Chapter 25

Electrical Potential

or ~~Potential~~ or V

— Potential energy (PE) is the energy of position in a field of force — in real physical contexts, I think it is always describable as some other kind of energy.

We'll see what this is for electrical PE in ch 26

— the ~~el~~ electric field allows an electrical PE to be defined

→ But not in all contexts.

— not when it is Faraday's law induced electric field.

→ (~~Not always, sometimes~~)

It is the energy of the E-field itself or its structure

25-2)

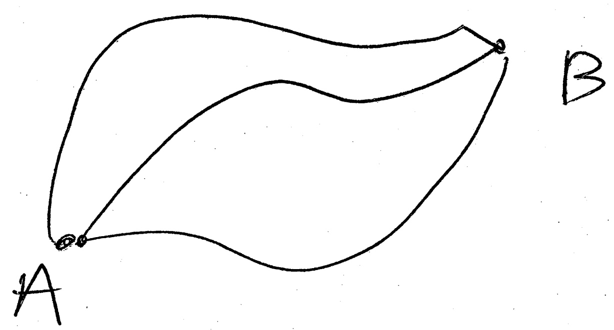
But for electric fields arising from charge, I think one can always define a PE.

~~[illegible]~~

Recall for a conservative force

$$\Delta PE = -W_{\text{done by the conservative force}}$$

ANd ~~T~~o be conservative,



W_{AB} is the same for any path.

which implies

$W_{AB} + W_{BA} = 0$ ~~or~~

~~or~~

or

~~the~~ work done going around any closed path is zero.

~~[struck]~~ Reverse is also true: if work is zero for any closed path then W_{AB} is path-independent.

In circuit this effect turns up as Kirchhoff's voltage law.

Energy for positive work done comes from the force field PE (somewhere)

hese ccount on the minus in $\Delta PE = -W$

and ~~go~~ for negative work done goes into the force field PE (somehow)

→ in Ch 26 we'll see "exactly" where the field ~~energy~~ PE is for the electric field.

It's in the E-field structure itself. But there are fundamental problems with point charges.

But one of the great boons of the PE concept, is you don't have to worry about where the PE "is" most of the time.

~~Note~~

25-4

The PE concept allows you to associate a PE ~~to~~ with each point in space.

— But only to within a constant.

The zero-point of PE can be chosen for convenience.

Only differences or changes in PE give physical effects.

§ 25.1 Electrical PE and Electrical Potential or V

In § 25.3 we'll show that

~~$\underline{E}$ field~~ the electric force of of electric charge is conservative.

- ~~Here~~ Here we just assume that that is so.

Say we have charge q ~~that~~ moves ~~thru~~ an electric field $\underline{E}$

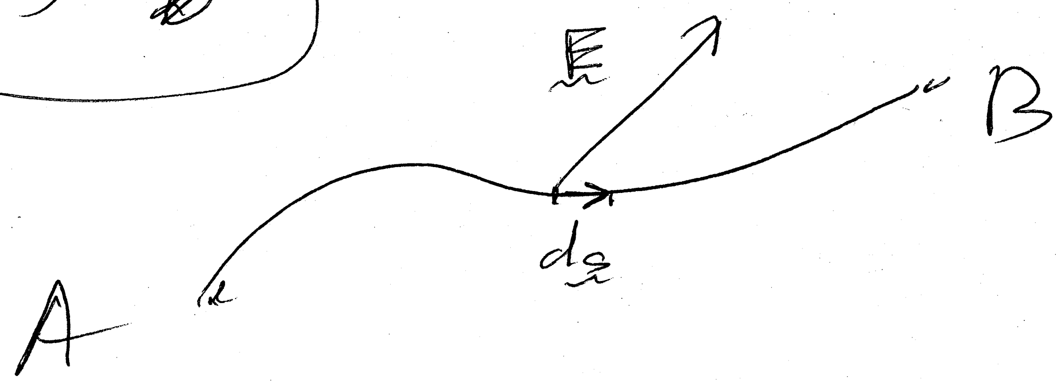
$\underline{F} = q\underline{E}$ is the electric force on q.

$$\Delta PE_{AB} = -W_{AB} \quad \left\{ \begin{array}{l} \text{Change in } q\text{'s} \\ \text{PE from} \\ \text{A to B} \end{array} \right.$$

$$= -\int_A^B q\underline{E} \cdot d\underline{s}$$

where $d\underline{s}$ is the differential ~~path~~ vector ~~between~~ on the path between A and B

25-6



We define the change in electrical potential from A to B by

$$\Delta V_{AB} = \frac{\Delta PE_{AB}}{q} = -\int_A^B \mathbf{E}\cdot d\mathbf{s}$$

So potential is PE per unit charge. It is a scalar like PE.

and is independent of the charge as the expression shows.

like PE the zero point of V can be chosen for convenience.

Potential turns out to be a very useful quantity in electrical matters.

— and it's pretty common since it's what one calls voltage often.

Units are volts symbol V

Roman V for units

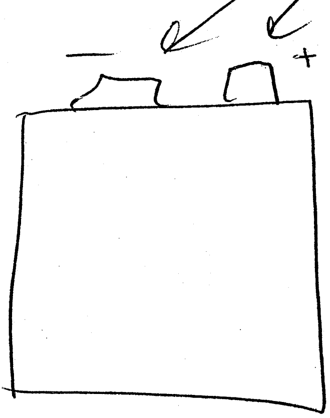
italic V for ~~the~~ quantity

$$1\,\mathrm{V} = \frac{\mathrm{J}}{\mathrm{C}} = \frac{\mathrm{N\,m}}{\mathrm{C}}$$

$\frac{\mathrm{N}}{\mathrm{C}}$ is the unit of electric field $= \frac{\mathrm{V}}{\mathrm{m}}$

25-8

Ex. 9V battery



Positive and negative ~~terminals~~.

If 1C of charge flow between ~~me~~ them, then 9 J of energy are acquired in the battery and deposited somewhere in a device.

Or another way as it will appear in Ch. 27 is that
$P = VI$
So if you draw a current I through a potential V, then the power generated in some form is $P = VI$

V
n
Power
per
unit
current.

Another unit of energy turns up often in our developments:

the electron-Volt = eV

$$1\,eV = 1e * 1V$$
$$= 1e * 1\,J/C$$

$$1e = 1.602\ldots \times 10^{-19}\,C$$

So $1\,eV = 1.602\ldots \times 10^{-19}\,J$

The eV is an atomic scale unit of energy.

— binding energies for atoms in molecules

and

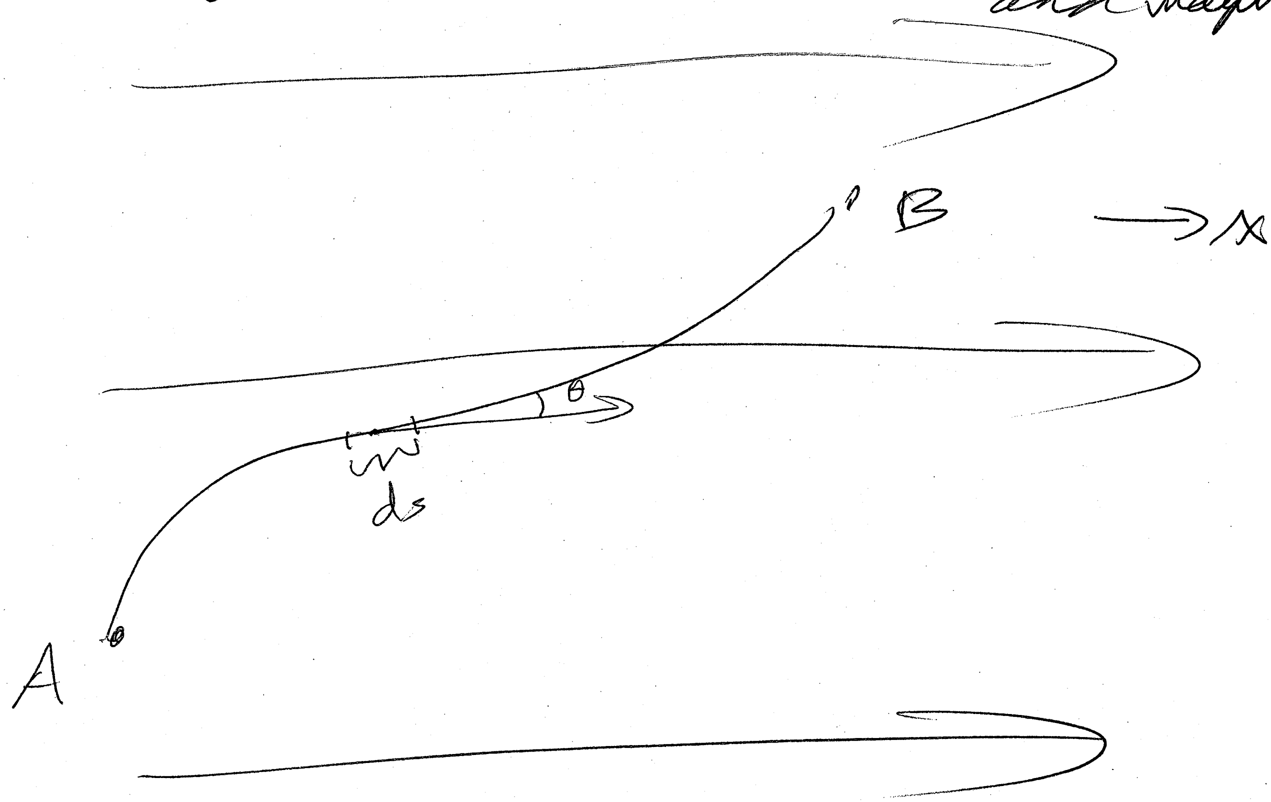
Valence electrons in atoms are of order eV.

Also when accelerating electrons or ions thru macroscopic potentials giving their energies in eV is often convenient.

~~25~~-10

§ 25.2 Potential of ~~of~~ a uniform $\underset{\sim}{E}$-field

$\underset{\sim}{E}$ constant in direction and magnitude.



$$\Delta V_{AB} = -\int_{\underset{\sim}{s}_A}^{\underset{\sim}{s}_B} \underset{\sim}{E} \cdot d\underset{\sim}{s}$$

Well $\underset{\sim}{E} \cdot d\underset{\sim}{s} = E \underbrace{\cos\theta \, ds}_{dx}$

$$\therefore \Delta V_{AV} = -\int_{x_A}^{x_B} E\,dx$$

$$= -E\,\Delta x_{AB}$$

since E is constant in magnitude.

So with our conventions

V decreases in $\hat{x}$ direction

V increases in $-\hat{x}$ direction

No forces except the electric force.

— a free charge $q > 0$ would be accelerated in the $+\hat{x}$

→ it's PE getting converted to KE

— a free charge $q < 0$ would be accelerated in the $-\hat{x}$

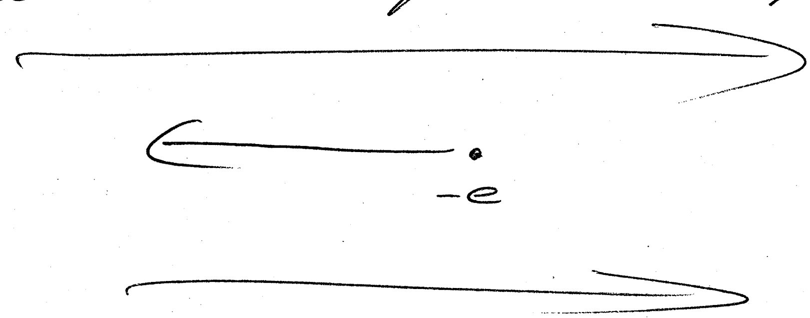
↔ it's PE being converted to KE.

25-12

Recall the work-kinetic energy theorem

$$\Delta KE = \underbrace{W_{net}}_{\text{work by all forces.}}$$

Ex Consider an electron accelerated thru 1V. What is it's final velocity if it started from rest?



$$W = -e \underbrace{E \Delta x}$$

~~but~~ but $\Delta x < 0$

and $E \Delta x = -1V$

$$= 1eV = 1.6 \times 10^{-19} J$$

We need to convert to MKS for a calculation of velocity.

$$v = \sqrt{\frac{2\,KE}{m}}$$

$$\doteq \sqrt{\frac{2 \cdot 1.6 \times 10^{-19}}{10^{-30}}}$$

$$= \sqrt{3.2 \times 10^{+11}} \doteq 6 \times 10^{5}\ \mathrm{m/s}$$

This actually kind of a typical free electron speed since energy conversions in atomic system are often of order eV's.

Note 6×10^{5} m/s is a very high speed but it is not very relativistic since

$$6 \times 10^{5} << 3 \times 10^{8}\ \mathrm{m/s} = c$$

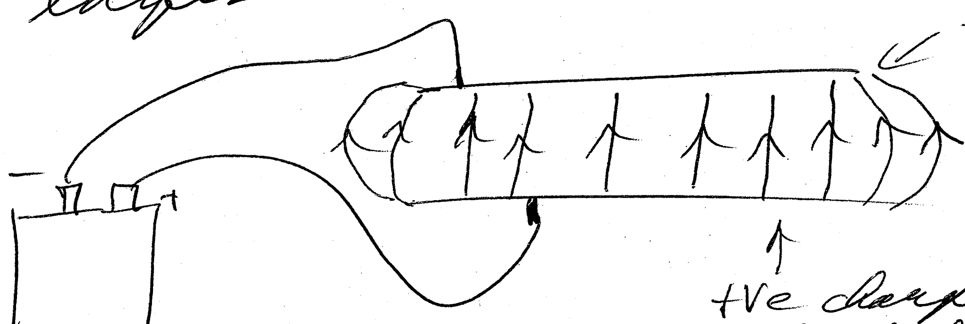
relativistic effect grow as $\beta = v/c$ and are tiny for $\beta = 1/500$

factor of ~500 smaller.

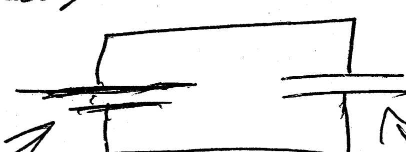
25-14)

Ex 25.1

Parallel plate capacitor (which we'll discuss in Ch26) approximates two infinite ~~sheets of~~ planes of uniform charge density far from the edges



or in circuit diagram schematic



If the battery is 12V, then there is 12V between the plates

— the plates are equipotentials (a concept we go into in §25.4, §25.5)

Far from the edges the $\mathbf{E}$-field should be uniform.

- If the plate separation is

$$d = .3\times 10^{-2}\ \mathrm{m}$$ small separation are usual for real capacitors

then

$$\Delta V = E d$$

or

$$E = \frac{\Delta V}{d} = \frac{12}{.3\times 10^{-2}}\ \frac{\mathrm{V}}{\mathrm{m}}$$

$$= 4\times 10^{3}\ \frac{\mathrm{V}}{\mathrm{m}}$$

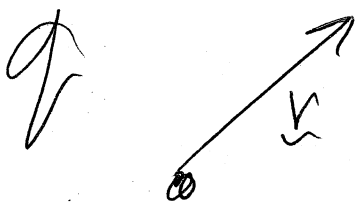
$\boxed{E_{\text{air breakdown}} = 3\times 10^{6}\ \frac{\mathrm{V}}{\mathrm{m}}}$ SJ-736

§25.3 <u>$\mathbf{E}$-field due to charge is conservative</u>

Consider a point

$\boxed{25\text{-}16}$

charge q isolated in space.



$$\mathbf{E} = \frac{kq}{r^2}\hat{r}$$

gives the electric field everywhere.

Assume this and prove the result is path independent.

$$\Delta V_{AB} = -\int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{E}\cdot d\mathbf{s}$$

Recall $\mathbf{E}$ is purely radial

$$\mathbf{E}\cdot d\mathbf{s} = \frac{kq}{r^2}\hat{r}\cdot d\mathbf{s}$$

$$= \frac{kq}{r^2}\, ds\cos\theta$$

where θ is the angle from the radial direction

$$ds \cos\theta = dr$$

$$\therefore \Delta V_{AB} = -\int_{r_A}^{r_B} \frac{kq}{r^2}\, dr$$

$$= \frac{kq}{r}\Bigg|_{r_A}^{r_B}$$

$$\Delta V_{AB} = kq\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

— the result is path independent as we foreshadowed.

So the field of one point charge is conservative.

By convention $r = \infty$ is set to zero potential and so the potential for

25-18

a point charge is

$V = \frac{kq}{r}$ where $\underline{E} = \frac{kq}{r^2}\hat{r}$

Note the key differences:

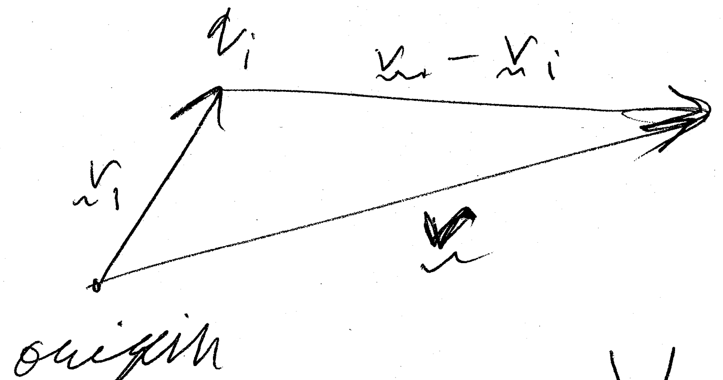
V is $\sim \frac{1}{r}$ and scalar

$\underline{E}$ is $\sim \frac{1}{r^2}$ and is a vector.

Potential is often easier to work with because it is a scalar.

To ~~obater~~ obtain the Potential of a system of charges one uses the superposition principle.

Say we have charge q_i



$$V_i(\underset{\sim}{r}) = \frac{k q_i}{|\underset{\sim}{r} - \underset{\sim}{r}_i|}$$

If one has a system of point charges

$$V(\underset{\sim}{r}) = \sum_i \frac{k q_i}{|\underset{\sim}{r} - \underset{\sim}{r}_i|}$$

Since potential is a scalar one can just add the potentials of each charge. — Each one is well defined with position & so the same i.

Serway writes this as

$$V = \sum_i \frac{k q_i}{r_i}$$ since it puts the the point of evaluation at the origin.

25 - 20

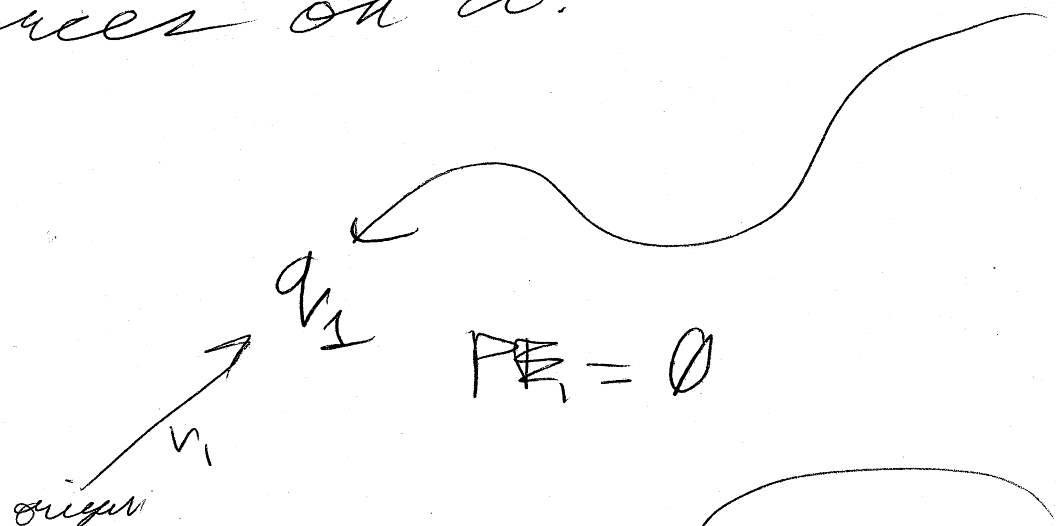
One can go from the sum to a continuous distribution of charges using integration
We do that in §25.5.

How Much energy does it take to assemble a charge distribution?
- from what starting point?
the conventional one is from all the charges at infinity relative to the ~~point~~ place of assembly and each other.

But one doesn't worry about assembling the point charges which is another big problem

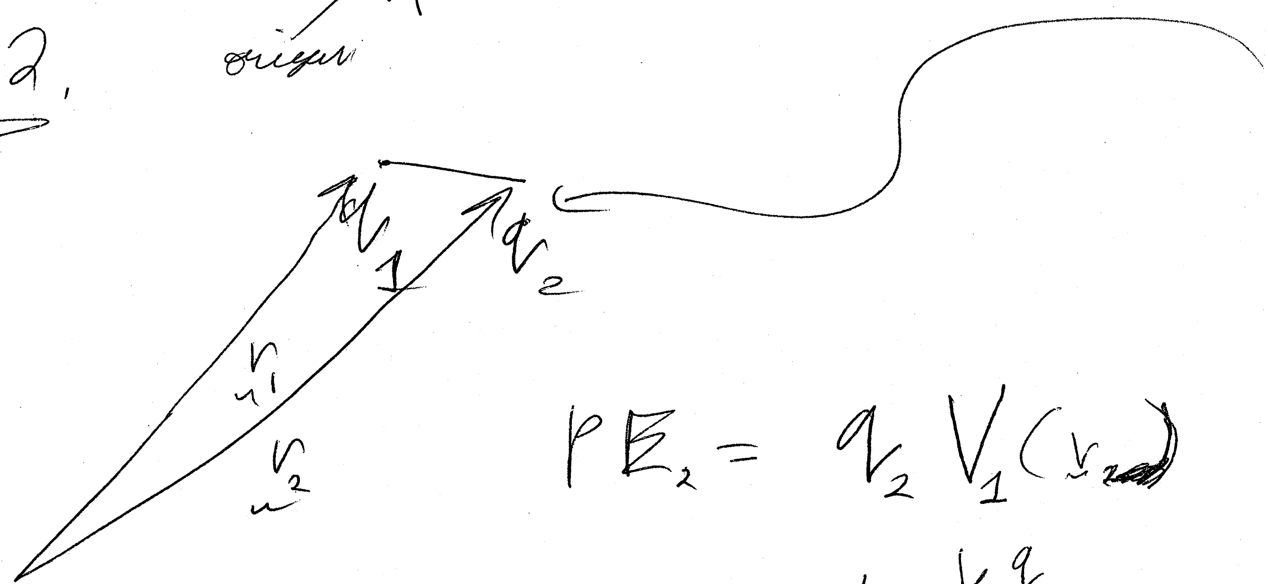
The first charge q_1 you can bring up for free since there are no other charges to exert forces on it.

1.



$$PE_1 = 0$$

2.



$$PE_2 = q_2 V_1(r_2)$$

$$= q_2 \frac{k q_1}{|r_2 - r_1|}$$

3.

$$PE_3 = q_2 \frac{k q_1}{|r_2 - r_1|} + q_3 \frac{k q_1}{|r_3 - r_1|} + q_3 \frac{k q_2}{|r_3 - r_2|}$$

25-22

I^{th}: $PE_I = PE_{I-1} + \Delta PE_I$

$$= PE_{I-1} + \sum_{i=1}^{I-1} q_I \frac{k q_i}{|r_I - r_i|}$$

$$= \sum_{j=1}^{I} q_j \sum_{i=1}^{j-1} \frac{k q_i}{|r_j - r_i|}$$

~~the sum~~ Visualize adding ~~the terms~~ ~~like so~~ in this kind of diagram.

$i \backslash j$	1	2	3	4 ...
1				
2	o			
3	o	o		
4	o	o	o	
⋮				

But one can symmetrize ~~the~~ formula by extending the sum over all spaces except the <u>diagonal</u> over ~~in the~~ table and dividing by 2 to avoid

I^{th} ~~term~~ case is moderately obviously

$$PE_I = \sum_{i=1}^{I} q_i \sum_{j=1}^{i-1} \frac{k q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

$$= \sum_{i=1}^{I} \sum_{j=1}^{i-1} \frac{k q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

You are counting ~~the~~ PE for every pair once.

One can give a more symmetric form

$$PE = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{k q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

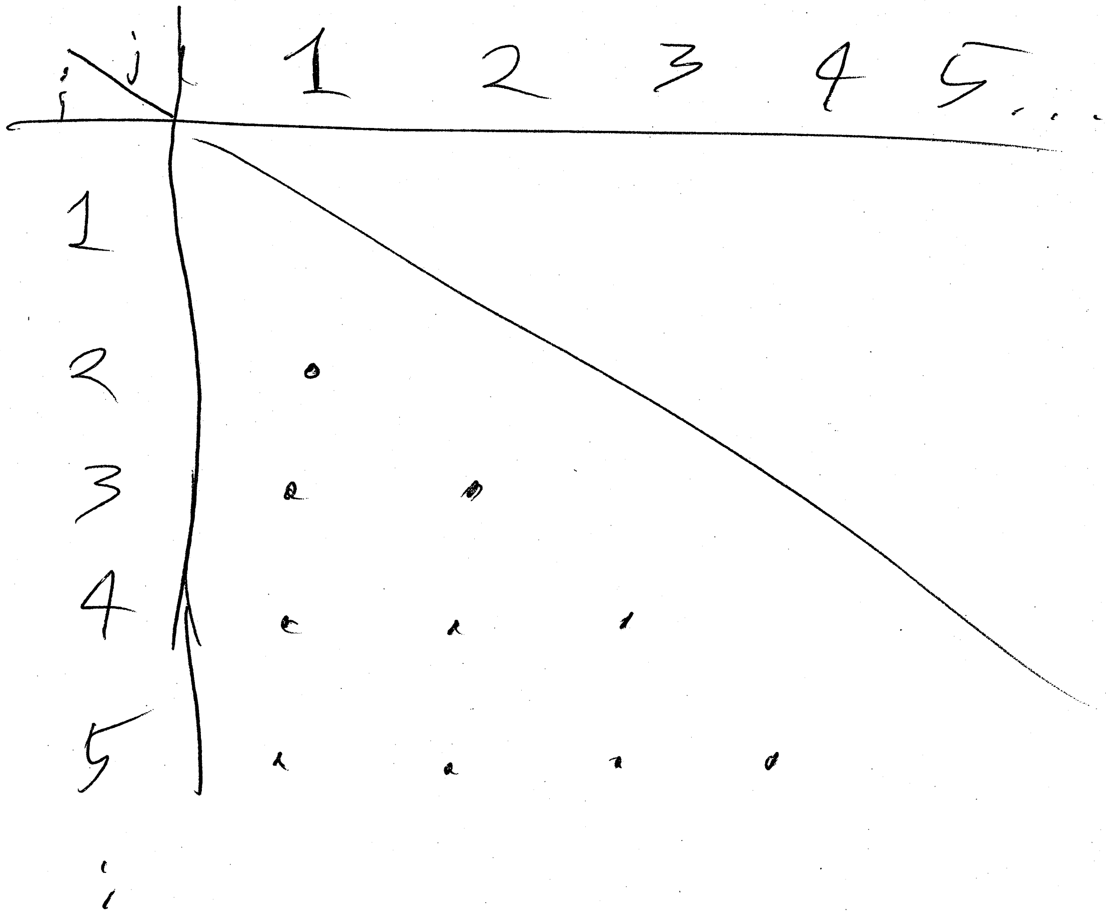
I think this is obvious (but I'd still like a simple math operation proof)

You count every pair twice.

The $\frac{1}{2}$ prevents double-counting

25-24

A visualization trick (which may not help much)



$$\sum_{i=1}^{I} \sum_{j=1}^{i-1} A_{ij} = \sum_{i=1}^{I} \sum_{j=i+1}^{I} A_{ij}$$

if $A_{kl} = A_{lk}$

As an integral form for a continuous charge distribution one could write (GrEM-93)

$$PE = \frac{1}{2}\int \rho(\underset{\sim}{x})\, V(\underset{\sim}{x})\, d\mathcal{V} = \frac{1}{2}\iint \frac{k\,\rho(\underset{\sim}{x})\,\rho(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\mathcal{V}'\, d\mathcal{V}$$

25-26

or alternatively
how much energy
did it take to bring
q_3 in from infinity

$$PE = q_3 V$$

$$\cong -3 \times 10^{-6} \times .7 \times 10^{4}$$

$$= -2 \times 10^{-2}$$

$$= -.02\ J$$

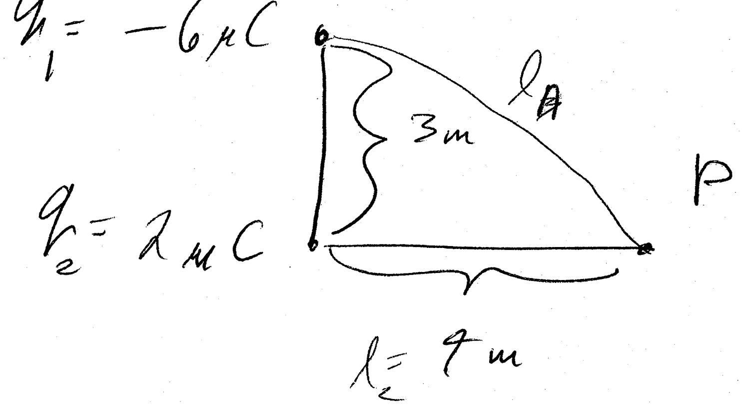
So the PE is negative
and energy could
be extracted bringing q_3
to this point.

~~If the~~ If it had just be
allowed to drift in as a free

Ex 25.3

micro-Coulombs are much more likely in ~~undergraduate~~ labs than coulombs

We hold ~~these~~ fixed



What is V at P?

$$V = \frac{k q_2}{l_2} + \frac{k q_1}{l_A}$$

$$\cong 10^{10} \cdot 10^{-6} \left(\frac{2}{4} + \frac{(-6)}{\sqrt{3^2 + 4^2}} \right)$$

$$= 10^4 \left(\frac{1}{2} - 1.2 \right)$$

$$= -.7 \times 10^4 \text{ V}$$

If $q_3 = 3\,\mu C$ is put at P, how much PE ~~did it~~ does it have charge, the PE [25-27

change from 0 at infinity to -.02 J at P would have gone into KE.

- The q_3 would try to get to q_1 actually. Pulled there by the electric force.

- ~~Tor~~ For macroscopic systems we don't have real point charges and we don't need to worry about infinite V or PE.

In the micro-realm, QM ~~prev~~ prevents infinite PE (not always in ways we thoroughly understand)

25 – 28

§25.4

Obtaining the $\underset{\sim}{E}$-field from the Potential

Recall

$$V_{AB} = -\int_A^B \underset{\sim}{E} \cdot d\underset{\sim}{s}$$

$\therefore$

$$dV = -\underset{\sim}{E} \cdot d\underset{\sim}{s}$$

$V = \frac{kq}{r}$

$\underset{\sim}{E} = \frac{kq}{r^2}\hat{r}$

as we already knew.

Special case

Consider a point charge

and a $d\underset{\sim}{s} = d\underset{\sim}{r}$

$\underset{\sim}{E} = \frac{kq}{r^2}\hat{r}$

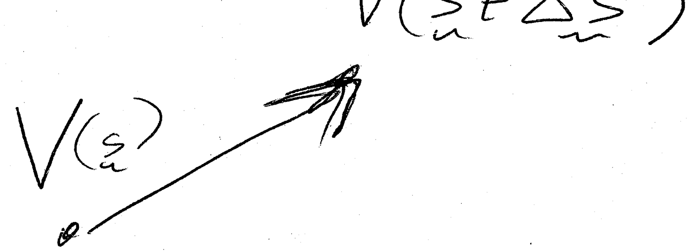
a radial displacement

$$\therefore \quad dV = -\frac{kq}{r^2}\,dr$$

$$\text{or} \quad E = -\frac{dV}{dr} \quad \text{or} \quad \underset{\sim}{E} = -\frac{dV}{dr}\hat{r}$$

General Case

— for this we need the <u>Gradient Operator</u> which SJ-70 2 give in a footnote



Say we want the general expression for $\frac{dV}{ds}$ where $\frac{dV}{ds}$ is the derivative of V in the $\underset{\sim}{s}$ direction at some point

Definition of derivative

$$\frac{dV}{ds} = \lim_{\Delta \underset{\sim}{s} \to 0} \frac{V(x+\Delta x, y+\Delta y, z+\Delta z) - V(x,y,z)}{\Delta s}$$

25-30

and $\Delta\underset{\sim}{s} = (\Delta x, \Delta y, \Delta z)$

Well we can Taylor expand

$V(x + \Delta x, y + \Delta y, z + \Delta z)$

$= V(x, y + \Delta y, z + \Delta z)$

dropping higher order terms as needed.

$+ \frac{\partial V}{\partial x}(x, y + \Delta y, z + \Delta z)(\Delta x)$

$\frac{\partial}{\partial x}$ is the Partial derivative operator.
– derivative with respect to x while y and z are considered constants.

if we expand in Δy and Δz we get higher order terms

$\left(\frac{\partial V}{\partial x}(x, y, z + \Delta z) + \frac{\partial^2 V}{\partial x \partial y}\Delta y\right)\Delta x$

If Δx and Δy are both tiny, $\Delta x \Delta y$ is very tiny.

$\approx \frac{\partial V}{\partial x}(x, y, z)\Delta x$

$$= V(x, y+\Delta y, z+\Delta z) + \frac{\partial V}{\partial x}(x,y,z)\,\Delta x$$

to 1st order

$$= V(x,y,z) + \frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial y}\Delta y + \frac{\partial V}{\partial z}\Delta z$$

Similarly where ~~the~~ partial derivatives are evaluated at x, y, z

$$\therefore \frac{dV}{ds} = \lim_{\Delta s \to 0} \frac{\frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial y}\Delta y + \frac{\partial V}{\partial z}\Delta z}{\Delta s}$$

$$= \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) \cdot \hat{n}$$

where $\hat{n} = \dfrac{(\Delta x, \Delta y, \Delta z)}{\Delta s}$

25-32

∇ is called the del ~~operator~~ symbol or gradient operator

Typographically it's the nabla symbol

$$\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

is the gradient of V. It's a vector.

In Cartesian coordinates

$$dV = \nabla V \cdot \hat{n}\, ds$$

$$dV = \nabla V \cdot d\underset{\sim}{s}$$ which is a customary format.

We now see that

$$\underset{\sim}{E} = -\nabla V$$

This is general, but if you are not using Cartesian coordinates,

then you need ∇
in other coordinates.

For spherical polar

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right)$$

(no proof)
Arf-81

If V is in a spherically symmetric system, it makes sense to use spherical ~~of~~ polar coordinates with the origin at the center of symmetry.

Since $V = V(r)$

$$\nabla V = \left(\frac{\partial V}{\partial r}, 0, 0 \right) = \frac{\partial V}{\partial r}\hat{r}$$

So if $V = \frac{kq}{r}$

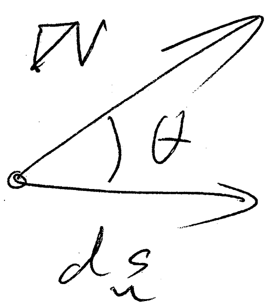
$$\underset{\sim}{E} = -\nabla V = \frac{kq}{r^2}\hat{r}$$

as before on p. 25-28

Fastest Increase & Decrease

$$dV = \nabla V \cdot d\underset{\sim}{s} = |\nabla V|\, ds \cos\theta$$

— with respect to <u>path length</u>

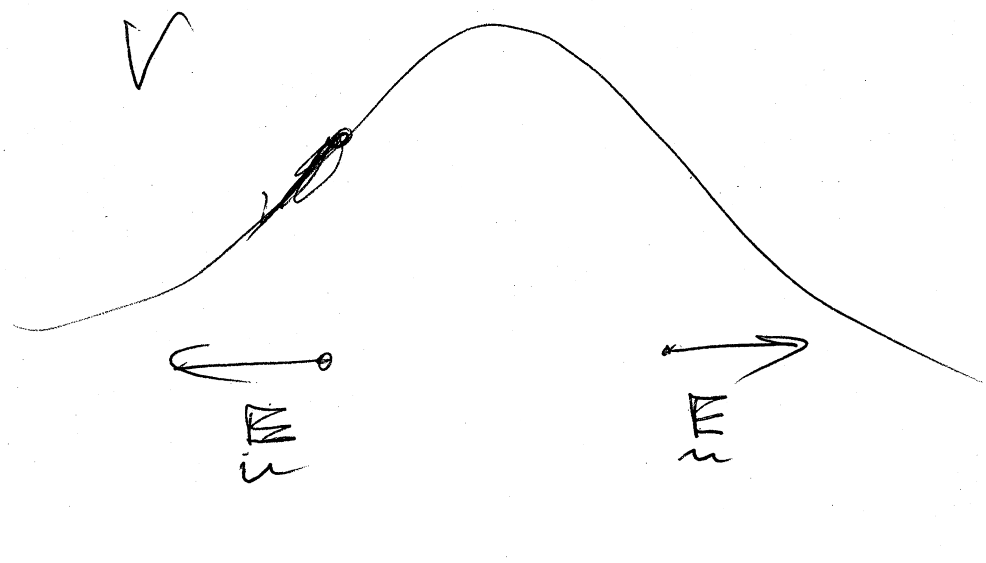


V increases fastest in gradient direction and decreases fastest opposite gradient direction.

Since $\underset{\sim}{E} = -\nabla V$

That minus sign means that the electric field points in the direction the potential has fastest decrease.

— from a pure math sense the minus may seem annoying, but physically it tells us that the force on a positive charge points down the potential hill



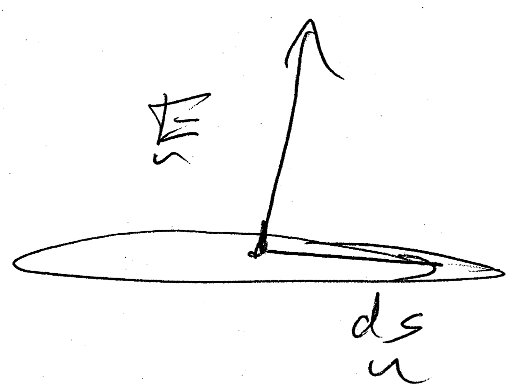
Physically this makes sense.

Equipotentials

$$dV = \nabla V \cdot d\underset{\sim}{s}$$

$$= -\underset{\sim}{E} \cdot d\underset{\sim}{s}$$

So on a path perpendicular to $\underset{\sim}{E}$, $dV = 0$.



Such paths lie in a surface perpendicular to $\underset{\sim}{E}$

These are equipotential surfaces since V is constant on them.

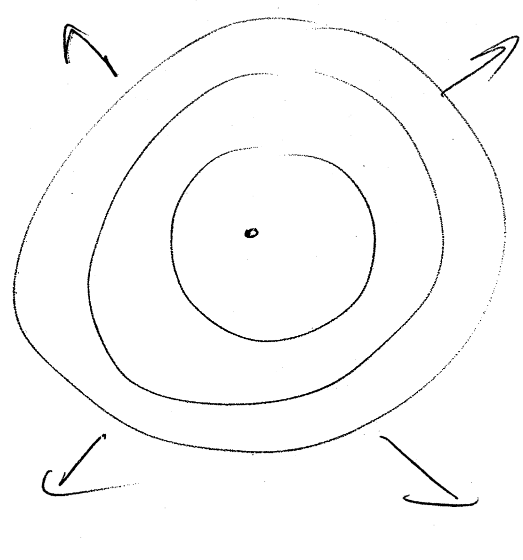
25 – 37

Equipotentials turn out to be useful in electrical things especially when dealing with conductors as we'll see in § 25.6

Example 1 Spherically symmetric ~~point charge~~ charge distribution

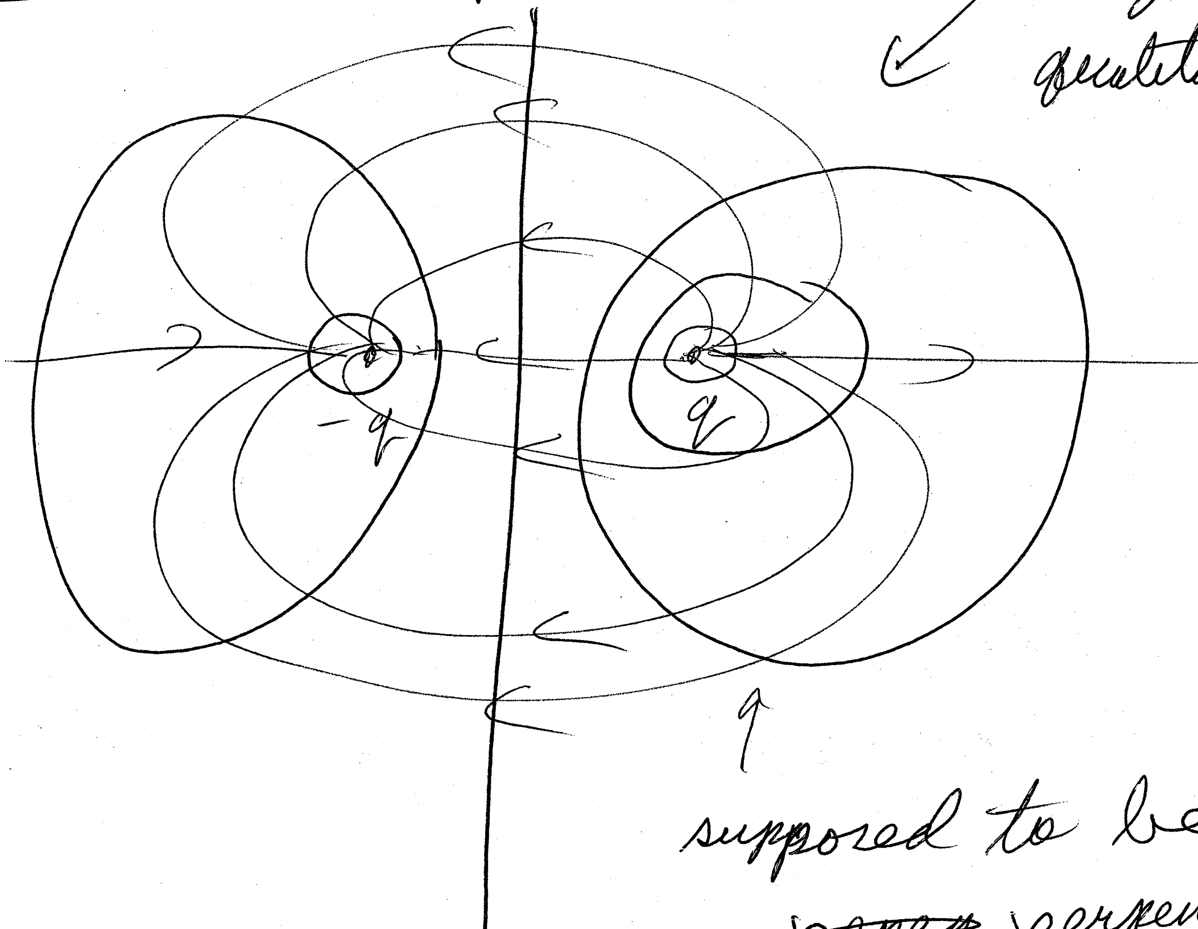
$\underline{E}$ is always radial by symmetry.

So the equipotentials are spheres concentric with the origin.



Ex 2 Electric Dipole

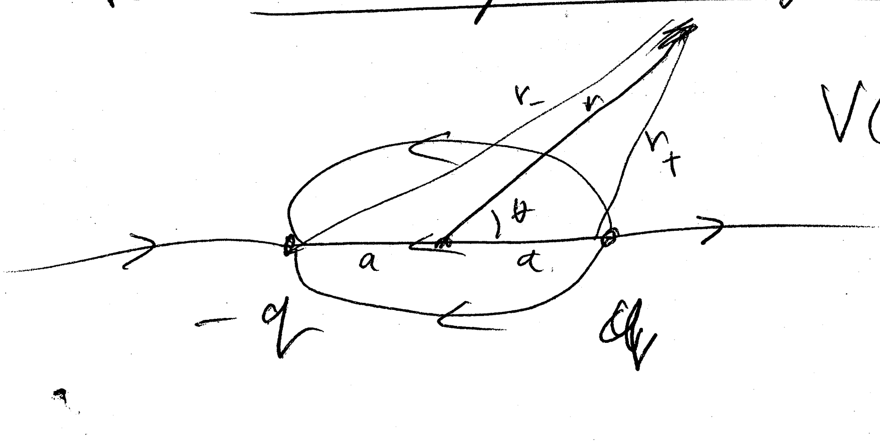
very qualitatively



supposed to be ~~perpen~~ perpendicular

Ex 25.4 (but I'm more general than SJ-702)

Potential of a dipole



$$V(r,\theta) = \frac{kq}{r_+} - \frac{kq}{r_-}$$

$$V(r,\theta) = kq\left(\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{1}{\sqrt{r^2+a^2+2ar\cos\theta}}\right)$$

using law of cosines
and $\cos(\pi - \theta)$
$= -\cos\theta$

This is the exact result.
Now for the far-field limit.

$$\frac{1}{\sqrt{r^2+a^2 \mp 2ar\cos\theta}} = \frac{1}{r}\,\frac{1}{\sqrt{1 \mp 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2}}$$

$$\simeq \frac{1}{r}\left(1 \pm \frac{a}{r}\cos\theta\right)$$

to 1st order in
Taylor series
in small $\frac{a}{r}$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x$$

around $x = 0$

$\boxed{25-40}$

$\therefore$ to 1st order in $\frac{a}{r}$

$$V(r,\theta) = \frac{kq}{r} \; \frac{2a}{r} \cos\theta$$

$$= \frac{k(2qa)}{r^2} \cos\theta$$

Recall $\underset{\sim}{p} = (2aq)\,\hat{a}$

is the dipole moment

(by definition)

$$V(r,\theta) = \frac{k\,p\cos\theta}{r^2} = \frac{k\,\underset{\sim}{p}\cdot\hat{r}}{r^2}$$

unlike a point charge (or a non-zero net charge), the dipole potential in the far-field limit goes as $\sim \frac{1}{r^2}$

$$\underset{\sim}{E} = -\nabla V$$

$$= -kP\left(\frac{-2}{r^3}\cos\theta,\ \frac{-\sin\theta}{r^3},\ 0\right)$$

using p 25-33 for ∇ in spherical polar coordinates.

$$= \frac{k}{r^3}\left(2P\cos\theta\,\hat{r} + P\sin\theta\,\hat{\theta}\right)$$

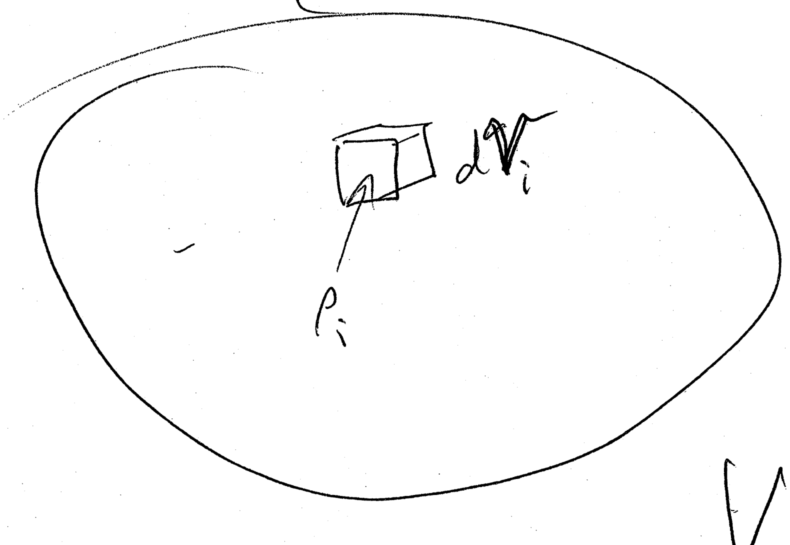
which was one of the forms we obtained for the far-field dipole E-field.

(see p. 23-75)

§ 25.5 Potential of a Continuous Charge Distribution

Recall $V(\underline{r}) = \sum_i \frac{k\, q_i}{|\underline{r} - \underline{r}_i|}$

for a discrete ~~distrib~~ distribution



a continuous distribution can be approximated as

$$V = \sum_i \frac{k\, \rho(\underline{r}_i)\, dV_i}{|\underline{r} - \underline{r}_i|}$$

then making the $dV_i \to 0$

and the miracle of calculus

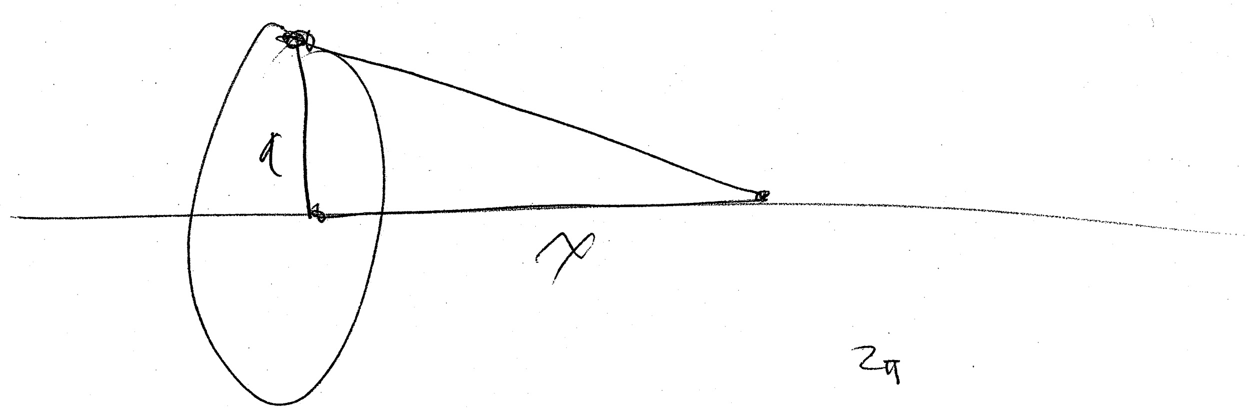
Gives

$$V(\underset{\sim}{r}) = \int_{\text{Volume}} \frac{k\,\rho(\underset{\sim}{r}')\,d\mathcal{V}'}{|r - \underset{\sim}{r}'|}$$

$\mathcal{V}$ is volume not V for potential.

If the geometry is nice there is no problem evaluating.

E 25.5 Potential on axis of a Ring of charge



$\lambda = \frac{Q}{2\pi r}$

linear charge density

$$V(x) = \int_0^{2\pi} \frac{k\lambda\,a\,d\theta}{\sqrt{x^2+a^2}}$$

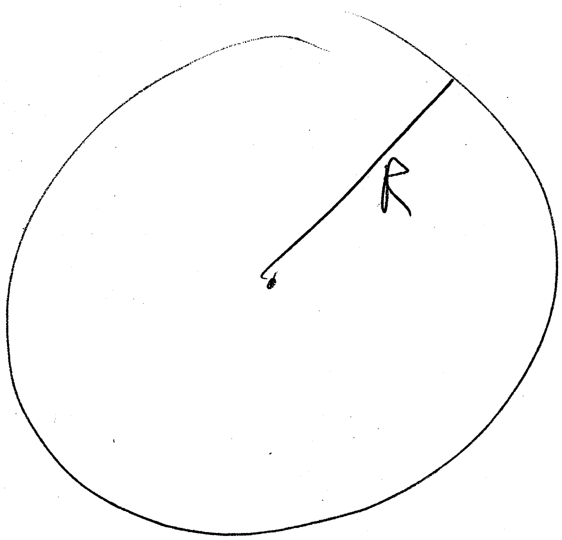
$$= \frac{kQ}{\sqrt{x^2+a^2}}$$

25-44

Off axis is horrible of course.

Example   Uniform Sphere of Charge

— Total charge Q



From Gauss's law

$$\underline{E} = \begin{cases} \dfrac{kQ}{r^2}\,\hat{r}, & r > R \\[2ex] \dfrac{kq_{encl}}{r^2}\,\hat{r} & r < R \\[2ex] = \dfrac{kQ\left(\frac{r}{R}\right)^3}{r^2}\,\hat{r} & \\[2ex] = \dfrac{kQ}{R^2}\left(\dfrac{r}{R}\right)\hat{r} & \end{cases}$$

The two expressions agree at $r = R$ as they should.

Recall from spherical symmetry

$$E = -\frac{dV}{dr}$$

So $V = \frac{kQ}{r}$ outside

where we set
$V(\infty) = 0$
as usual.

Inside

$$V = -\int E\,dr$$

$$= -\frac{kQ}{R^2}\frac{r^2}{2R} + \text{Constant}$$

$$= -\frac{kQ}{R}\frac{1}{2}\left(\frac{r}{R}\right)^2 + \text{Constant}$$

At the surface

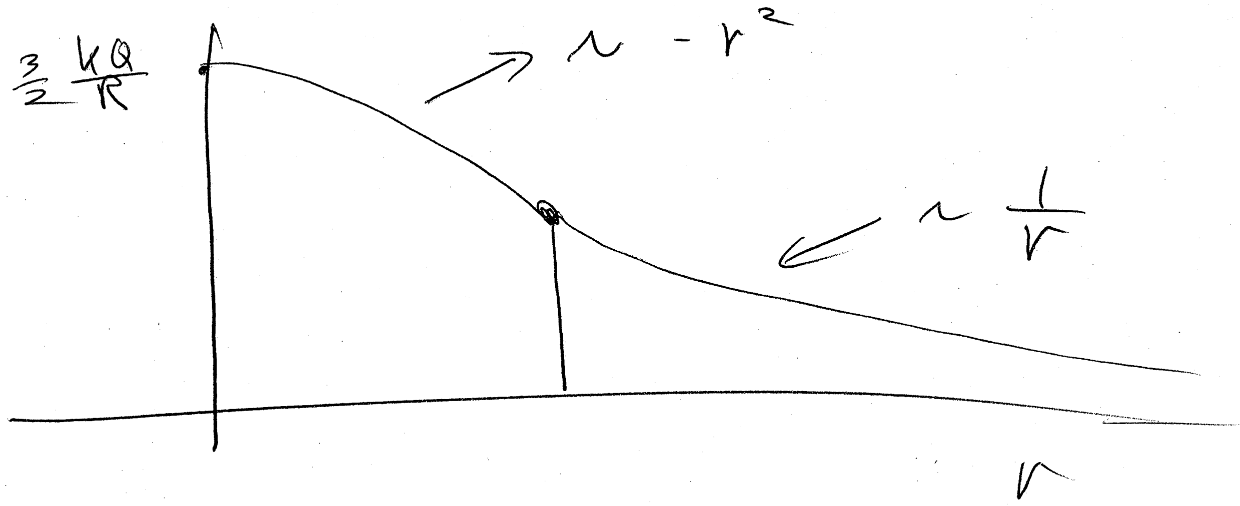
$V_{inside} = V_{outside}$ and this sets the constant.

$$-\frac{kQ}{R}\frac{1}{2} + C = \frac{kQ}{R}$$

$$C = \frac{3}{2}\frac{kQ}{R}$$

$$V_{\text{inside}}(r) = \frac{kQ}{R}\left(\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right)$$

Potential as a function of r



Example PE to assemble a sphere of uniform charge

↳ bringing the charge up from infinity.

Recall

$$PE = \frac{1}{2}\sum_{ij} \frac{k q_i q_j}{|r_i - r_j|} = \frac{1}{2}\sum_{ij} q_i V_j$$

(from p. 25-23)

In the continuous limit

$$PE = \frac{1}{2} \int_{\text{Volume}} \rho(\underset{\sim}{r})\, V(\underset{\sim}{r})\, d\mathcal{V}$$

$$d\mathcal{V} = 4\pi r^2\, dr$$

for a sphere

$$\rho(\underset{\sim}{r}) = \frac{Q}{\frac{4}{3}\pi R^3}$$ which is a constant

$$PE = \frac{3}{2}\frac{Q}{R^3} \int_0^R \frac{kQ}{R}\left(\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right) r^2\, dr$$

$$= \frac{3}{2}\frac{Q^2}{R^4} \int_0^R \left(\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right) r^2\, dr$$

$$= \frac{3}{2}\frac{Q^2}{R} \int_0^1 \left(\frac{3}{2} - \frac{1}{2}x^2\right) x^2\, dx$$

where $x \equiv \frac{r}{R}$

$\boxed{25-48}$

$$= \frac{3}{2}\frac{kQ^2}{R}\Big[\underbrace{\underbrace{\frac{1}{2}x^3 - \frac{1}{2}\cdot\frac{1}{5}x^5}_{\frac{1}{2}\ -\ \frac{1}{10}}}_{\frac{4}{10}\ =\ \frac{2}{5}}\Big]\Big|_0^1$$

$$= \frac{3}{5}\frac{kQ^2}{R}$$

Say $Q = 1\ C$

and $R = 1\ m$

$PE \simeq \frac{3}{5}\cdot 10^{10}\ J$ which is huge

For comparison

$1\ kW\text{-}h$

$= 1kW\cdot h\ \frac{3600s}{h}$

$= 3600\ kJ$

$= 3.6\times10^6 J$

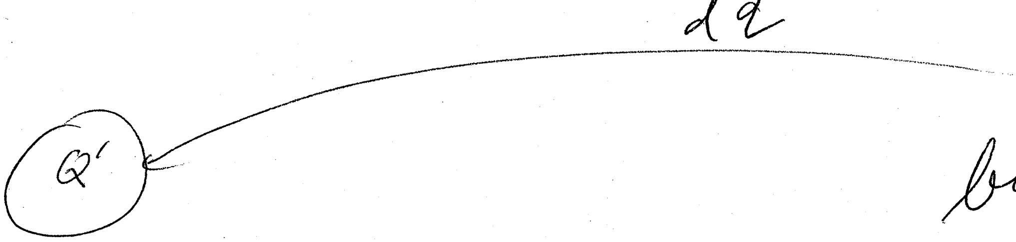
So it takes $\frac{6\times10^9}{3.6\times10^6} \approx 2000 kW$ of energy to create this sphere of charge

And so this another way to see ~~that~~ assembly ~~why assembly~~ a coulomb of net charge is hard.

Optional

The same calculation can be done in a more picturesque way.

– assemble the charge distribution by bringing up bits of charge from infinity and ~~so~~ spreading them on a growing sphere.



$$PE = \int_0^R \underbrace{\frac{kQ\left(\frac{r}{R}\right)^3}{r}}_{\text{potential at surface of charge present}} \underbrace{\left(\frac{Q}{\frac{4\pi}{3}R^3}\, 4\pi r^2\, dr\right)}_{\text{charge brought up from infinity}}$$

~~kQ^2~~

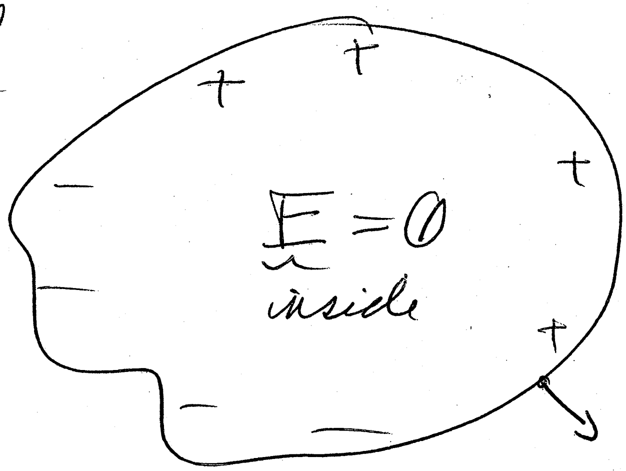
$$= 3k\frac{Q^2}{R^6}\int_0^R r^4\, dr = \frac{3}{5}\frac{kQ^2}{R^6}\, r^5\Big|_0^R$$

$$PE = \frac{3}{5}\frac{kQ^2}{R}$$ again as it should.

The two calculations should give the same result.

§ 25.6 Potential of an Electrostatic Conductor

Recall



At surface

$$\underset{\sim}{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

where $\hat{n}$ is normal to the surface and σ is the surface charge density

As we proved
from the electrostatic
condition of a conductor,

$\underset{\sim}{E}$ at surface is normal
to the surface.
and outward

$\therefore$ $dV = -\underset{\sim}{E} \cdot d\underset{\sim}{s}$

where $d\underset{\sim}{s}$ is along
the surface is zero

So the surface is an equipotential.

— no matter what shape.

Now going inward and inside
$dV = -\underset{\sim}{E} \cdot d\underset{\sim}{s}$ has $\underset{\sim}{E} = 0$.

$\therefore$ $dV = 0$ for any displacement

25-52

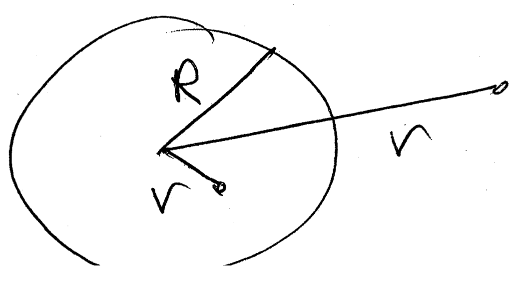
inside the conductor.

So the whole conductor is at one potential value.

— this turns out to be an important result since we are always using conductors in electrical applications

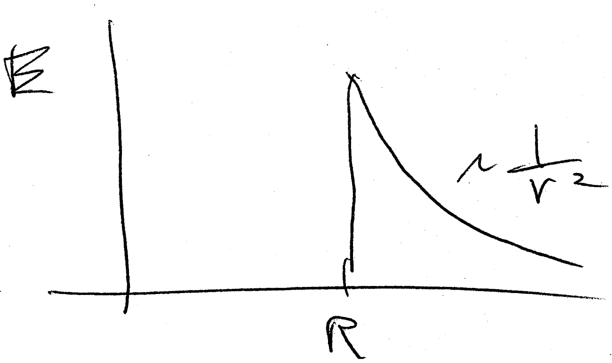
— It turns up for capacitors and circuits as we'll soon see.

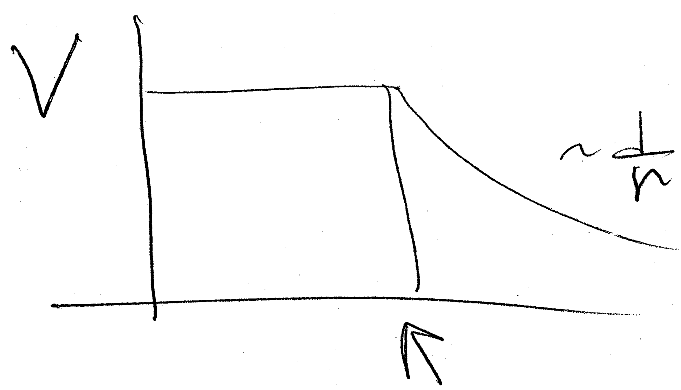
Ex Spherical conductor with net charge Q.



$$\underset{\sim}{E} = \begin{cases} \dfrac{kQ}{r^2}\hat{r} & \text{outside} \\ 0 & \text{inside} \end{cases}$$

$$V = \begin{cases} \dfrac{kQ}{r} & \text{outside} \\ \dfrac{kQ}{R} & \text{inside} \end{cases}$$



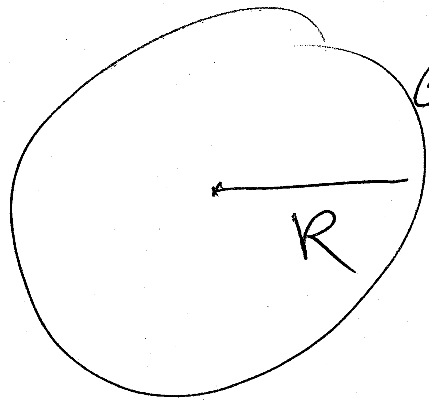


So the result is different from our uniform sphere ~~of~~ of charge. (see p. 25-46)

[Uniform sphere of charge is pretty much a theoretical object. Hard ~~to~~ to construct. A charged conducting sphere is easy.]

25-54

What is PE of assembly in this case



bring charge dq up from infinity and smear it evenly over the surface.

$$dPE = \frac{kq}{R}\,dq$$

$$PE = \int_0^Q \frac{kq}{R}\,dq$$

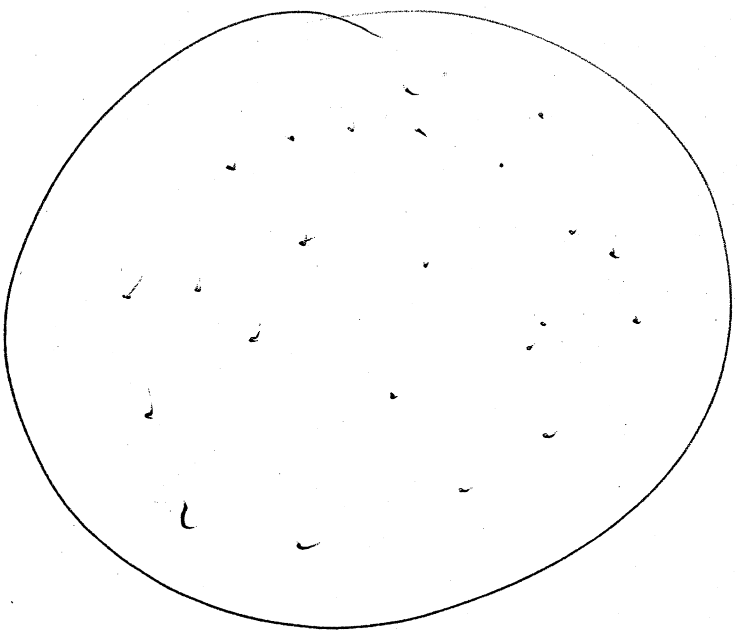
$$= \frac{kQ^2}{2R} = \frac{1}{2}\frac{kQ^2}{R}$$

which is a bit smaller than our uniform sphere result of $\frac{3}{5}\frac{kQ^2}{R}$

(see pg 25-48)

We can understand why this must be so

Say we had a sphere of uniform charge



— then we freed the charge except the boundary is impenetrable.

~~the charge~~

For definiteness and without loss of generality, ~~the~~ let us say the charge is positive.

— the E-field inside is either outward pointing or zero.

2V-56

as long as spherical symmetry is maintained

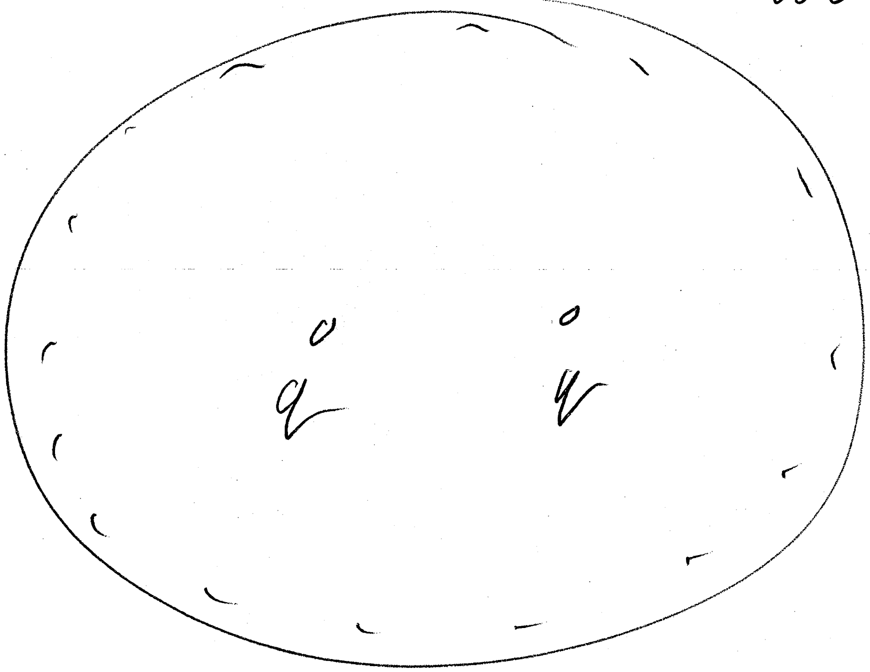
— ~~so~~ we expect the charge to be pushed to the surface. where ~~any imbalan~~ it would form a uniform surface density.

Now what if spherical symmetry is NOT maintained ~~do~~ due to perturbations.

— nevertheless the final result is the stable equilibrium and must be reached.

→ Say two charges were

left inside and the surface was uniform



$\underset{\sim}{E} = 0$ inside ~~the~~ by Gauss's law except for the mutual repulsion of the charges

So they'd repel each other till they hit the walls.

If one charge was left and $\underset{\sim}{E} = 0$ inside.

it would just float around

25-58

but that is a negligible anomaly if there are many charges

What happens to the energy?

Well if there is no dissipation to waste heat, the charges wouldn't settle down on the surface, but randomly bounce around.

but we assume dissipation to waste heat ~~so~~ turns the KE they acquire into from the electric force to waste heat

High Surface $\underset{\sim}{E}$-field and charge density at points of high curvature

- we mentioned this in Ch 24 (p. 24-42) without proof.

- It's only a tendency that can be overruled by strange geometries and applied fields.

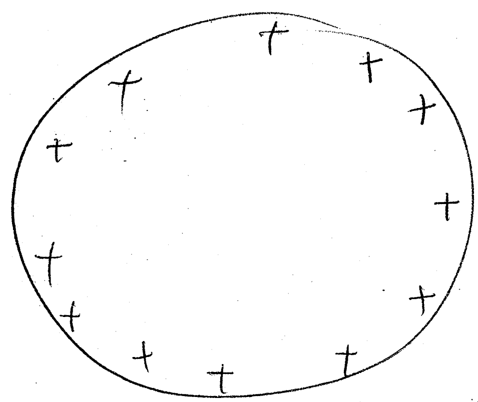
- But it's a useful rule of thumb

→ the corners and edges of conductors are most dangerous for shocks → that where $\underset{\sim}{E}$ is highest and the air might break down and conduct.

Since it is only a tendency no exact proof is possible. One can only look a representative cases and see what happens.

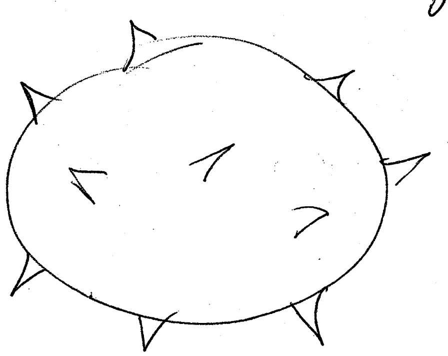
Inconclusive Case

Take a charged ~~sph~~ conducting sphere



The charge is uniformly spread on the surface.

Now let little points grow on it



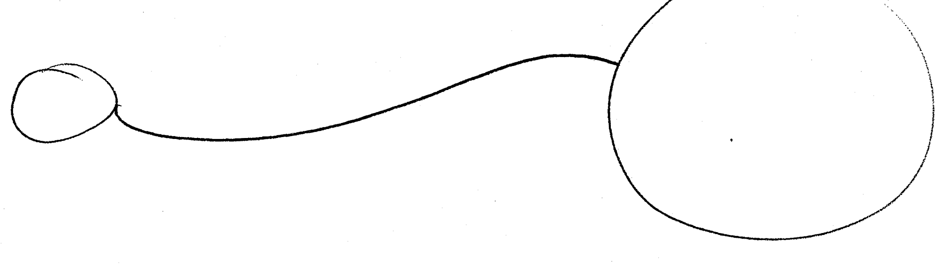
It seems reasonable to guess that the charge will cluster a bit in the prongs to spread out more

<u>but</u> if it clusters in the prongs, does ~~it~~ it create~~s~~ little concentrations that are more self-repulsive than before?

— Just qualitatively the case seems undecidable.

<u>Conclusive case</u>

Two conducting spheres joined by a thin conducting wire



26-62

The spheres are so well separated and the wire so thin that both spheres can be regarded as having spherical symmetry with negligible perturbation by other sphere and wire

[This argument itself probably needs a quantitative justification, but it sounds pretty plausible.] — one could always make the wire longer and thinner one could think.

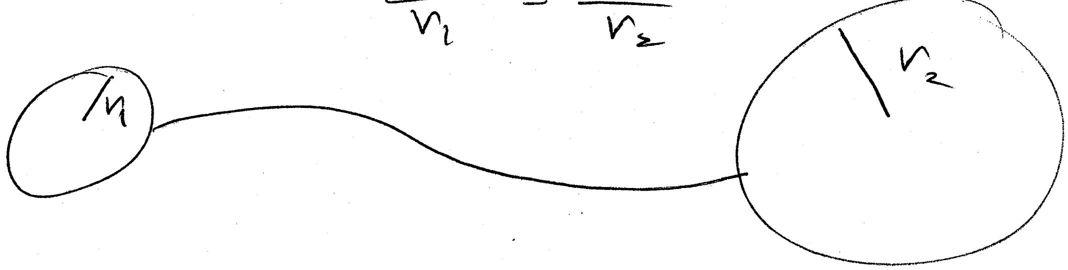
The system has a net charge and is allowed to come to electrostatic equilibrium.

This being the case
the whole system is ~~an~~
one equipotential on the
surface.

$$\therefore \quad V_1 = \frac{k q_1}{r_1} = \frac{k q_2}{r_2} = V_2$$

on surface

or ① $\frac{q_1}{r_1} = \frac{q_2}{r_2}$



but $E_1 = \frac{k q_1}{r_1^2}$, $E_2 = \frac{k q_2}{r_2^2}$ on surface

② $$\frac{E_1 r_1^2}{q_1} = k = \frac{E_2 r_2^2}{q_2}$$

Recall $E = \frac{\sigma}{\varepsilon_0}$ at conductor surface.

Multiply ① * ②

$$E_1 r_1 = E_2 r_2 \quad \text{or} \quad E_2 = E_1 \left(\frac{r_1}{r_2}\right)$$

which implies $\sigma_1 r_1 = \sigma_2 r_2$ $\quad \sigma_2 = \sigma_1 \left(\frac{r_1}{r_2}\right)$

25-64

~~if~~ ~~So the field~~

~~E_1 increase~~

goes up ... goes down

So $E_2 \uparrow$ as $r_2 \downarrow$

~~and E_1~~

and $\sigma_2 \uparrow$ as $r_2 \downarrow$

and vice versa.

So the higher the curvature (smaller r_2), the larger E_2 and σ_2

This is a very special case, but experience shows it to be representative

Relatively high

$\underline{E}$ and σ tend to occur

near regions of ~~sharp~~

high curvature (corners,

edges, prongs)

— just a tendency, but

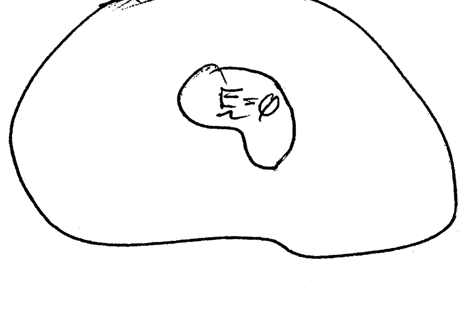
a useful rule of thumb.

Optional

Cavity in Conductors

in ~~Equistatic~~

Electrostatic equilibrium



$\underline{E} = 0$ inside

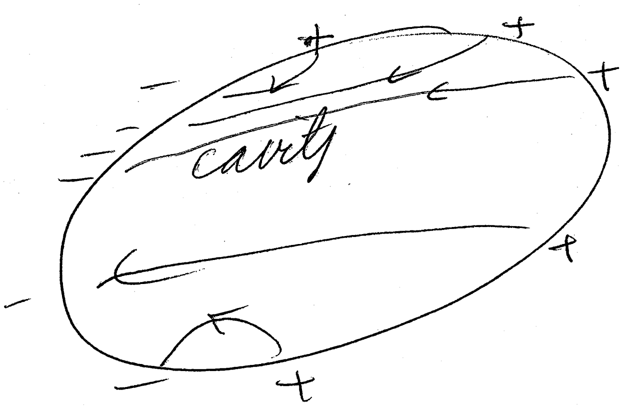
and we proved

25-66

~~on~~ p 24-52 — 56

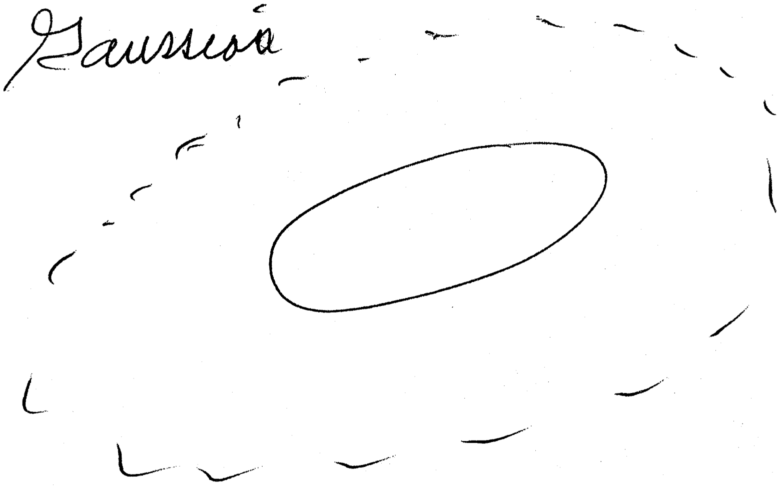
provided ~~the~~ solution for a conductor with a given $V = 0$ on surface (a boundary condition ~~is any~~ with no net charge inside is unique.

ST-709 give I think a faulty proof.



The whole interior of the ~~conductor~~ solid conductor has $\underset{\sim}{E} = 0$ and is an equipotential and so is the interior cavity surface since charge would flow if it weren't

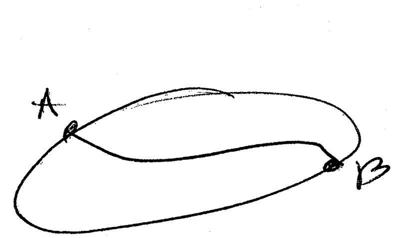
Gaussian surface



$\underline{E} = 0$ on surface

$\therefore q_{encl} = 0$

but there could one imagine be a charge separation on the cavity walls and internal $\underline{E}$-field from arguments so far.



SJ-709

argue

$$\Delta V = -\int_A^B \underline{E} \cdot d\underline{s} = 0$$

25-68)

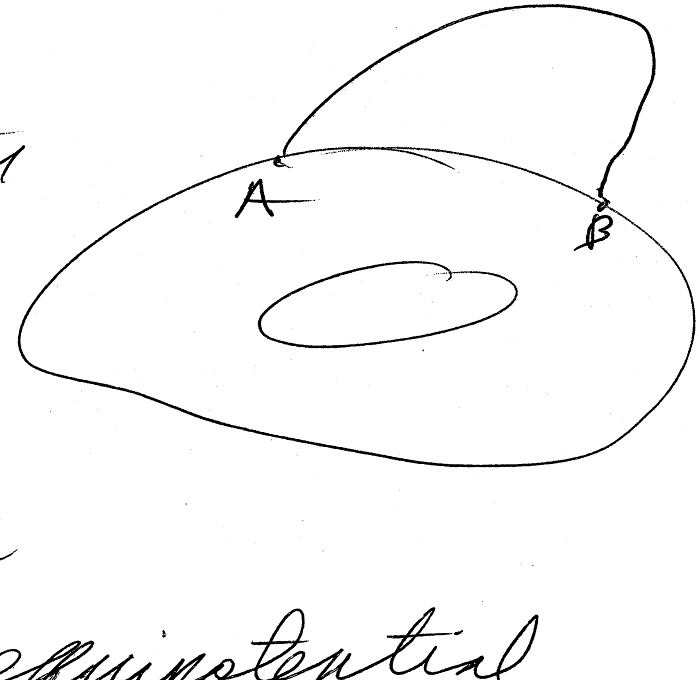
for any path through cavity since the cavity wall must be an ~~equipotent~~ equipotential

true.

$\therefore$ $\underset{\sim}{E} = 0$ inside since the above is true for any path

False

after all the outer surface of conductor



is an equipotential

and $\therefore \Delta V_{AB} = -\int \underset{\sim}{E} \cdot d\underset{\sim}{s} = 0$

for any path,

but $\underline{E} \neq 0$ outside

in general.

So SJ-709's argument

fails.

Our p24-52-56

argument is valid with

uniqueness of solution

(GrEM-126)

Optional proof of the

uniqueness of solution

- won't be gone over or tested

- for those who know some vector calculus, it's pretty easy for others it may not

25-70

be completely clear.

First recall Gauss's law

$$\oint \underset{\sim}{E} \cdot d\underset{\sim}{A} = \frac{q_{encl}}{\varepsilon_0}$$

by Gauss's theorem (Arf-48)

$$= \int_{Volume} \frac{\rho}{\varepsilon_0} \, d\mathcal{V}$$

integral of charge density over interior volume.

$$= \int_{Volume} \nabla \cdot \underset{\sim}{E} \, d\mathcal{V}$$

$\nabla \cdot$ is the divergence operator

$\underset{\sim}{E} = -\nabla V$ recall $\therefore \nabla \cdot E = -\nabla^2 V$ where ∇^2 is the Laplacian operator

$$\therefore -\int_{Volum} \nabla^2 V \, d\mathcal{V} = \int_{Volume} \frac{\rho}{\varepsilon_0} \, d\mathcal{V}$$

$$\therefore \int_{volume} \left(\nabla^2 V + \frac{\rho}{\varepsilon_0} \right) d\mathcal{V} = 0$$

This result must
be true for any volume.

$$\therefore \quad \nabla^2 V + \frac{\rho}{\varepsilon_0} = 0 \quad \left\{ \text{a differential equation.} \right.$$

This is called Poisson's
equation (GrEM-110)

Second Say you had a region
of space with no net
charge anywhere in it.

Then $\nabla^2 V = 0$ (Laplace's
equation
GrEM-111)

You now specify
V_{BC} ~~the potential everywhere~~
on the surface of the
region

Call this V_{BC} $\left\{ \begin{array}{l} \text{BC} \\ \text{for boundary} \\ \text{condition.} \end{array} \right.$

26-72)

The solution for V in the interior is set by V_{BC} and Laplace's equation.

For a conductor in an electrostatic case $V_{surface} = $ Const $\therefore V_1 - V_2 = C$ on surface $\therefore$ only one solution aside from constant inside

<u>Third</u> The solution is unique.

Say we do have two solutions V_1 and V_2

$V_1 = V_2$ on BC — or a bit more general $V_1 - V_2 = $ Const on surface.

~~Then~~ $\cancel{\nabla^2(V_1 - V_2)}$

Then define $V_3 = V_1 - V_2$ which is 0 or C on BC

$$\nabla^2 V_3$$

$$= \nabla^2(V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2$$

$$= 0 - 0 = 0$$

We can do this since the Laplacian is a linear operator.

$\therefore$ V_3 is a solution for V in the region with BC $V_3 = 0$.

<u>Fourth</u> Since $V_3 = 0$ or c ~~a constant~~ on the boundary surface, there must maxima and/or minima of V_3 in the interior

<u>or</u>

$V_3 = 0$ or c everywhere in the interior in which case its a trivial solution and not counted

~~sand~~ $V_1 = V_2$ everywhere or $V_1 = V_2 + C$

and there is only one solution.

Fifth

$$\nabla^2 V_3 = 0$$

throughout interior.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

A stationary point has

$$\frac{\partial V}{\partial x} = 0$$

$$\frac{\partial V}{\partial y} = 0$$

$$\frac{\partial V}{\partial z} = 0$$

– for the stationary point to be a maximum

$$\frac{\partial^2 V}{\partial x^2}, \frac{\partial^2 V}{\partial y^2}, \frac{\partial^2 V}{\partial z^2}$$

must all be < 0

– for a stationary point to be a minimum,

$$\frac{\partial^2 V}{\partial x^2}, \frac{\partial^2 V}{\partial y^2}, \frac{\partial^2 V}{\partial z^2}$$ must

all be > 0 (GrEM-117)

but $\nabla^2 V = 0$

can't hold if all the 2^{nd} derivatives have the same sign.

$\therefore$ a valid solution cannot have maxima or minima

25-76

in the interior.

This is only possible for $V_3 = 0$ or c on boundary.

if $V_3 = 0$ or c everywhere in the region.

So $V_3 = 0$ or c everywhere

$V_1 = V_2$ or $V_1 = V_2 + c$ everywhere.

There is only one solution for specified V_{BC} and $\nabla^2 V = 0$ in interior

The solution is unique.

This completes the proof