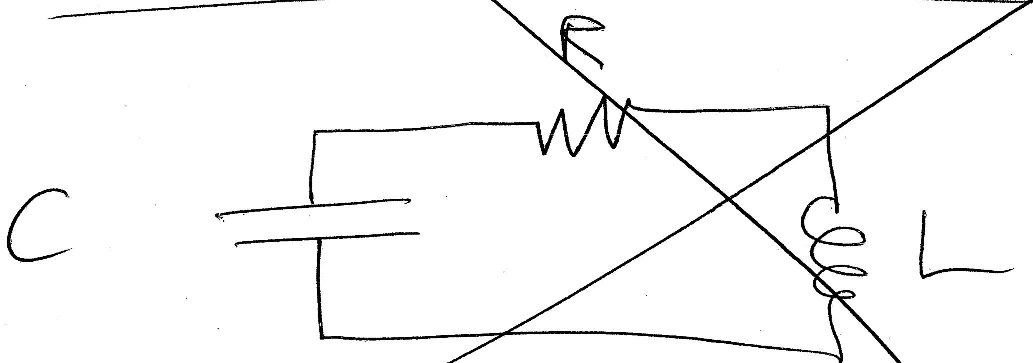
Now $f = \frac{1}{P} = \frac{1}{2\pi\sqrt{LC}}$

is frequency in Hertz.

$$\therefore\ f = \frac{1}{2\times 10^{-4}\,s} = .5 \times 10^{4}\,Hz$$

$$= 5000\,Hz$$

~~§ 32.6 RLC circuit (omit?)~~



32-38

Applying Kirchhoff's voltage law

$$\underbrace{\frac{q}{C}}_{\text{a rise}} - \underbrace{IR - L\frac{dI}{dt}}_{\text{two drops.}} = 0$$

or $L\frac{d^2q}{dt^2} + \frac{dq}{dt}R + \frac{q}{C} = 0$

or differentiating once and re-arranging

$I = -\frac{dq}{dt}$

$$L\frac{d^2I}{dt^2} + \frac{dI}{dt}R + \frac{I}{C} = 0$$

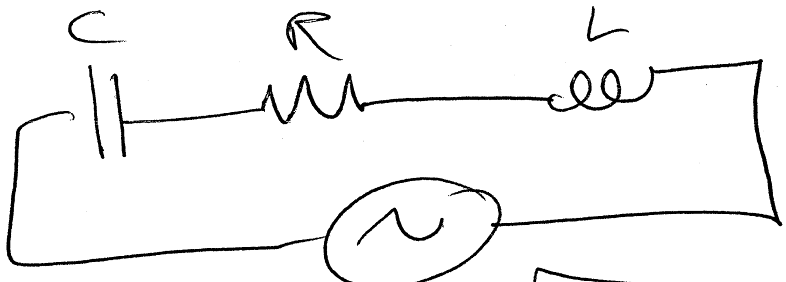
If $C \to \infty$, like removing ~~the~~ the capacitor and replacing with a connection and we have an RL circuit

§ 32.6 RLC circuit

RLC circuits contain resistor, inductor, capacitor.

The bare bones one ~~has~~ just those

One can add an AC emf



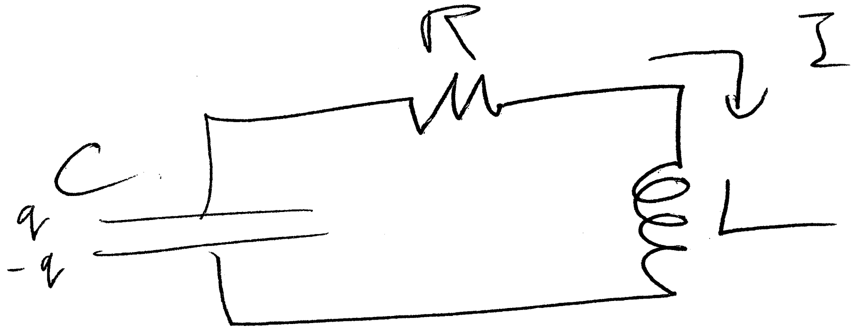
symbol

(SJ-932)

This is an interesting ~~case~~ because multiple AC frequencies can be input to RLC circuit at one time and RLC (usually the C) can be adjusted to respond strongly only to a narrow range of AC frequencies. This allows "tuning" as for a radio (SJ-939)

But here we'll only consider the barebones case.

To understand more complex cases (which I don't think we'll do in this course), one has to understand the barebones RLC circuit



Apply Kirchhoff voltage law

Going around the loop counterclockwise

$$\underbrace{\frac{q}{C}} - \underbrace{IR} - \underbrace{L\frac{dI}{dt}} = 0$$

$\frac{q}{C}$: a rise for $q > 0$ — drop for $q < 0$

IR: a drop for $I > 0$; rise for $I < 0$

$L\frac{dI}{dt}$: a drop for $\frac{dI}{dt} > 0$; rise for $\frac{dI}{dt} < 0$

32-42

Now $-\frac{dq}{dt} = I$

If q is decreasing of the top plate, ~~there~~ is a ~~positive~~ current

and if q is increasing on the top plate, ~~there~~ is a negative current.

So all our conventions ~~seem~~ are consistent.

$$-\frac{d^2q}{dt^2} = \frac{dI}{dt}$$

$$\therefore \frac{q}{C} + q'R + Lq'' = 0$$

or in conventional form

32-44

which is "always" true for the context ~~or~~ ideally

by ~~from~~ physical law
and partially by initial
or boundary conditions.

How do we get the solutions?
We guess a trial solution
based on experience
and find out what
constraints the DE puts
on the guessed constants.

But before we do that
one can say the above
DE is exactly analogous

$$Lq'' + q'R + \frac{q}{C} = 0$$ 32-43

This is a linear 2^{nd} order differential equation (DE) for q.

Math guarantees two independent solutions
— i.e., one solution can't be made out of the other.

— One also gets two constants of integration set by initial conditions

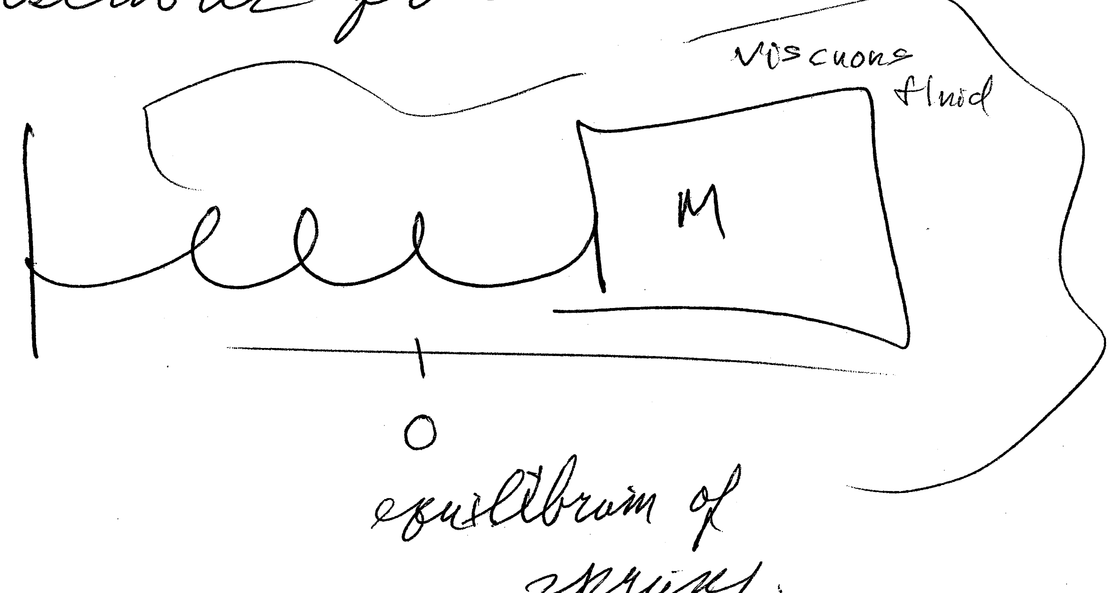
To be definite
$q = q_0$ at $t = 0$
$I_0 = -\frac{dq}{dt} = 0$ at $t = 0$

This is frequently how things go in physics. A solution is partially determined

~~Part of a solution comes~~ is deter

~~to~~ a problem in ~~QM~~ mechanics.

~~the~~ damped harmonic oscillator problem where the damping force is a linear viscous force.



$$\underset{\sim}{F}_{net} = ma$$

$$\therefore \quad m\frac{d^2x}{dt^2} = \underbrace{-kx}_{\substack{\text{linear}\\ \text{restoring force}\\ \text{or Hooke's}\\ \text{law force}}} \underbrace{-\,b\frac{dx}{dt}}_{\substack{\text{a linear}\\ \text{viscous force}\\ \text{that always}\\ \text{opposes}\\ \text{the direction}\\ \text{of motion.}}}$$

32-46

Comparing the two DE's

$L \longrightarrow m$

acts like
~~the~~ inertia
and tries to
prevent "acceleration"

$\frac{1}{C} \longrightarrow k$

acts like
a force constant

$R \longrightarrow b$

acts like
a coefficient
of "viscous force.
~~drag~~.

— the mechanical motion could
be an oscillation from

a displacement
but $b \neq 0$ would
cause it to be damped
out.
Similarly for our DE
q will turn out to
oscillate for certain
conditions, but for $R \neq 0$
the oscillation must come
to "rest" with $q=0, I=0$
— It must damp out.

Solution (we go a bit beyond text)

Consider the trial solution
$$q = q_0 e^{\alpha t}$$

32-48a

where q_0 and α are undetermined constants.

Substitute into our DE and we get

$$L\alpha^2 + \alpha R + \frac{1}{C} = 0$$

(cancelling out $q_0 e^{\alpha t}$)

Just an algebraic quadratic equation with solution

$$\alpha_\pm = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

from

$$\frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

The general solution with two undetermined constants is

$$q = Ae^{\alpha_+ t} + Be^{\alpha_- t}$$

Actually it's helpful to look at the exponential constants $\alpha_\pm$ in terms of a time scale for decay.

$$\tau = -\frac{1}{\alpha_\pm} = \frac{1}{\frac{R}{2L} \mp \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}}$$

$$= \begin{cases} \dfrac{2L/R}{1 \mp \sqrt{1 - \left(\frac{2\sqrt{\frac{L}{C}}}{R}\right)^2}} \\ \dfrac{2L}{R} \end{cases}$$

for $R < R_{crit} = 2\sqrt{\frac{L}{C}}$

where as we'll show there is an oscillation and the complex τ can be interpreted as giving a complex oscillation

32-~~4~~8c

$\frac{2L}{R}$ for $R = R_{crit} = 2\sqrt{\frac{L}{C}}$

or $\tau = \frac{2L}{R} = \sqrt{LC}$

L/R for $R >> R_{crit}$

lower case solution

$$\frac{2L/R}{1 - \left(1 - \frac{1}{2}\frac{4L/C}{R^2}\right)}$$

for $R >> R_{crit}$ upper case solution

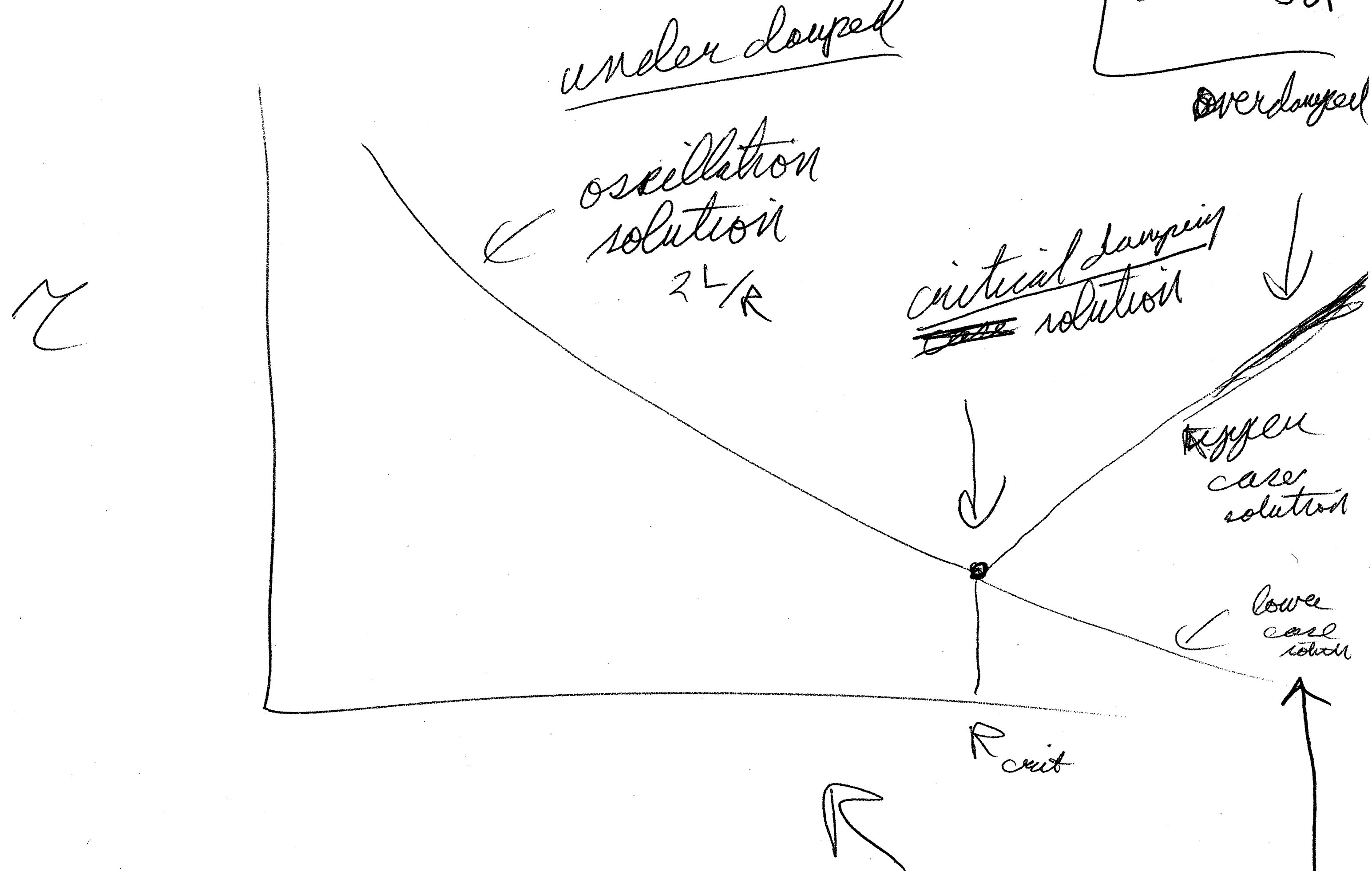
$= RC >> L/R$

for $R \gg R_{crit}$

and ~~is the~~ dominant long time time scale for decay.

Unless ~~the coefficient~~ of this solution is exactly zero.

32-48d



The critical damping case gives τ minimum (not counting

— this can be proven mathematically but I leave that to optional notes.
(that the R_{crit} gives τ_{min})

τ_{min} gives the fastest decay as long as the upper case overdamped solution is not set to zero.

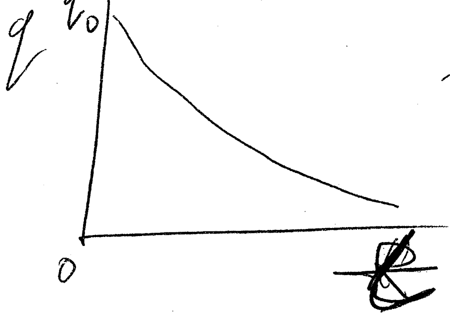
32-48e

Overdamping is ~~the~~ $R > R_{crit}$

~~molasses~~

"molasses" solution

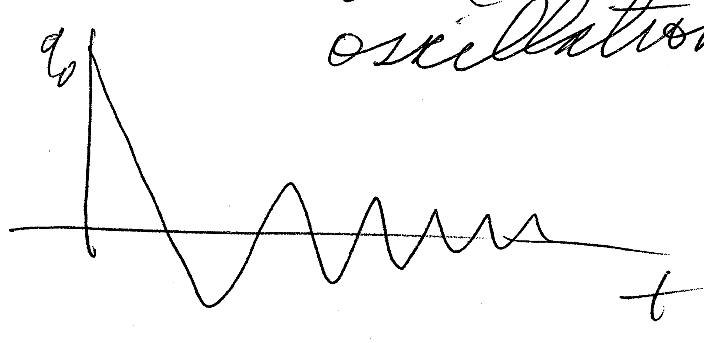
— the system can't come to equilibrium quickly because the resistance slows the everything down.



Underdamped $R < R_{crit}$

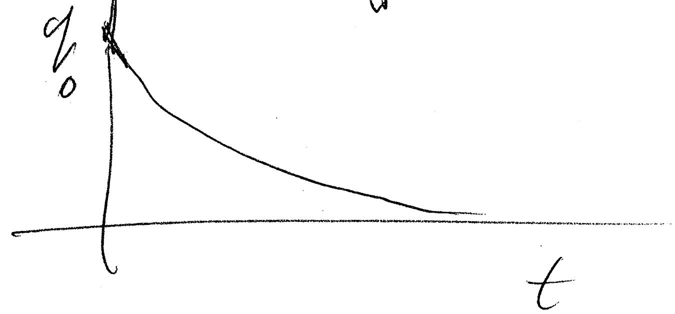
— here there is an overshoot because resistance is so small

and oscillation



<u>Critically damped</u> $R = R_{crit}$

is "just right" as Goldilocks would say for ~~faster~~ fastest evolution to $q = 0$.



— Of course formally $q = 0$ is only reached at $t = \infty$ in all cases, but critical damping gives the fastest approach and so the fastest in practical effect.

32-48g

The discriminant

is

$$d = \left(\frac{R}{2L}\right)^2 - \frac{1}{LC} \quad \text{in general}$$

> 0 then there is just a decaying solution. - overdamped case

$= 0$ critically damped case

(For fixed L & C about the fastest case for ending the motion.)

< 0 — complex solutions but the real parts are also solutions and these give oscillatory solutions - underdamped case

32-50

a) Overdamped $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} > 0$

or $R > R_{crit}$

$$q = q_+ e^{\alpha_+ t} + q_- e^{\alpha_- t}$$

$$\alpha_\pm = -\frac{R}{2L} \pm \underbrace{\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}}_{\text{always} < \left(\frac{R}{2L}\right)}$$

recall

and so $\alpha_\pm < 0$

and both solutions are decaying exponentials.

$$-\alpha_+ < -\alpha_-$$

$$-\frac{1}{\alpha_+} > -\frac{1}{\alpha_-}$$

and so the α_+ case is the more slowly declining

32-51

solution and
must dominate at
late times.

If we ~~take initial~~ conditions

$\cancel{q_0 = 0}$ $\quad q_0 = q_+ + q_-$

and $0 = I_0 = -\frac{dq}{dt}\Big|_0$

$= q_+ \alpha_+ + q_- \alpha_-$

then we can solve for q_+ and q_-

For the record

$q_- = q_0 - q_+$

$\therefore \quad 0 = q_+ \alpha_+ + (q_0 - q_+)\alpha_-$

$= q_+(\alpha_+ - \alpha_-) + q_0 \alpha_-$

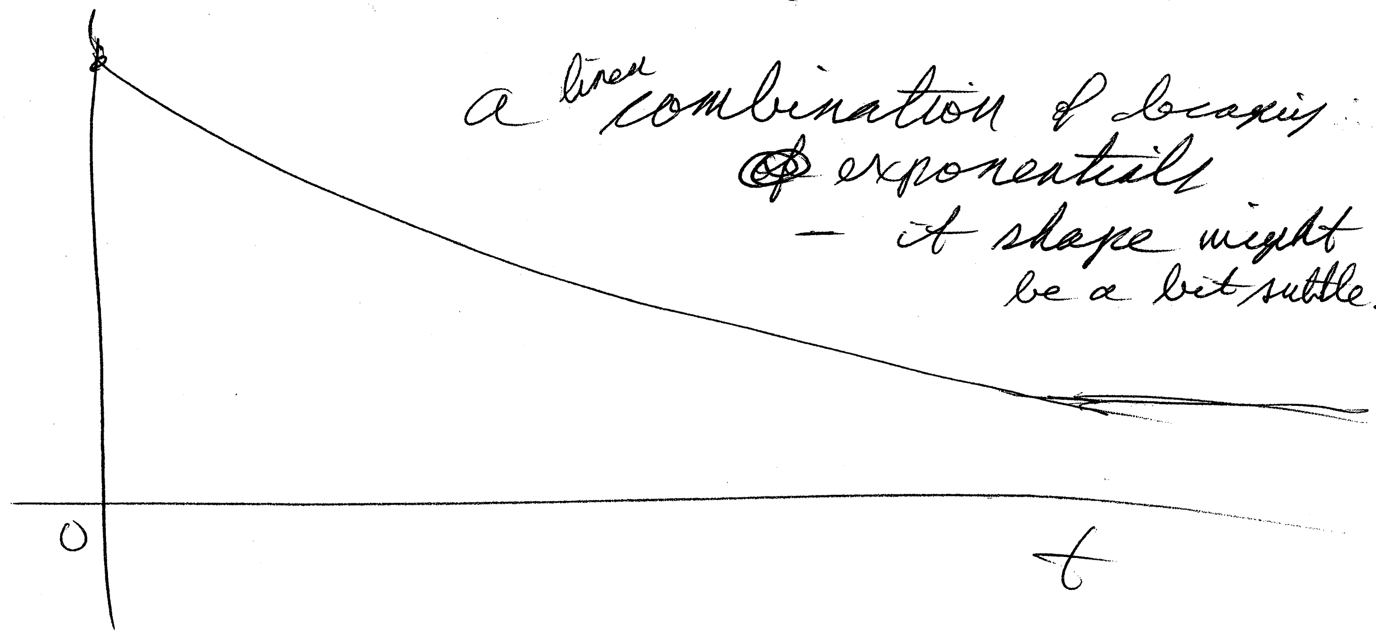
$q_+ = \frac{q_0 \alpha_-}{\alpha_- - \alpha_+} > 0$

$q_- = \frac{-q_0 \alpha_+}{\alpha_- - \alpha_+} \cdot$ [redacted] 0

since
$\alpha_- - \alpha_+ < 0$
and
α_- and
α_+ are
both negative

32-52)

When overdamped, it can take a long time to reach zero because of "Molasses effect" — slow decline in spring to equilibrium.



a linear combination of decaying exponentials — it shape might be a bit subtle.

b) Critically damped

$$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = \emptyset$$

or $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$ or $\frac{2L}{R} = \sqrt{LC}$

In this case our two solutions collapse to one

$R_{crit} = 2\sqrt{\frac{L}{C}}$

$$q = q_0 e^{-t/\tau}$$

~~$\tau = 2L/R$~~

$$\tau = 2L/R = \sqrt{LC}$$

Actually there must always be a 2nd solution — but our trial solution failed to find it in this special case.

(tr)ial (s)olution is a (t)rick. (T)o guarantee (i)t will (a)lways (w)ork.

The other solution is $te^{-t/\tau}$

~~[illegible]~~ Optional Reading

Math detail from Art 403-404

$$q_2 = q_1 \int^t \frac{e^{-\int^{t'} \frac{R}{L} dt'}}{q_1^2} dt$$

$$= e^{-t/\tau} \int^t \frac{e^{-\frac{R}{L}t'}}{e^{-2t'/\tau}} dt \qquad e^{-\frac{2t'}{\tau}}$$

$$= e^{-t/\tau} t$$

~~Thus the full general solution is~~

~~$q =$~~

32-54

We can confirm this is a solution

$$q' = e^{-t/\tau} + \left(-\frac{1}{\tau}\right)e^{-t/\tau}t$$

$$q'' = \left(-\frac{1}{\tau}\right)e^{-t/\tau} + \left(\frac{1}{\tau^2}\right)e^{-t/\tau}t$$

$$+ \left(-\frac{1}{\tau}\right)e^{-t/\tau}$$

$$= -\frac{2}{\tau}e^{-t/\tau} + \frac{e^{-t/\tau}t}{\tau^2}$$

$$Lq'' + q'R + \frac{q}{C} = 0$$

$$q'' + q'\frac{2}{\tau} + \frac{q}{LC} = 0$$

$$q'' + q'\frac{2}{\tau} + \frac{q}{\tau^2} = 0$$

$$\left(-\frac{2}{\tau}e^{-t/\tau} + \frac{e^{-t/\tau}t}{\tau^2}\right) + \left(\frac{2e^{-t/\tau}}{\tau} - \frac{2}{\tau^2}e^{-t/\tau}t\right) + \frac{e^{-t/\tau}}{\tau^2}t$$

$= 0$ confirmed

So the full solution is

$$q = q_0 e^{-t/\tau} + q_1 t e^{-t/\tau}$$

But we have to match the initial conditions.

$$q = q_0 \text{ at } t = 0$$

is already satisfied

$$I_0 = -\frac{dq}{dt}\Big|_0 = 0$$

$\frac{dq}{dt} = \left(-\frac{q_0}{\tau} + \frac{q_0}{\tau} - \frac{q_0 t}{\tau^2}\right)e^{-t/\tau} = q_0 e^{-t/\tau} \left(-\frac{t}{\tau}\right) < 0$ and declines monotonically.

$$\frac{dq}{dt} = -\frac{1}{\tau} q_0 e^{-t/\tau} + q_1 e^{-t/\tau} - \frac{1}{\tau} q_1 t e^{-t/\tau}$$

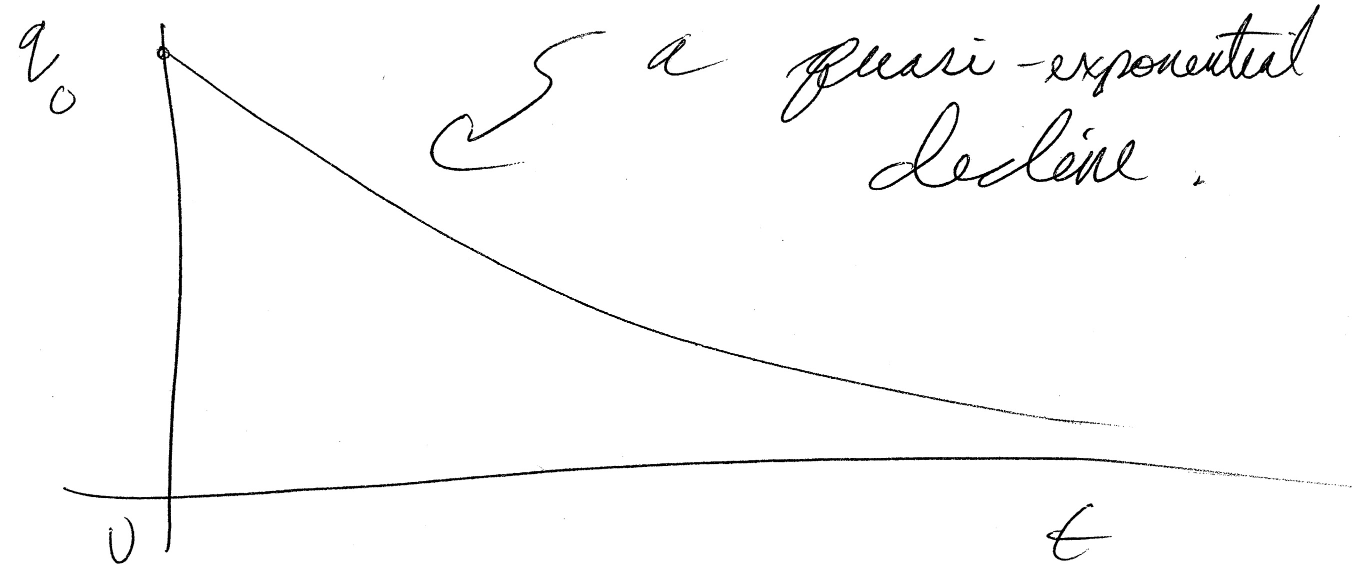
$$\frac{dq}{dt}\Big|_{t=0} = -\frac{1}{\tau} q_0 + q_1 = 0$$

$$q_1 = \frac{q_0}{\tau}$$

$$q = q_0 \left(1 + \frac{t}{\tau}\right) e^{-t/\tau}$$

32-56

– Overdamped is slower decline by molasses effect.
– underdamped ~~is~~ slower decline by oscillations



a quasi-exponential decline.

c) Underdamped case

$$\alpha_{\pm} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

but now the discriminant is negative and α is a complex number.

So $q = q_+ e^{\alpha_+ t} + q_- e^{\alpha_- t}$

is an ~~real~~ actual q solution to the DE, but not a "real" solution which means not physical,

Or in the perspective of p. 32-69, the real part of the complex solution

But appropriate linear combinations of the two complex solutions gives two real solutions (really real solutions).

We omit details (for the moment)

$$q = (A\cos\omega t + B\sin\omega t)e^{-\frac{t}{\tau}}$$

where $\tau = \dfrac{2L}{R}$

and $\omega = \sqrt{\dfrac{1}{LC} - \left(\dfrac{R}{2L}\right)^2} > 0$

32-~~5~~8

A and B are set by our Initial conditions.

$$A = q_0 \text{ clearly.}$$

and $$\frac{dq}{dt} = (q_0 \cos\omega t + B \sin\omega t)e^{-t/\tau}\left(-\frac{1}{\tau}\right)$$

$$+ (\cancel{-}q_0\omega \sin\omega t + B\omega \cos\omega t)e^{-t/\tau}$$

For $t = 0$ ~~= 0 for t~~

$$0 = q_0\left(-\frac{1}{\tau}\right) + B\omega$$

$$B = \frac{q_0}{\tau\omega} = \frac{q_0}{\sqrt{\frac{4L}{R^2C} - 1}}$$

$$\left[\frac{L}{R^2C}\right] = \frac{V/(A/s)}{(V/A)^2 \frac{C}{V}} = \frac{As}{C} = 1$$

unitless as it should be.

32-59

If $\frac{4L}{R^2C} >> 1$,

then $B << q_0$

and this seems to be the case the text assumes.

or another way to ~~to~~ look at it $\omega = 2\pi f$

$= 2\pi / T$ $\left\{ \text{period of oscillation} \right.$

So if $T << 2\pi\tau$ or ω big

~~then~~ $B << q_0$

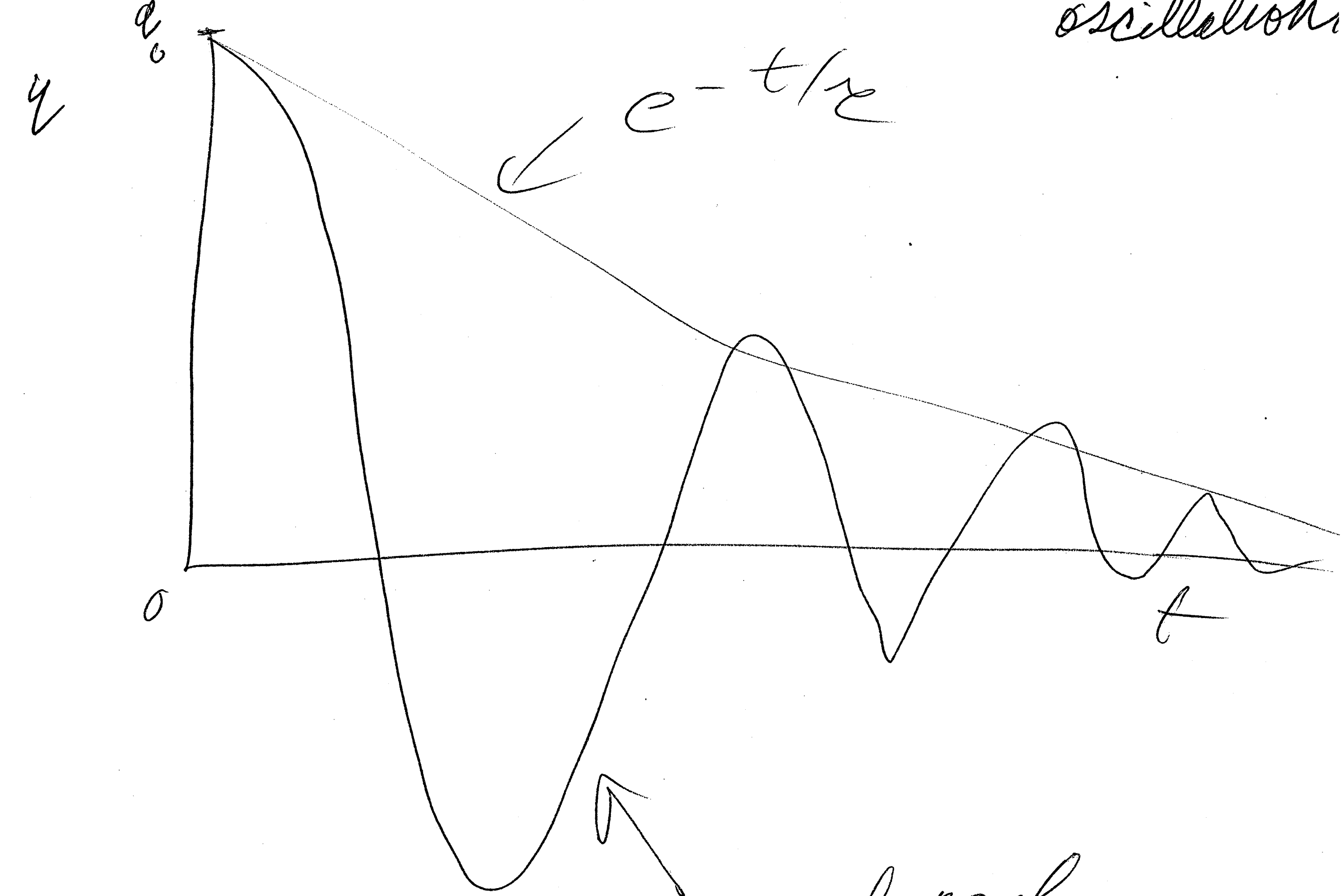
So

$$q = \left(q_0 \cos\omega t + \underbrace{\frac{q_0}{\tau\omega} \sin\omega t}\right) e^{-t/\tau}$$

but for high frequency the term is ~~negligible~~ small.

32-60

In the underdamping case the decline is slow to zero because of oscillation.



enveloped ~~the~~ oscillations - damped oscillations.

R is essentially the damping parameter.

As R is reduced
for fixed L & C

32-61

$$\alpha_{\pm} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

the damping declines
from ~~under~~ underdamped
to critically damped
for $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0$

or $R = 2\sqrt{\frac{L}{C}} = R_{crit}$

to underdamped
for $R < 2\sqrt{\frac{L}{C}} = R_{crit}$

32-62

Mechanical analogies
- not exact ones

- Spring closed are usually
overdamped.
one doesn't want them
to close too quickly
and bop one on the nose.

- But for some reason
restaurant doors to the
kitchen that swing both
ways are a ~~little~~ overdamped
and oscillate a bit.
(I don't know why)

32-63

Optional further details

$$\alpha_\pm = -\frac{1}{\tau_{RL}} \pm \sqrt{\frac{1}{\tau_{RL}^2} - \frac{1}{\tau_{LC}^2}}$$

$$\tau_{RL} = \frac{\cancel{2}L}{R}$$

$$[\tau_{RL}] = \left[\frac{\cancel{V}/(A/s)}{V/A}\right] = s \quad \checkmark$$

$$\tau_{LC} = \sqrt{LC}$$

$$\cancel{\left[\frac{V/(A/s)}{}\right]} \quad [\tau_{LC}] = \sqrt{\frac{V}{A/s}\frac{C}{V}}$$

$$= \sqrt{s^2}$$

$$= s \quad \checkmark$$

$$\alpha_\pm = \begin{cases} -\frac{1}{\tau_{RL}}\left[1 \mp \sqrt{1 - \left(\frac{\tau_{RL}}{\tau_{LC}}\right)^2}\right] \\ -\frac{2}{\tau_{RL}} = -\frac{1}{\tau_{RL}/2} \quad \text{for lower case and } \frac{\tau_{RL}}{\tau_{LC}} << 1 \\ -\frac{1}{\tau_{RL}}\left(\frac{1}{2}\left(\frac{\tau_{RL}}{\tau_{LC}}\right)^2\right) \\ = -\frac{1}{2}\frac{\tau_{RL}}{\tau_{LC}^2} \quad \text{for upper case and } \frac{\tau_{RL}}{\tau_{LC}} << 1 \end{cases}$$

32-64)

and so decline time scale
(e-folding time

are

$$\frac{\tau_{RL}}{2} \quad \text{and} \quad \tau_{LC}\left[2\left(\frac{\tau_{LC}}{\tau_{RL}}\right)\right]$$

$$= \frac{L}{R} = \tau_{RL*}$$

$>> 1$

and so this upper case solution dominates at late time.

$$= 2\sqrt{LC}\,\frac{\sqrt{LC}}{2L/R}$$

$$= RC = \tau_{RC}$$

$$[\tau_{RC}] = \frac{V}{A}\cdot\frac{C}{V}$$

$$= s \ \checkmark$$

If $\tau_{RL} = \tau_{LC}$ or $R_{crit} = 2\sqrt{\frac{L}{C}}$

then

$\frac{\tau_{RL}}{\cancel{2}}$ or $\frac{\tau_{LC}}{\cancel{2}}$ is the ~~same~~ e-folding time.

If $\tau_{RL} < \tau_{LC}$,

then $\tau_{RL} = \frac{2L}{R}$ is

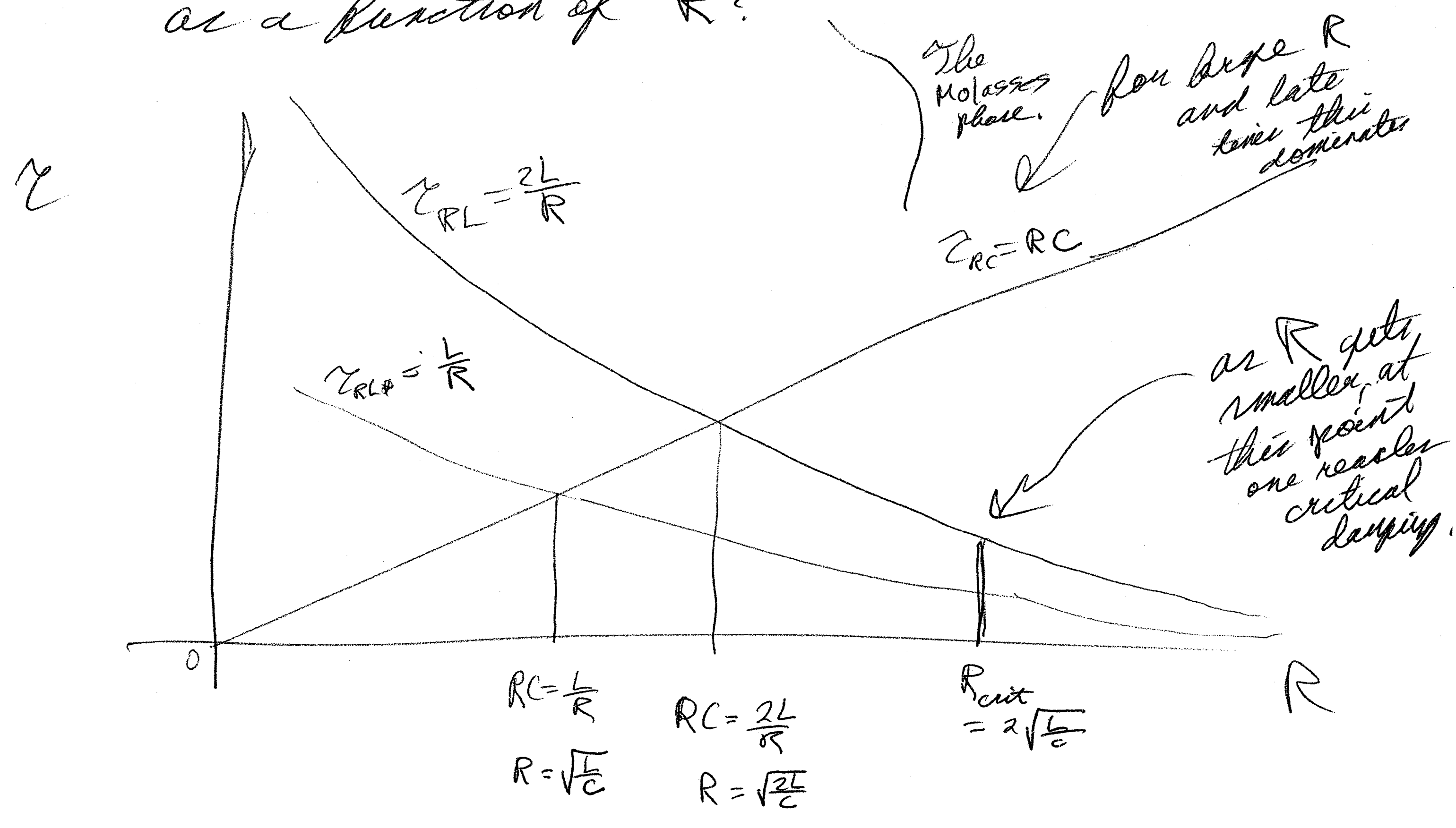
again the e-folding time,

but there is also the

oscillation with frequency

$$w = \sqrt{\frac{1}{\tau_{LC}^2} - \frac{1}{\tau_{RL}^2}}$$

How do all ~~time scal~~ e-folding times behave as a function of R?



$\boxed{32\text{-}66}$

$$\tau_{general} = \frac{1}{\frac{1}{\tau_{RL}} \mp \underbrace{\sqrt{\frac{1}{\tau_{RL}^2} - \frac{1}{\tau_{LC}^2}}}_{\substack{\text{when } \tau_{RL} = \tau_{LC} \\ \text{these are equal} \\ \text{and } \tau_{gen} = \tau_{RL}}}}$$

for $R \geq R_{crit}$

$$= \frac{\overbrace{\tau_{RL}}^{\frac{2L}{R}}}{1 \mp \sqrt{1 - \left(\frac{\tau_{RL}}{\tau_{LC}}\right)^2}}$$

$$\frac{\tau_{RL}}{\tau_{LC}} = \frac{2L/R}{\sqrt{LC}} = \frac{2\sqrt{\frac{L}{C}}}{R} = \frac{R_{crit}}{R}$$

$$= \frac{2L}{R_{crit}} \frac{x}{1 \mp \sqrt{1 - x^2}}$$

[illegible]

$$x = \frac{R_{crit}}{R}$$

$$\frac{2L}{2\sqrt{L/C}} \Rightarrow \sqrt{LC} = \tau_{LC}$$

32-67

Where is the true stationary point?

$$f = \frac{x}{1-\sqrt{1-x^2}}$$

$$\frac{\partial f}{\partial x} = \frac{1}{1-\sqrt{1-x^2}} - \frac{x\left(-\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right)}{\left(1-\sqrt{1-x^2}\right)^2}$$

$$= \frac{1}{\left(1-\sqrt{1-x^2}\right)^2}\left[1-\sqrt{1-x^2} - x^2/\sqrt{1-x^2}\right]$$

$$= \frac{1}{(\ldots)}\frac{1}{\sqrt{1-x^2}}\left[\underbrace{\sqrt{1-x^2}-(1-x^2)+x^2}_{\sqrt{1-x^2}-1}\right]$$

for a stationary point

$$\sqrt{1-x^2}-1 = 0$$

$$1-x^2 = 1$$

$$x^2 = 0, \text{ or } R=\infty$$

but that's a bad point.

32-68

$$f(x) \simeq \frac{x}{1-\left(1-\frac{1}{2}x^2\right)} \quad \text{for small } x.$$

$$= \frac{2}{x}$$

$\therefore f$

So as $x \to 0$, $R \to \infty$

the τ_{gen} increases without ~~outgoin~~ bound from $x = 1$

or $x_{crit.} = 1$

So there are no stationary points.

So the actual minimum time scale is at $x_{crit} = 1$

or $R = 1$

Now what about ~~getting those~~ real oscillatory solutions?

Well the complex solution we obtained by trial solution is

$$q = Ae^{-t/\tau}e^{i\omega t} + Be^{-t/\tau}e^{-i\omega t}$$

where $\tau = \frac{2L}{R}$

and $\omega = \sqrt{\underbrace{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}_{>0}} > 0$

A and B are abitrary so say we set $B \rightarrow +iB$

and ~~take the~~ real part of the solution which is also

32-70)

a solution of DE

— the real & imaginary parts of DE must separately solve the DE $\Longrightarrow$ this is just part & parcel of complex number theory.

$e^{i\omega t} = \cos\omega t + i\sin\omega t$

by Euler's formula
Art - 264

$\therefore\ q = (A\cos\omega t + B\sin\omega t)e^{-t/\tau}$

Just as on p. 32-57.

S32.4 Mutual Inductance

Somehow I omitted this subject in it's proper place.

But there is not much new.

— Self-inductance is the induced emf in a circuit or circuit device induced by the ~~circuit itself~~ the current in the circuit or device

Mutual-inductance is when the induced emf in a circuit or device is due to

32-72

to a current elsewhere.

— usually one can consider it as the interaction between circuits.

This often first happens in a small way without any intention

— But in some cases ~~transformers~~ most obviously — it is a design feature

For Fixed in space circuit paths.

$$\mathcal{E}_2 \propto \frac{d\Phi_{12}}{dt} \propto \frac{dI_1}{dt}$$

Φ_{12} is flux in 2 due to 1

32-73

— The proportionality constant is M

$$\mathcal{E}_2 = - M_{12} \frac{dI_1}{dt}$$

conventional minus

— the sense of $\mathcal{E}_2$ can be deduced from Lenz's law.

The reverse situation occurs

$$\mathcal{E}_1 = - M_{21} \frac{dI_2}{dt}$$

and it can be proven

(but I don't know how and it seems to be tricky)

32-74

that $M_{12} = M_{21}$,

and so one can just write M for one ~~case~~ system.

No proof - we have faith

M depends on geometry and on magnetic materials that are used.

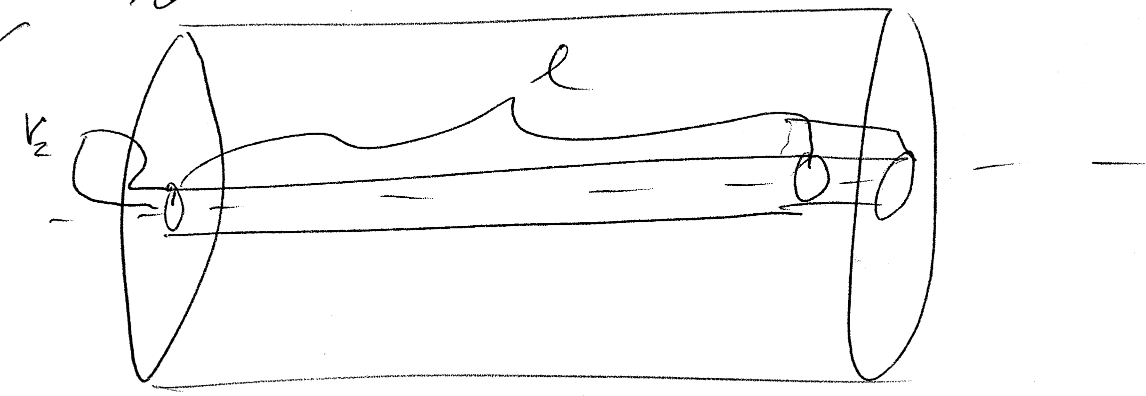
in transformers soft iron cores usually

They do have a big and important effect.

We'll just consider one example case.

Two con-axial solenoids that we treat as ideal

32-74



this means $B = \mu_0 n I$

n: turns per unit length

I: current

B is uniform inside and zero outside due to any one of the solenoids

This is an infinite solenoid result, – but we use it for finite ones in the ideal approximation

$$\Phi_{12} = \underbrace{N_2}_{\text{Number of turns in solenoid 2}} \; \underbrace{A_2}_{\pi r_2^2} \; B_1$$

$B_1 = \mu_0 n_1 I_1$

32-7b)

$$\Phi_{12} = \frac{N_2}{l_2} \underbrace{A_2 l}_{V_2} \mu_0 n_1 I_1$$

V_2 the volume of 2

$$= \mu_0 n_1 n_2 V_2 I_1$$

$$\mathcal{E}_2 = -\frac{d\Phi_{12}}{dt} = -\mu_0 n_1 n_2 V_2 \frac{dI_1}{dt}$$

$$\therefore \quad M_{12} = \mu_0 n_1 n_2 V_2 \quad \text{(TM-977)}$$

What of M_{21} ?

$$\Phi_{21} = N_1 A_2 B_2$$

B_2: $\mu_0 n_2 I_2$

B_2 only extends over A_2 and is zero outside

$$= n_1 l A_2 \mu_0 n_2 I_2$$

$$= \mu_0 n_1 n_2 V_2 I_2$$

$$\mathcal{E}_1 = -\underbrace{\mu_0 n_1 n_2 V_2}\frac{dI_2}{dt}$$

$$M_{21} = \mu_0 n_1 n_2 V_2$$

$$= M_{12}$$

So we've confirmed the general result in ~~this~~ this special case.

32-78

— Mutual inductance is tricky since one can have feedbacks

1 induces a current in 2

which induces an emf in 1

which affects the current in 1

which changes the induced current in ~~2~~, and so on

Which could be regarded a part of self-induction? I'm not sure.

— Nature presumably sorts this out quickly.

— But I'm not sure we know how to calculate all this easily in