MOMENTUM

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ABSTRACT

Lecture notes on what the title says and what the keywords say.

Subject headings: momentum — momentum conservation — impulse-momentum theorem — collisions — inelastic collisions — completely inelastic collision — elastic collisions — mass-varying systems — rocket propulsion

1. INTRODUCTION

Under the heading of nothing forbids us, we will define a new dynamical variable momentum.

Momentum is also called linear momentum, but the qualifier "linear" is really only needed when one wants to emphasize the distinction between momentum and angular momentum. Angular momentum is the rotational-dynamics analog of momentum and is introduced in the lecture *Rotational Dynamics*.

For a classical point particle, the definition of momentum is

$$\vec{p} = m\vec{v} , \tag{1}$$

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where $\vec{p}$ is momentum, m is the particle mass, and $\vec{v}$ is the particle velocity.

Momentum is a **VECTOR DYNAMICAL VARIABLE**.

The MKS momentum unit is

$$\text{unit}[\vec{p}] = \text{unit}[m\vec{v}] = \text{kg m/s} \ . \tag{2}$$

where unit[ ] is my own idiosyncratic unit function. The MKS momentum unit has no special name or symbol. In fact, since the MKS momentum unit is the only common momentum unit in use, one often doesn't bother specifying the unit in speech or simple calculations—it is simply understood. But in **TEST ANSWERS**, specify it.

For a system of classical point particles, one naturally defines the system momentum (or total momentum) to be

$$\vec{p} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i = m \frac{\sum_i m_i \vec{v}_i}{m} = m\vec{v} \ , \tag{3}$$

where the sum is over the individual particle momenta $\vec{p}_i$ of the particles that make up the system, m_i is a particle mass, $\vec{v}_i$ is a particle velocity, m is total mass, and $\vec{v}$ is now the center-of-mass velocity.

Remember a system of particles can be anything—e.g., a single classical point particle, a collection of free classical point particles, a rigid object, a flexible object, a collection of objects flying around and colliding or not colliding, a sample of liquid, a sample of gas—anything.

The microscopic momenta of actual particles are caused by thermal motion. This motion is random on the macroscopic scale and cancels out completely in equation (3). Therefore, one only needs to attribute the macroscopic average momentum of particles to the particles that make up any macroscopically small, but microscopically large subsystem of a system. Those actual particles viewed in this way act like the classical point particles of physics myth.

For macroscopic objects, calculating the total system energy is simple since all the particles of macroscopically small, but microscopically large subsystems are flying in formation. One often makes the continuum approximation and treats macroscopic objects bing made of a continuum of classical substance.

The center-of-mass kinetic energy can be written in terms of system momentum magnitude using $\vec{v} = \vec{p}/m$. Behold:

$$KE = \frac{1}{2}mv^2 = \frac{p^2}{2m} \ . \tag{4}$$

Now consider Newton's 2nd law for a system:

$$\vec{F}_{\rm net} = m\vec{a} \ , \tag{5}$$

where $\vec{F}_{\rm net}$ is the net force on the system (and also the net external force), m is the system mass, and $\vec{a}$ is the center-of-mass acceleration. The 2nd law applies to a classical point particle, of course, since a classical point particle is a system.

Let's assume m is constant. Then

$$\vec{F}_{\rm net} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v}\,)}{dt} = \frac{d\vec{p}}{dt} \ . \tag{6}$$

Thus, we have

$$\vec{F}_{\rm net} = \frac{d\vec{p}}{dt} \tag{7}$$

which is a new version of Newton's 2nd law. This is the form that Newton originally gave—but only sort of since he wasn't using modern notation—and he used the Latin word for "motion" instead of momentum (Wikipedia: Newton's laws of motion).

Equation (7) is actually more general than the $\vec{F}_{\rm net} = m\vec{a}$ form that we have been using since it applies when the mass of the system is varying. We havn't allowed for mass variation in our derivation of equation (7), but we will show how that is done in § 4 which covers mass-varying systems.

For the moment, we only consider systems with constant mass. Even for constant-mass systems equation (7) is of interest because it brings out a new feature of Newtonian dynamics: i.e., what $\vec{F}_{\rm net} = 0$ implies for momentum.

It implies

$$\vec{p} = \text{a constant} \ , \tag{8}$$

and since $\vec{p} = m\vec{v}$ (where $\vec{v}$ is center-of-mass velocity recall), we also find that

$$\vec{v} = \text{a constant} \tag{9}$$

which in turn implies that the center-of-mass kinetic energy

$$KE = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \text{a constant} \ . \tag{10}$$

In words, if the net force on a system (i.e., a constant-mass system) is zero, then momentum is conserved—or in other words, we have **CONSERVATION OF MOMENTUM**. Also conserved are center-of-mass velocity and the center-of-mass kinetic energy.

One often says one has **CONSERVATION OF MOMENTUM** if the net external force is zero. But having net external force is only a necessary condition, not a sufficient condition: the net internal force must be zero too. But if one understands that Newton's 3rd law implies the net internal force is zero, then saying one has conservation of momentum when the net external force is zero is OK—and is probably clearer, since saying the net force is zero tends to leave the impression that there are no internal forces acting—and in general there are and those can be interesting cases.

By the way, having the net external force zero does **NOT** mean that there are no external forces. It just means that they sum to zero. But the fact that they sum to zero, does not mean they do nothing. They may be affecting the internal motions of the system. In simple examples of conservation of momentum, we usually assume there are no external forces at all.

By the way also, recall that the 3rd law is not always valid even in classical physics (e.g., Goldstein et al. 2002, p. 7–8). We discussed this point in the lecture *Newtonian Physics I*. The cases where the 3rd law fails are beyond the scope of this course. The situation isn't so scandalous however. Generalizations of the 3rd law that account for the exceptions to the ordinary 3rd law do exist.

CONSERVATION OF MOMENTUM is useful in solving various kinds of problems. At our level, ideal collision problems are a main example and we will consider those in § 3.

Let's consider just a special case of conservation of momentum.

Say we had two objects: object 1 with mass m_1 and object 2 with mass m_2. Now say these two objects interact and the net external force is zero during the interaction. The objects collectively and individually don't gain for lose mass. By conservation of momentum, one has

$$\vec{p} = \vec{p}\,' \qquad \text{or} \qquad \vec{p}_1 + \vec{p}_2 = \vec{p}_{1'} + \vec{p}_{2'} \ , \tag{11}$$

where the unprimed subscripts indicate pre-interaction and the primed ones indicate post-interaction. We can rewrite the conservation-of-momentum equation in the form

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_{1'} + m_2\vec{v}_{2'} \ , \tag{12}$$

where the velocities are center-of-mass velocities for the two objects.

This **CONSERVATION OF MOMENTUM** equation (12) by itself allows us to solve for one unknown only. Any one of the variables in equation (12) can be the unknown variable. Often, however, one would like to predict the outcome velocities $\vec{v}\,'_1$ and $\vec{v}\,'_2$: i.e., one would like to predict the future evolution of the system. But we **CANNOT** predict the entire outcome of the interaction nor the course of the interaction from the initial conditions (i.e., $\vec{v}_1$ and $\vec{v}_2$) given equation (12) alone. We'd have to have a detailed understanding of the interaction to solve for the whole outcome and the whole course of the interaction.

But given the initial conditions and one of the post-interaction velocities, we could solve for the other one without knowing anything about the detailed interaction. Like conservation-of-mechanical-energy cases, conservation-of-momentum cases often allows us to find partial information or make partial predictions of the future very easily without a full understanding of the system.

Let's do a simple example calculation.

1.1. Example: Archer and Arrow

A tall archer is standing on frictionless ice and shoots an arrow in the horizontal direction. Initially everything is at **REST**. The archer mass is 50 kg and the arrow mass is 0.50 kg.

What are the external forces on the archer-arrow system and can we neglect them?

There is gravity and air drag (AKA air resistance) on both the archer and arrow.

We can neglect them. The gravity on the archer is canceled by the normal force of the ice on her. We assume that the gravity on the arrow and air drag on archer and arrow can be neglected since the time of the time of interaction—which is the arrow shooting event—is very short. These forces cause only a small change in individual momenta and the total system momentum. In the ideal limit of a vanishingly short interaction, the net force on the archer-arrow system is exactly zero, and thus momentum is exactly conserved. We assume this ideal limit which is not a bad approximation actually.

So what are the archer and arrow post-interaction velocities? Initially, they are both zero recall.

Well we can't tell with what we know so far which are the masses and that the initial velocities are zero.

We need one post-interaction velocity to find the other.

Let the arrow post-interaction velocity be $50\hat{x}\,\mathrm{m/s}$.

What the archer's post-interaction velocity?

You have 30 seconds. Go.

Well now using equation (12) with initial total momentum zero and letting the arrow be object 1 and the archer object 2, we find

$$\begin{aligned} 0 &= m_1\vec{v}_{1'} + m_2\vec{v}_{2'} \\ \vec{v}_{2'} &= -\frac{m_1}{m_2}\vec{v}_{1'} \\ &= -\frac{0.5}{50} \times 50\hat{x} \\ &= -0.5\hat{x}\,\mathrm{m/s}\ . \end{aligned} \tag{13}$$

So the archer moves opposite to the arrow at a much slower speed.

The result isn't too surprising.

The last equation also shows that the momenta of the two objects are equal in magnitude and opposite in direction.

2. IMPULSE-MOMENTUM THEOREM

We define the impulse of a force $\vec{F}$ acting for a time Δt by

$$\vec{I} = \int_{\Delta t} \vec{F}\, dt \ , \tag{14}$$

where the integral is over time Δt. The integral, loosely speaking, is an infinite sum of the differentials $\vec{F}\, dt$.

The concept of impulse is most useful when the force $\vec{F}$ is very strong compared to other forces present and acts for a much shorter time and this time of activity is usually Δt itself.

Now recall our momentum version of Newton's 2nd law equation (7) $\vec{F}_{\rm net} = d\vec{p}/dt$, where we still assuming constant mass for the system. We integrate the 2nd law over time Δt to get

$$\Delta\vec{p} = \int_{\Delta t} \vec{F}_{\rm net}\, dt \tag{15}$$

or

$$\Delta\vec{p} = \vec{I}_{\rm net} \ , \tag{16}$$

where $\vec{I}_{\rm net}$ is the net impulse.

Equation (16) $\Delta\vec{p} = \vec{I}_{\rm net}$ is the impulse-momentum theorem (e.g., Tipler &Mosca 2008, p. 256).

The impulse-momentum theorem is yet another version of Newton's 2nd law.

Why do we need yet another version of the 2nd law?

Well it's more useful in some special cases than the other versions.

The cases of greatest use is when there is a very strong force $\vec{F}$ that acts over Δt and that dominates the evolution of the system during Δt. Other forces can be considered negligible during Δt although they can be important over longer time scales.

Such cases are often collision events.

The collision approximation is to assume that only the collision forces act over the time of the collision Δt. This is an idealization in general although it can be exactly true in some special cases.

With the collision approximation, the impulse-momentum theorem becomes

$$\Delta\vec{p} = \vec{I}_{\rm col} \ , \tag{17}$$

where $\vec{I}_{\rm col}$ is the collision force impulse.

Let's to an example.

2.1. Example: Hit Baseball

A baseball of mass 0.14 kg moving is hit by a baseball bat. The motion of both objects over the collision time is assumed to be horizontal. So the system is 1-dimensional.

The initial ball velocity is $v = -40\,{\rm m/s}$ and its final velocity is $v' = 56\,{\rm m/s}$. The time of the collision is $\Delta t = 1.5\times 10^{-3}\,{\rm s}$.

By the by, 40 m/s in US customary units is 89.5 mi/h which is a poorish major-league fast ball speed. Nolan Ryan could throw at over 100 mi/h—and sometimes even over the plate.

What it the time-averaged force of the bat on the baseball?

From the collision-approximation impulse-momentum theorem specialized to one dimension, we find

$$\Delta p = I_{\rm col} = \int_{\Delta t} F_{\rm col}\,dt = F_{\rm col,avg}\Delta t \ , \tag{18}$$

where $F_{\rm col,avg}$ is the time-averaged collision force.

Thus

$$F_{\rm col,avg} = \frac{\Delta p}{\Delta t} = \frac{m(v' - v)}{\Delta t} = \frac{0.14 \times 96}{1.5 \times 10^{-3}} \approx 10^4\,{\rm N}\ . \tag{19}$$

A more exact calculation gives 9000 N which is about 2000 lb of force. So if these numbers are at all right—and I only adapted them from Cutnell & Johnson (2007, p. 199) and have no idea if they are at all realistic—then being hit by a swung bat can be dangerous especially if you are speeding into it—but you-all already knew that.

Was our assumption of the collision approximation justified?

Well we neglected gravity and air drag on the baseball.

The magnitude of the gravity force on the baseball is only about 1.4 N.

The terminal speed of baseball acted on by gravity and air resistance alone is about 42 m/s (e.g., Halliday et al. 2001, p. 105). At the terminal speed, the air drag force is equal to the gravitational force. Since the air drag force depends on speed and 42 m/s is of order of our baseball speeds, the air drag force magnitude is of order the same size as the gravitational force magnitude.

Now since $1.4\,{\rm N} << 9000\,{\rm N}$, we conclude that gravity and air drag were negligible during the collision. Our assumption of the collision approximation is well justified.

Of course, over the long-term evolution of the ball from pitcher to final location, gravity and air drag are important.

3. COLLISIONS IN ONE DIMENSION

We will only consider collisions in one dimension.

This saves us the tedium of full vector formalism.

We don't need—or at least don't want—more practice with full vector formalism.

Somehow some constraint keeps the colliding object centers of mass on one axis.

We will also make the collision approximation: i.e.,. we assume that only the collision forces act over the time of the collision Δt.

This means that we have conservation of momentum for the overall system of colliding objects since the net external force is implied zero by assuming the collision approximation.

We will also only consider two-object collisions. Specializing to one dimension, equation (12) (see § 1) for the conservation of momentum of two objects in an interaction becomes

$$m_1 v_1 + m_2 v_2 = m_1 v_{1'} + m_2 v_{2'} \ . \tag{20}$$

where the particles are labeled by 1 and 2, the unprimed velocities are pre-interaction, and the primed velocities are post-interaction. The velocities are center-of-mass velocities for the two objects.

There are 6 dynamical variables only to consider since we consider only the pre-collision and post-collision situation and not the interaction event itself. These variables are the ones appearing in equation (20), of course.

For simple analysis, it's usual to divide 1-dimensional collisions for two objects into three types: inelastic collisions, completely inelastic collisions, and elastic collisions.

We'll look at each type in turn.

But we only analyze these types using conservation principles: conservation of momentum and a special conservation of kinetic energy rule. There is a distinction between the two conservation rules. We impose conservation of momentum of the overall system in all our collision types. This means that the center-of-mass kinetic energy of the overall system is conserved in all ideal collision cases. This result follows from our discussion in § 1 where we showed that if system momentum is conserved, so is system center-of-mass velocity and center-of-mass kinetic energy. But this conservation of center-of-mass kinetic energy of the overall system is **NOT** the special conservation of kinetic energy rule.

The special conservation of kinetic energy rule defines elastic collisions only. This special rule is that the sum of colliding object center-of-mass kinetic energies is conserved in the sense of being the same before and after the collision event: there is usually no conservation of the sum of colliding object center-of-mass kinetic energies during the event. The sum of colliding object center-of-mass kinetic energies is **NOT** the same as the overall system center-of-mass kinetic energy.

We will not go into details of collision forces which are complex in general. By using conservation rules, we avoid needing to know about those details. But as mentioned in *NEWTONIAN DYNAMICS II*, conservation rules only give partial information. So we get only partial information, not complete information. But we don't have to work hard or know very much to get that partial information.

3.1. Inelastic Collisions

In inelastic collisions, the sum of the center-of-mass kinetic energies of the colliding objects is **NOT** conserved and the objects do **NOT** stick together.

Actually, one can offer a different refined definition of inelastic collisions. The refinement is a replacement for saying the objects do **NOT** stick together. Instead one says that no matter what interactions (including sticking together) go on between the objects post-collision, one continues to treat them as separate objects with their own centers of mass. The refined definition may be practically useful in some cases and conceptually useful in that it makes it clear that one can consider the separate behavior of objects that are still interacting post-collision.

But for intro physics, the first definition of elastic collisions is the usual one since we primarily deal just with inelastic collision examples where the objects don't interact at all post-collision. So we will only use the first definition in our discussion and examples, unless otherwise stated.

Now in completely inelastic collisions, the sum of the center-of-mass kinetic energies of the colliding objects is **NOT** conserved and the objects **DO** stick together.

Actually, one can also offer a different refined definition of inelastic collisions. The refinement is a replacement for saying the objects **DO** stick together. Instead one says that no matter what interactions (including sticking together) go on between the objects post-collision, one treats them as a combined system with with a single combined system center of mass. The refined definition may be practically useful in some cases and conceptually useful in that it makes it clear that one can consider the combined behavior of objects that post-collision are not stuck together and maybe not interacting at all.

But for intro physics, the first definition is usual one since we primarily deal just with completely inelastic collision examples where the objects are definitely stuck together post-collision. So we will only use the first definition in our discussion and examples, unless otherwise stated.

From the unrefined-definition point of view, the two kinds of collision are physically distinct. From the refined-definition point of view, the two kinds of collisions are different distinct in how one analyzes them.

From either point of view, neither of elastic and completely inelastic collisions are special cases of the other according to our definitions.

The forces of collision for both inelastic and completely inelastic collisions for real solid objects in many cases are the elastic forces of the objects. But internal resistances dissipate some kinetic energy and elastic potential energy to waste heat in those cases. This makes such collisions inelastic. Collisions involving only field forces like the macroscopic Coulomb force and gravity may not have much dissipation and can be better approximations to elastic collisions: i.e., collisions where the sum of the center-of-mass kinetic energies of the colliding objects is conserved. There are many kinds of collisions to consider if one is being very general—which we are not.

Inelastic collisions are considered in this section and completely inelastic collisions in § 3.2.

For the inelastic collisions, we have the general constraint of the conservation of momentum expressed by equation

$$m_1 v_1 + m_2 v_2 = m_1 v_{1'} + m_2 v_{2'} \ , \tag{21}$$

where the unprimed indices indicate pre-collision and the primed indices, post-collision. Note that there are 6 variables.

Since the collisions are inelastic, we have no constraint on the sum of center-of-mass kinetic energies, except that

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \neq \frac{1}{2} m_1 v_{1'}^2 + \frac{1}{2} m_2 v_{2'}^2 \ . \tag{22}$$

A simple inequality is seldom of much use in solving for unknowns.

Without any detailed information about the collision forces, we must initially know 5 of the 6 variables in order to know everything about the pre- and post-collision situation. The conservation of moment gives the 6th variable.

In particular, we can't predict the whole outcome (i.e., $v_{1'}$ and $v_{2'}$) given the initial conditions. Someone has to give us one of the outcome velocities.

3.1.1. Example: A Mysterious Inelastic Collision

Just as an example, say object 1 has mass $m_1 = 1.0\,\mathrm{kg}$ and velocity $v_1 = 2.0\,\mathrm{m/s}$ and object 2 has mass $m_2 = 2.0\,\mathrm{kg}$ and velocity $v_2 = 0.0\,\mathrm{m/s}$. Someone tells us that $v_{2'} = 6.0\,\mathrm{m/s}$.

What is $v_{1'}$?

You have 1 minute working individually or in groups. Go.

Solving equation (21) for $v_{1'}$ gives

$$v_{1'} = \frac{m_1 v_1 + m_2(v_2 - v_{2'})}{m_1} = \frac{1 \times 2 + 2 \times (0 - 6)}{1} = -10\,\mathrm{m/s}\ . \tag{23}$$

Object 1 recoils from the collision.

Is there something odd about this result?

Look at the sizes of the post-collision velocities (i.e., $v_{1'} = -10\,\mathrm{m/s}$ and $v_{2'} = 6.0\,\mathrm{m/s}$) compared to the pre-collision velocities (i.e., $v_{1'} = 2\,\mathrm{m/s}$ and $v_{2'} = 0.0\,\mathrm{m/s}$).

What would you say about the sum of object individual kinetic energies?

Well it looks like the sum of the object kinetic energies increased during the collision. It has:

$$KE = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = 2.0\,\mathrm{J} \tag{24}$$

$$KE' = \frac{1}{2}m_1 v_{1'}^2 + \frac{1}{2}m_2 v_{2'}^2 = 50 + 36 = 86\,\mathrm{J}\ . \tag{25}$$

Question: Can the sum of the object kinetic energies really increase during an inelastic collision?

a) Yes.

b) No.

c) Maybe.

Yes, it's (a).

If there is some kind of explosion, the sum of the colliding objects kinetic energies can increase. The explosion can be a transformation of chemical energy or stored elastic potential energy into kinetic energy.

In retrospect, our archer-and-arrow example (§ 1.1) can be recognized as an inelastic collision with an increase in the sum of colliding object kinetic energies. Chemical energy in the archer became elastic potential energy in the bow which in turn became the kinetic energies of the archer and arrow.

But note the overall center-of-mass kinetic energy cannot change as we discussed in § 3. In the archer-and-arrow example (§ 1.1), for example, the overall system center-of-mass kinetic energy was zero before and after the arrow was shot.

3.2. Completely Inelastic Collisions

As stated in § 3.1 in completely inelastic collisions, the sum of the center-of-mass kinetic energies of the colliding objects is **NOT** conserved and the objects **DO** stick together.

Also as stated in § 3.1, there is a refined definition of completely inelastic collisions. We won't repeat the discussion of the refined definition here.

And also as stated § 3.1, the first definition of completely inelastic collision is the usual one for intro physics and we use that one in our discussion and examples, unless otherwise stated.

For completely inelastic collisions, the overall system center-of-mass velocity v is the post-collision velocity of relevance since we are considering the two objects as having combined into one object. The overall system center-of-mass velocity is given by

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v \ , \tag{26}$$

of course.

For completely inelastic collisions, the whole outcome in a sense can be solved for given the initial conditions because of the extra constraint that we only make use of overall system center-of-mass velocity as the relevant final center-of-mass velocity.

An interesting fact true in all dimensions and for any number of colliding objects that collide completely inelastically (i.e., the end system is considered one object) is that the sum of individual object center-of-mass kinetic energies from before the collisions is always greater than or equal to the center-of-mass kinetic energy of the post-collision combined object. The equality only holds if all the individual object center-of-mass velocities were equal before the collision process started. We assume the system of objects is closed in the total collision event which means no particles enter or leave and there is no net external force.

The proof of the above statement is actually simple, and so we will leave it as an exercise for the students—probably it's in a homework. In fact, the proof is very similar, but not identical, to the proof given in the lecture *ENERGY* for the sum of the center-of-mass kinetic energies of any set of subsystems of systems always being greater than or equal to the system center-of-mass kinetic energy. The only distinction in the proofs is that in the latter, the system is considered at one instant in time and in the former, the subsystem center-of-mass kinetic energies are evaluated before the collision and the system center-of-mass kinetic energy is a constant through the collision event.

Where does it go, the lost kinetic energy from colliding objects in a general completely inelastic collision?

Total energy is always conserved for a closed system, and so the lost kinetic energy must go somewhere. But where actually depends on the details of the system and the collision process.

First, it doesn't necessarily have to go anywhere. It could still be kinetic energy of the parts of the combined object. This would be the case if all the individual events in the collision process conserved kinetic energy and the final combined object is just a combined object because we so regard it and the original objects are actually all present and non-interacting after the process.

But in cases were the original object was physically combined into a bound combined objects, there must be losses of translational kinetic energy: i.e., energy of straight-line motion.

One obvious sink for the lost energy is into waste heat. When objects collide, there is some compression and possibly oscillation. Internal resistance in the objects can turn all the energy of compression and oscillation into waste heat.

If the some of the oscillating subsystems are ideal, some of the lost energy can end up as the conserved combination of macroscopic mechanical energy of oscillation (i.e., a combination of oscillation kinetic energy and potential energy).

Some of the lost energy could end up as rotational kinetic energy. This case is very often realized when the colliding objects actually stick together.

Some energy could have gone into emitted sound energy. Of course, we actually assumed there was no medium when we assumed no external forces. But in less ideal completely inelastic collisions, sound energy is possible sink for the lost energy. Of course, sound energy gets turned into waste heat pretty quickly usually.

Electromagnetic radiation could carry away some of the lost energy. This electromagnetic radiation's source would be the waste heat produced in the collision or perhaps some non-thermal process in collision.

Let's turn to a non-trivial completely inelastic collision process.

3.2.1. Example: Ballistic Pendulum

A ballistic pendulum is a device for measuring bullet speeds. It was the first practical way to measure bullet speeds (Wikipedia: Ballistic Pendulum). Nowadays, fast photography has probably largely replaced it for that job.

The setup is a follows. A bullet is fired at a block of wood typically suspended by two cords. The block is rectangular and level. The cords are perpendicular to the bullet path. The bullet embeds in the block and the block swings up to some maximum height that is measured.

The bullet-block collision is a completely inelastic collision that happens on a short time scale where the collision approximation is assumed valid. The upswing of the block is a process in which ideally mechanical energy is conserved. Initially, the bullet-block mechanical energy is all kinetic taking the initial level as zero potential energy and when the bullet-block reaches its maximum height all the energy is potential energy. The height measurement gives the potential energy.

Say the bullet's mass is m_{bullet} and the block's mass is m_{block} and their sum is m.

What's the post-collision bullet-block velocity v in terms of the bullet's pre-collision velocity v_{bullet}?

It's a completely inelastic collision remember.

You have 30 seconds. Go.

To find the bullet's pre-collision velocity $v_{\rm bullet}$, we note that the post-collision bullet-block velocity v

Well

$$v = \frac{m_{\rm bullet}}{m} v_{\rm bullet} \ , \tag{27}$$

where we have used equation (26) for completely inelastic collisions and where the block is, of course, taken as being initially at rest.

We can now find bullet's pre-collision velocity $v_{\rm bullet}$ in terms of post-collision bullet-block velocity v.

Do that. You have 10 seconds. Go.

Behold:

$$v_{\rm bullet} = \frac{m}{m_{\rm bullet}} v \ . \tag{28}$$

From the work-energy theorem

$$\Delta E = W_{\rm non} \ , \tag{29}$$

we find that

$$mgy_{\rm max} = KE_{\rm post-collision} = \frac{1}{2} m v^2 \ . \tag{30}$$

Solving for the bullet velocity gives

$$v_{\rm bullet} = \frac{m}{m_{\rm bullet}} \sqrt{2gy_{\rm max}} \ . \tag{31}$$

We see that using the completely inelastic collision sort of amplifies the velocity measurable by conservation of mechanical energy by a factor of

$$\frac{m}{m_{\rm bullet}} = \frac{m_{\rm bullet} + m_{\rm block}}{m_{\rm bullet}} \ . \tag{32}$$

Say $y_{\rm max} = 0.5\,{\rm m}$, $m_{\rm bullet} = 0.05\,{\rm kg}$, and $m_{\rm block} = 5.0\,{\rm kg}$.

What it the bullet speed?

You have 30 seconds. Go.

Behold:

$$v_{\rm bullet} = \frac{m}{m_{\rm bullet}}\sqrt{2gy_{\rm max}} \approx \left(\frac{0.05+5.0}{0.05}\right) \times \sqrt{2\times 10\times 0.5} \approx 300\,{\rm m/s}\ . \tag{33}$$

This is sort of a moderate bullet speed for a rifle. It's actually subsonic since the speed of sound in air at 1 atm and 20°C is 343 m/s (e.g., Halliday et al. 2001, p. 400).

3.3. Elastic Collisions

In elastic collisions, the sum of colliding object kinetic energies is conserved and the objects **DO NOT** stick together.

They cannot stick together since then the sum of the individual object center-of-mass kinetic energies is not conserved as we asserted in § 3.2. The center-of-mass kinetic energy of the combined object is always less than or equal to the sum of center of mass kinetic energies of the original objects.

There is an exception, of course. They can stick together and conserve the sum of colliding object kinetic energies if the initial colliding object velocities were equal. But then for an actual combining collision as in our first definition of a completely inelastic collision, the colliding objects had to be in contact before the "collision" and the "collision" caused no exchange of momenta, but only fused the objects together. For our second definition, we merely think of the objects as one combined object after the collision event whether they ever contacted each other or not, or ever fused together or not.

Elastic collisions can be regarded as a special case of inelastic collisions, of course. They are inelastic collisions which happen to conserve the sum of colliding object kinetic energies.

But elastic collisions are special enough to be regarded as a separate category for our level of analysis.

Hereafter in this section, we will just say conservation of kinetic energy instead of conservation of colliding object kinetic energies for the sake of brevity.

We have two equations of constraint in this case: the conservation of momentum equation

$$m_1v_1 + m_2v_2 = m_1v_{1'} + m_2v_{2'} \tag{34}$$

and the conservation of kinetic energy equation

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_{1'}^2 + \frac{1}{2}m_2v_{2'}^2 \ . \tag{35}$$

A qualification is needed about conservation of kinetic energy. We only require and only have that it is the same before and after the interaction. During the interaction, some or all of the kinetic energy is transformed into some other form of energy which in many actual cases is elastic potential energy of the colliding objects. But all this energy is transformed back into kinetic energy by the end of the interaction.

For elastic collisions of two objects in one dimension, we can solve in general for the full outcome (i.e., $v_{1'}$ and $v_{2'}$) since we have two equations for the two unknowns (i.e., $v_{1'}$ and $v_{2'}$).

But solving for the outcome formulae is tricky since one of the equations (eq. (35)) is non-linear in the unknowns.

But if the solution is tricky, there are tricks.

The main trick—after which everything is straightforward—is to rewrite equations (34) and (35) so that all of the object 1 variables appear on one side of the equal sign in each case and all the object 2 variables appear on the other side in each case.

The rewriting preserves some of the symmetry of equations (34) and (35). Preserving symmetries in solving systems of equations is often a good thing since it can lead to semi-obvious simplifications—which is the case here.

The rewritten equations are

$$m_1(v_1 - v_{1'}) = -m_2(v_2 - v_{2'}) \tag{36}$$

and

$$m_1(v_1^2 - v_{1'}^2) = -m_2(v_2^2 - v_{2'}^2) \ . \tag{37}$$

Now recall the old difference of squares result: for any a and b,

$$(a+b)(a-b) = a^2 - ab + ba - b^2 = a^2 - b^2 \ . \tag{38}$$

Now dividing equation (37) by equation (36) and using the difference of squares result, we obtain

$$v_1 + v_{1'} = v_2 + v_{2'} \ . \tag{39}$$

Note by the way that we assume that $v_{1'} \neq v_1$ and $v_{2'} \neq v_2$ or else we'd have been dividing by zero which leads to an undefined result.

To sidetrack for a moment, equation (39) can be rearranged to get

$$v_{2'} - v_{1'} = -(v_2 - v_1) \tag{40}$$

which shows that the relative velocity of the objects has the same magnitude before and after the collisions, but velocity changes in sign. This is an important result in its own right:

$$v_{2'\ \text{relative to}\ 1'} = -v_{2\ \text{relative to}\ 1} \ . \tag{41}$$

To maintrack again, multiply equation (39) by m_2 and subtract equation (36) from it, and one obtains

$$m_2(v_1 + v_{1'}) - m_1(v_1 - v_{1'}) \quad = \quad m_2(v_2 + v_{2'}) + m_2(v_2 - v_{2'})$$

$$(m_2 - m_1)v_1 + (m_1 + m_2)v_{1'} \;\;=\;\; 2m_2 v_2 \; . \tag{42}$$

Solving for $v_{1'}$ gives

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2 v_2}{m_1 + m_2} \tag{43}$$

(e.g., Halliday et al. 2001, p. 207).

Absolutely, positively, equations (34) and (35) only apply to 1-dimensional elastic collisions. Student have an awful temptation on tests to use them for any collision. Do **NOT** do that.

Now the conservation of momentum and kinetic energy equations (i.e., eq. (34) and (35)) are symmetric in the indices 1 and 2. So if one does the same operations starting from those equations, but interchanging the roles of objects 1 and 2 in the operations, one must arrive at the equation

$$v_{2'} = \frac{(m_2 - m_1)v_2 + 2m_1 v_1}{m_1 + m_2} \; . \tag{44}$$

(e.g., Halliday et al. 2001, p. 207).

Just to elaborate on our solution for $v_{2'}$ by symmetry. Say we interchanged the labels for objects at the start and then did all the steps as before. We would have gotten equation (43) where the new 1 meant the old 2 and the new 2 meant the old 1. So we'd really have the solution for the outcome of the object originally labeled by 2. Now we just interchange the labels again. The old 2 is now the newest 2, and our solution for its outcome velocity $v_{2'}$ is just equation (44) as we've aforesaid.

Equations (43) and and (44) give us the complete outcome of the elastic collision in terms of the initial conditions. To summarize compactly, we have

$$v_{1'} \;\;=\;\; \frac{(m_1 - m_2)v_1 + 2m_2 v_2}{m_1 + m_2} \; , \tag{45}$$

$$v_{2'} \;\;=\;\; \frac{(m_2 - m_1)v_2 + 2m_1 v_1}{m_1 + m_2} \tag{46}$$

(e.g., Halliday et al. 2001, p. 207).

Now there is in fact another set of solutions for the outcome velocities that we missed in the derivation of equations (45) and and (46) since we couldn't allow undefined divisions by zero.

What are those other set of solutions?

You have 10 seconds to write them down. Go.

Well the other set of solutions—which yours truly calls the ghost solutions—are

$$v_{1'} = v_1 , \tag{47}$$

$$v_{2'} = v_2 . \tag{48}$$

Of course, these are the solutions if the objects do not collide—they are going in opposite directions or are going in the same direction, but havn't interacted yet—but that is a trivial case.

Can they ever be the solutions of an actual collision in any sense?

Yes. The objects collide, but there are no collision forces. Say a ball and ring (with inner radius larger than the ball's) collide in the sense that their centers of mass pass through each other. No forces act, but there is a collision—a ghost collision.

Rather than do boring numerical examples, let's look at a few special cases of the solutions equations (45) and (46).

We won't do all special cases. That would cause mucho ennui.

3.4. Special Case: $m_1 = m_2$

Say $m_1 = m_2$.

What do equations (45) and (46) specialize to?

You have 10 seconds. Go.

Behod—er, behold:

$$v_{1'} = v_2 \qquad \text{and} \qquad v_{2'} = v_1 \ . \tag{49}$$

We see that in this case that the objects just interchange velocities.

To specialize just a bit more, say $v_1 \neq 0$ and $v_2 = 0$. Then we have

$$v_{1'} = 0 \qquad \text{and} \qquad v_{2'} = v_1 \ . \tag{50}$$

So object 1 just stops and object 2 carries on filling object 1's role.

Here we can demonstrate this with using two balls of a Newton's cradle (Wikipedia: Newton's Cradle) which yours truly tends to call a ball pendulum. I'm assuming a Newton's cradle is around somewhere in the demo room.

Usually Newton's cradle consists of 5 identical metal balls that are hung as pendulums each from by two strings which are perpendicular to the pendulum swing direction. The two strings force a pendulum ball to swing only in one plane (the cradle plane) and make all the collisions of the pendulum balls 1-dimensional collisions along the axis of the cradle. More pedantically, one can say that the two strings make the balls equilibrium in the cradle plane a stable equilibrium. When the cradle is all at rest all the balls are just forcelessly touching since their pivot points are offset along the cradle axis in order to arrange this setup.

In the two ball mode of operation, the two balls collide in an approximately elastic collision after one ball is released from an upswing position. One doesn't usually think of metal balls as being very "elastic", but they are in the sense of nearly conserving kinetic energy through a collision. Because second ball starts at rest, it acquires all the momentum of the first ball which is stopped.

The second ball upswings and then down swings and collides with the first ball which then acquires all the momentum and upswings again.

The upswings and down swings of the balls conserve mechanical energy approximately, but not momentum. The combined gravity and tension force can change momentum. (Note a centripetal force can change momentum even when it does no net work by being perpendicular to the direction of motion.) Since gravity is a conservative force and the tension force ideally does no work, mechanical energy is ideally conserved.

The motion one gets is a swinging oscillation with collisions at the bottom of the swings.

Ideally no energy is lost and the oscillation is perpetual. Actually energy is lost to air drag, friction in the pivot points, and the slightly inelastic nature of the collisions. This energy is dissipated to waste heat although some of it passes through a sound energy stage first.

If you use all the balls starting with one in upswing position, you get an oscillation that passes through 4 collisions on each half cycle.

More complicated cases. For example having two balls on one side in the upswing position initially or having two balls on either side in the upswing position initially. The behavior of these cases is easier to demonstrate than to describe, so yours truly will demonstrate. But the cases can be analyzed easily too and be understood using the principles of conservation of momentum, kinetic energy, and mechanical energy.

But we won't go on to that.

The students are encouraged to go into it for fun.

3.5. Special Case: $m_2 = \infty$ and $v_2 = 0$

If $m_2 = \infty$ and $v_2 = 0$, equations (45) and (46) specialize to

$$v_{1'} = -v_1 \qquad \text{and} \qquad v_{2'} = 0 \ . \tag{51}$$

Here we have the stoppable object collided with the immovable object. The stoppable object just bounces off and conserves speed. The immovable object is unmoved.

But what has happened to conservation of momentum. The total momentum is $m_1 v_1$ before the collision and $-m_1 v_1$ after.

Well sending object 2's mass to infinity was unrealistic. Actually it can only get to be very much more massive than object 1. So what really happens is that object 1's speed is slightly reduced and object 2 does gain some velocity in the initial direction of object 1. Momentum does actually end up being conserved.

But the way we've taken the limit of infinite mass causes the conservation of momentum to break down in our idealized result. The conservation of energy (i.e., conservation of the sum of colliding object kinetic energies) is maintained.

An infinite mass object becomes a source and sink for momentum in our formalism.

Actually, collisions in multi-dimensions with effectively infinite mass objects happen all the time in life and nature. You stub your toe on a couch, walk into a wall, etc. What about elastic collisions with effectively infinite mass objects. Well they happen a lot in space. Small objects (e.g., asteroids, space probes) interact with large objects (e.g., planets) elastically through the gravitational force. There is no big mechanism to dissipate mechanical energy to other forms in these cases along as the interaction is just gravitational. Small dissipations to heat occur with tidal effects. Such elastic collisions are often used to accelerate space probes and are called gravity assists or slingshot maneuvers (Wikipedia: Gravitational assist). A probe is "bounced off" a planet and picks up extra kinetic energy. The bouncing process is really a partial orbit around the planet.

4. MASS-VARYING SYSTEMS

The mass-varying systems arn't so hard to understand in essence—but it is rocket science.

Say that you have a system of particles: the system could be anything—e.g., an ideal classical point particle, a realistic classical particle (i.e., a very tiny object in some sense), a collection of free particles, a rigid object, a flexible object, a bunch of objects flying around and colliding or not colliding, a sample of liquid, a sample of gas—anything.

The system has mass m, but this m is **NOT** constant with time.

But we will assume that one can make the continuum approximation and add/subtract mass from the system in a continuous way. This let's us use calculus and avoid host of difficulties beyond our scope and ambition.

Let's say the particles of the system are acted on by an external force that includes all forces, except it doesn't include what below we call **MOMENTUM FLUX FORCES**. This external force can be made up of, e.g., gravity, the electric force, and normal forces.

Let's label this force $\vec{F}_{\text{other}}$ in this derivation.

But also there is mass being added to the system.

We will cop-out—er, avoid difficult generality by saying the mass is added at a single velocity $\vec{v}_{\text{flux}}$, where the subscript flux stands for flux of mass and flux of momentum.

There is a flux of momentum when you add mass with momentum.

The limitation to a single velocity can be lifted but is beyond our scope—it's a long bad road to perdition, in fact.

We are being general about how we add the mass to the system.

Maybe it is added when it crosses some defined barrier.

Maybe it is added just because we count it as part of the the system at a certain time.

Let the rate of mass added be dm/dt.

Now dm/dt can be positive or negative. The negative addition of mass is actually a subtraction, of course. For rocket propulsion (which we consider below), the mass is actually subtracted and is considered no longer part of the system when it has reached its final ejection velocity relative to the rocket.

In differential time dt, the change in the momentum of the system (i.e., the total momentum) is

$$d\vec{p} = \vec{F}_{\text{other}}\,dt + \vec{v}_{\text{flux}}\frac{dm}{dt}\,dt \ , \tag{52}$$

where the first term is the known effect of forces other than momentum flux force—and we know this from the formulation of Newton's 2nd law without the momentum flux force—and the second momentum flux force itself times dt. The momentum flux force is the rate of momentum into the system carried by the mass entering the system.

Because the time is differentially small, we don't have to worry about the change in momentum due the non-momentum-flux force on the added or subtracted mass. That change can be expected to be of order

$$\left|\left(\frac{\vec{F}_{\text{other}}}{m}\right)\,dm\,dt\right| = \left|\left(\frac{\vec{F}_{\text{other}}}{m}\right)\frac{dm}{dt}\right|\,(dt)^2 \ , \tag{53}$$

where we assume that $\vec{F}_{\text{other}}$ is at least mass proportional to some crude approximation. The change is 2nd order in dt, and so vanishes in a differential sense.

The rate of change of momentum for the system is now seen to be

$$\frac{d\vec{p}}{dt} = \vec{F}_{\text{other}} + \vec{v}_{\text{flux}}\frac{dm}{dt} \ . \tag{54}$$

Now the term $\vec{v}_{\text{flux}}dm/dt$ (which we've already called the momentum flux force) is actually considered a kind of force. Yours truly calls it a **MOMENTUM FLUX FORCE**. Despite having been in physics since 1977, I not sure of the conventional name for this force. Some textbooks may call it the momentum transport force. Maybe it has no common name despite being a rather common quantity.

Since we count the **MOMENTUM FLUX FORCE** as a force, the net external force on the system is

$$\vec{F}_{\text{net}} = \vec{F}_{\text{other}} + \vec{v}_{\text{flux}}\frac{dm}{dt} \ . \tag{55}$$

Now we have the equation

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \tag{56}$$

which we derived before assuming a constant mass system in § 1 (see eq. (7)). As mentioned in § 1, this is the form that Newton originally used—only sort of since he wasn't using modern notation—and he used the Latin word for "motion" instead of momentum (Wikipedia: Newton's laws of motion).

Our derivation here of equation (56) is more general in that it allows for systems of changing mass. Note we needed to introduce the concept of **MOMENTUM FLUX FORCE** in our derivation.

But note that now

$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{v}\,)}{dt} = m\frac{d\vec{v}}{dt} + \frac{dm}{dt}\vec{v} = m\vec{a} + \vec{v}\frac{dm}{dt} \ , \tag{57}$$

where we have used the product rule, $\vec{v}$ is the center-of-mass velocity, and $\vec{a}$ is the center-of-mass acceleration.

The product rule does hold for scalar multiplication with a vector. For a proof, consider $f\vec{A}$, where f is a scalar function and $\vec{A}$ a vector function. Assume that the unit vectors $\hat{x}_i$ for the space are constant. The proof is

$$\begin{aligned} \frac{d(f\vec{A})}{dt} &= \sum_i \frac{d(fA_i)}{dt}\hat{x}_i = \sum_i \left(f\frac{dA_i}{dt} + \frac{df}{dt}A_i \right) \hat{x}_i \\ &= f\frac{d\vec{A}}{dt} + \frac{df}{dt}\vec{A} \ , \end{aligned} \tag{58}$$

where the last expression is the product rule expansion itself. What if the unit vectors are not constant? In classical physics, we can always express a vector in terms of constant Cartesian unit vectors. Thus, the proof with constant unit vectors, I think, implies that the result holds in general since vectors shouldn't depend on how they are described.

We can now find an equation for the (center-of-mass) acceleration $\vec{a}$.

Do so. You have 1 minute working individually or in groups. Go.

Behold:

$$\begin{aligned} \vec{F}_{\rm net} &= \vec{F}_{\rm other} + \vec{v}_{\rm flux}\frac{dm}{dt} = m\vec{a} + \frac{dm}{dt}\vec{v} = \frac{d\vec{p}}{dt} \\ \vec{a} &= \frac{\vec{F}_{\rm other}}{m} + \frac{(\vec{v}_{\rm flux} - \vec{v}\,)}{m}\frac{dm}{dt} \ . \end{aligned} \tag{59}$$

Of course, equation (59) is **NOT** a solution for acceleration in general. It's a differential equation for acceleration in general.

A couple of special cases of equation (59) should be considered. The first is when $dm/dt = 0$. The second is when $\vec{v}_{\rm flux} - \vec{v} = 0$. In both cases, one recovers the familiar result

$$\vec{a} = \frac{\vec{F}_{\rm other}}{m} \tag{60}$$

which is just Newton's 2nd law without the **MOMENTUM FLUX FORCE**.

The case with $\vec{v}_{\rm flux} - \vec{v} = 0$ is interesting because you are varying the mass of the system (assuming dm/dt is not zero also), but there is no acceleration if $\vec{F}_{\rm other} = 0$. Mathematically, this is because the **MOMENTUM FLUX FORCE** exactly cancels the $(dm/dt)\vec{v}$ term of equation (57). Another way to look at the situation is that the momentum per unit mass of the added mass is the same as the momentum per unit mass of the system, and so the momentum to mass ratio of the system stays constant. This ratio is just the center-of-mass velocity itself, and so there is no center-of-mass acceleration.

A trivial example of the case with $\vec{v}_{\text{flux}} - \vec{v} = 0$ is where you have a rocket flying at a constant velocity through empty space, but you regard a subsystem as an ever diminishing piece of the rocket. The subsystem is formally losing mass, but nothing dynamically is happening to the overall system of the rocket.

A similar trivial example is that you are still flying through space in a rocket with no external non-moment-flux forces acting on the rocket. You put items outside of the rocket through an open window, but with no push one way or the other. The rocket is actually losing mass, but the momentum per unit mass of the rocket is constant. The items placed outside are just flying in formation with the rocket. This sort of thing actually happens with astronauts go on space walks from orbiting spacecraft. Here there is an external force gravity, but affects both spacecraft and astronaut alike—they are both in free fall. The astronaut and rocket continue to fly in formation as long as there are no pushes between them. Actually astronauts are tethered to prevent small relative velocities due to small pushes from causing the astronauts to drift off.

4.1. Rocket Propulsion

In a rocket, rocket fuel burns and expands.

The fuel is collimated to flow out the back end of the rocket.

The burnt fuel pushes on the rocket and accelerates it forward while the rocket by

Newton's 3rd law pushes the burnt fuel away in rearward direction.

So fuel and rocket push each other apart.

This is how the rocket is propelled through space.

In empty space, there is nothing external for the rocket to push or pull on, and so the rocket can only push on that part of itself that it ejects. On Earth pushing and pulling on things is the usual way to accelerate things and keep then moving against resistive forces.

For the combined system of rocket and burnt fuel in empty space, there are no external forces and the total momentum stays constant. We are neglecting gravity for simplicity. It's always around to some degree or other depending on where in space you are. But in empty space over the time scales of rocket accelerations, the accelerations due to gravity can be (although not always are) negligible. Of course, launching from a planet is not in empty space, and gravity and air drag are not negligible in this case.

If the combined system were at rest (and thus has zero momentum) in some inertial reference frame before fuel ejection, then momentum stays zero and the center of mass of the combined system never ever moves no matter how long the fuel ejection goes on for.

But the rocket part of the combined system does accelerate away from the initial point.

So rocket propulsion is a conservation-of-momentum effect.

However, a simple-minded, direct conservation of momentum approach to solving the propulsion problem is clonky—but intro textbooks love it for some reason—maybe because of its total obscurity.

It's better to treat the rocket as a mass varying system and use the **MOMENTUM FLUX FORCE**. In this case, $\vec{F}_{\text{other}} = 0$, and so equation (59) specializes to

$$\vec{a} = \frac{(\vec{v}_{\text{flux}} - \vec{v}\,)}{m}\frac{dm}{dt} \ . \tag{61}$$

Let's assume a 1-dimensional system for simplicity and drop the vector notation. We put the positive direction in the direction of rocket motion. Now we have the equation of motion

$$a = \frac{(v_{\rm flux} - v)}{m}\frac{dm}{dt} \tag{62}$$

for the rocket. Note that $v_{\rm flux} - v$ is less than zero since the fuel is moving at a lower velocity than the rocket rocket (given our choice of positive direction)—unless the rocket is firing its retrorockets—but it isn't doing that in our example.

Now $v - v_{\rm flux}$ is actually the relative speed of the ejected fuel with respect to the rocket. This relative speed is a parameter of the rocket mode of operation, and so is fixed by the rocket controls. It is a constant if those controls so dictate. We will assume it is a constant and give it the symbol $v_{\rm ex}$: it is a positive number or zero number since it is a speed, not a velocity.

Equation of motion now becomes

$$a = -\frac{v_{\rm ex}}{m}\frac{dm}{dt} \ , \tag{63}$$

where since $dm/dt < 0$ for a rocket, the acceleration is positive. We note that the quantity

$$v_{\rm ex}\left|\frac{dm}{dt}\right| \tag{64}$$

is called the thrust or thrust force (e.g., Halliday et al. 2001, p. 183).

The rocket equation of motion equation (63) can actually be integrated to find v as a function of m and constants of integration initial mass m_0 and initial velocity v_0.

Do the integration.

You have 1 minute working individually or groups. Go.

Behold:

$$a = -\frac{v_{\rm ex}}{m}\frac{dm}{dt}$$

$$\begin{aligned} \frac{dv}{dt} &= -\frac{v_{\rm ex}}{m}\frac{dm}{dt} \\ dv &= -\frac{v_{\rm ex}}{m}\,dm \\ \int_{v_0}^{v} dv' &= -\int_{m_0}^{m}\frac{v_{\rm ex}}{m'}\,dm' \\ v - v_0 &= -v_{\rm ex}\left[\ln(m) - \ln(m_0)\right] \\ v &= v_0 + v_{\rm ex}\ln\left(\frac{m_0}{m}\right) \ , \end{aligned} \tag{65}$$

where m_0 and v_0 are initial, respectively, initial mass and velocity as we foreshadowed.

If one specified the mass evolution of the rocket (i.e., specified $m(t)$), then one could specify the velocity evolution using equation (65).

We note that if $m \to 0$, then the rocket velocity according to equation (65) diverges logarithmically to infinity.

Question: This doesn't actually happen. Why not?

a) If the mass goes to zero, then there is nothing left of the rocket. It's awfully hard to make the rocket mass go anywhere near zero in one sense since there must be some solid structure: everything can't be just fuel. But on the other hand, rockets we launch from Earth to space have an awful lot of fuel stages to get a relatively tiny payload to space. Low-Earth orbital velocity is about 8 km/s which is really fast by terrestrial standards: of order 24 times the speed of sound in air.

b) Relativistic effects prevent the velocity of matter objects from reaching the speed of light.

c) It happens all the time. This question is a crock.

The answers are (a) and (b). Within classical physics (a) is the full answer. But this answer allows arbitrarily high velocities: just not infinite velocities. But relativistic effects prevent the rocket from accelerating to arbitrarily high velocities. The rocket speed cannot reach the speed of light in any inertial reference frame. What about non-inertial reference frames? Textbooks grow coy on such tricky questions—the attitude seems to be that there is a special place in—in a dark and dreary place for people who ask such questions.

4.1.1. Example: Rocket in Space

Say we have a rocket in empty, gravity-free space with $v_0 = 3.0 \times 10^3\,\mathrm{m/s}$ and $v_{\mathrm{ex}} = 5.0 \times 10^3\,\mathrm{m/s}$.

What is v when $m = (1/2)m_0$?

You have 1 minutes working individually or in groups. Go.

Behold:

$$\begin{aligned} v &= v_0 + v_{\mathrm{ex}} \ln\left(\frac{m_0}{m}\right) \\ &= 3.0 \times 10^3 + 5.0 \times 10^3 \times \ln(2) \\ &= 3.0 \times 10^3 + 5.0 \times 10^3 \times 0.69314718\ldots \\ &\approx 3.0 \times 10^3 + 3.5 \times 10^3 \\ &\approx 6.5 \times 10^3\,\mathrm{m/s}\ . \end{aligned} \tag{66}$$

What is the thrust force if $dm/dt = -50\,\mathrm{kg/s}$? Well

$$\begin{aligned} F_{\mathrm{thrust}} &= v_{\mathrm{ex}} \left|\frac{dm}{dt}\right| \\ &= 5.0 \times 10^3 \times 50 \\ &= 2.5 \times 10^5\,\mathrm{N}\ . \end{aligned} \tag{67}$$

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REFERENCES

Arfken, G. 1970, Mathematical Methods for Physicists (New York: Academic Press)

Barger, V. D., & Olson, M. G. 1987, Classical Electricity and Magnetism (Boston: Allyn and Bacon, Inc.)

Cutnell, J. D., & Johnson, K. W. 2007, Physics, 7th Edition (Hoboken, New Jersey: John Wiley & Sons, Inc.)

Enge, H. A. 1966, Introduction to Nuclear Physics (Reading, Massachusetts: Addison-Wesley Publishing Company)

French, A. P. 1971, Newtonian Mechanics: The M.I.T. Introductory Physics Series (New York: W.W. Norton & Company)

Goldstein, H., Poole, C., & Safko, J. 2002, Classical Mechanics, 3rd Edition (San Francisco: Addison-Wesley)

Griffiths, D. J. 1999, Introduction to Electrodynamics (Upper Saddle River, New Jersey: Prentice Hall)

Halliday, D., Resnick, R., & Walker, J. 2001, Fundamentals of Physics, 6th Edition (New York: John Wiley & Sons, Inc.)

Ohanian, H. C. 1988, Classical Electrodynamics (Boston: Allyn and Bacon, Inc.)

Serway, R. A. & Jewett, J. W., Jr. 2008, Physics for Scientists and Engineers, 7th Edition (Belmont, California: Thomson)

Tipler, P. A., & Mosca, G. 2008, Physics for Scientists and Engineers, 6th Edition (New York: W.H. Freeman and Company)

Weber, H. J., & Arfken, G. B. 2004, Essential Mathematical Methods for Physicists (Amsterdam: Elsevier Academic Press)

Wolfson, R. & Pasachoff, J. M. 1990, Physics: Extended with Modern Physics (London: Scott, Foresman/Little, Brown Higher Education)

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