# ENERGY

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## ABSTRACT

A lecture on what the title says and what the keywords say.


*Subject headings:* energy — conservation of energy — energy forms — $E = mc^2$ — work — kinetic energy — work-kinetic-energy theorem — potential energy — conservative forces — work-energy theorem — mechanical energy — conservation of mechanical energy — forces of constraint — equilibria — power — energy units — human body energy

## 1. INTRODUCTION

In this lecture, we introduce energy.

Energy is immensely important in all of physics, not just Newtonian physics. It is also immensely important in chemistry, biology, engineering, economics, and human society.

But what is energy?

Well everyone, it seems, admits that it is very hard to define. Elementary physics textbooks, for example.

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I think many people would agree that there is no adequate one-sentence definition.

But I will give one-sentence definition that I find useful:

"Energy is the transformable and conserved universal essence of structure."

This is just my own idiosyncratic definition that others may dispute, but it works for me.

It's not adequate—for one thing what the heck does it mean.

First, energy is transformable because it comes in many forms and every form is transformable into every other form. Sometimes the energy transformations are spontaneous and sometimes they are easy and sometimes they are hard to do and sometimes practically impossible for humans at least on a large scale. There is, however, no pure energy. It always has some form. We will give important examples of energy forms in § 1.2 below.

The forms are concretely defined by each having their own formula: the amount of that energy form is calculated from the formula using other physical variables. Some of the variables that we calculate energy from have already been encountered in intro physics such as position, velocity, angular velocity, mass, and the gravitational field. Others such as the rotational inertia and electromagnetic field turn up later in this course or other courses.

Second, "conserved" in the physics jargon means not created or destroyed. Energy counting all forms is conserved. But forms of energy can be non-conserved, of course, since all forms of energy are transformable. There are limited energy conservation rules for subsets of energy forms that apply in special cases. In this lecture, for example, we consider the conservation of mechanical energy (which is the sum of kinetic and potential energy) in § 5.1. We also always use the classical energy conservation rule, except when discussing the equation $E = mc^2$ in § 1.4.

I say "essence of structure", because this seems to be a valid description. Saying a system has so much energy of some form is a description of the structure of the system. But note more energy of some form doesn't mean more structure of some kind: amount of structure is frequently a subjective judgment. But changes in system structure are usually, but not always accompanied by a change in some form of energy.

I thought about saying "measure of structure" in my definition. But because of conservation of energy, it seems that energy is a kind of stuff or substance in some sense. "Essence" implies a measurable stuff, where the stuff has not got a lot of complicating features. Vaguely this sounds like energy to me. But having made the definition using the word "essence", it seems pointless to reiterate "essence" —one can just say "energy" to mean energy and that's what we do hereafter.

Note I use the word structure broadly. A motion is a structure in the context of this discussion.

Nature is full of structures or arrangements if you prefer. Many of the most basic of these structures are well described by physical variables and those are the structures studied by physics. As mentioned above, amounts of the various forms of energy can be calculated from those physical variables. The amounts of the energy forms alone often offer at least a short description of a system (i.e., of its structure) that is adequate for many purposes. For example, just in giving a schematic description of a system. The changes in a system can almost always be described by describing the energy transformations that go on and this is often the best schematic description.

A few more thoughts on a definition of energy.

Energy is sometimes defined as the capacity for change or, to be consistent with my definition, transformation. If you have energy in some form in a system, then the system can be changed by changing that energy into some other form or by adding to or subtracting from the system energy in some form. This corresponds well to the qualitative usage of the term energy in everyday life. If you have energy in the qualitative sense, you can do things which usually involves changing things. But energy in the physics sense is quantitative. Since there are energy conservation rules, the amounts of energy in its various forms put limits on how much change can occur.

In fact, using energy conservation rules alone often allows one to have information, but usually not full information, about a system's evolution. This is a major use of energy in this lecture.

A completely adequate definition of energy probably has to include a full recital of all the things that energy is used for in physics. We make a start on that in this lecture.

In modern physics, a full description of a system almost always includes an energy inventory. At a level beyond intro physics, the full treatment of a system's evolution can be done in terms of energy.

I use the word "universal" in my definition because the energy concept is universally useful in physics: i.e., in all the systems we deal with. I also use "universal" since all energy forms are energy since any form can be transformed into any other form.

Also all forms of energy have the same physical dimensions. But not all things with the same dimensions as energy are energy: torque has the same dimensions as energy, but isn't energy. Torque comes up in the lecture *ROTATIONAL DYNAMICS*.

Not to be coy, the dimensions of energy are

$$\mathrm{ML^2/T^2} \ , \tag{1}$$

where recall that the Roman letters M, L, T stand, respectively, for the dimensions of mass, length, and time. The MKS unit of energy is called the joule (symbol J) and is, of course, defined by

$$J = \mathrm{kg\,m^2/s^2} \ . \tag{2}$$

Of course, there are kilojoules, megajoules, etc.

The joule was named for James Joule (1818–1889) who extended the energy concept to heat energy (properly internal energy) and coined the word thermodynamics for the physics of internal energy and temperature. A word on pronunciation is in order:

> The unit of energy is the joule
> and this rhymes with drool,
> but it should rhyme with bowel
> to be correct for James Joule.

Even Joule's contemporaries were a bit uncertain of the pronunciation of his name—which was also the name of his brand of beer—he was brewer—"I'll have a pint of Joule" one might say.

The joule is a derived unit. But that does not mean that the energy is not a fundamental quantity. It may be the most fundamental quantity. But it is a rather abstract quantity at least in most ordinary physical applications. We will see that there is a concrete way of thinking about energy when we consider the equation $E = mc^2$ below in § 1.4. This equation shows that we can think of energy as mass and determine it from measurements of mass (or mass changes) alone, in principle. In practice, mass measurements are often far from the best means to determine most of the forms of energy. In the classical limit, one can't determine energy from mass measurements alone at all. In the classical limit, energy is determined from a calculation with more directly observable quantities like mass (but not alone), displacement, and velocity.

## 1.1. Conservation of Energy

As mentioned in § 1, energy counting all forms is conserved: i.e., energy is not created or destroyed though its forms do change.

The conservation of energy is a basic principle of all of physics.

Conservation of energy implies, among other things, that the total energy of a closed system stays constant no matter what changes occur in the system. In fact, energy is conserved in every little interaction in the system which implies it is conserved overall.

Energy may change form in many different ways, but the total stays the same.

And energy does transform from one form to another. In fact, any form of energy can transformed to any other form as emphasized in § 1. Whether that transformation happens depends on the system. Some energy transformations cannot happen because of the nature of a system. But if this is system changed appropriately, they could.

Is conservation of energy a postulate of physics or a derived result and is it exactly true?

We will treat it as an axiom. There is a theorem called Noether's theorem that derives conservation of energy from other physical principles, but yours truly is essentially ignorant of Noether's theorem and any limitations it may have. There are ambiguities about the conservation of energy at high level in physics (e.g., Greene 2004, p. 532). We won't concern ourselves with them. Experimentally, energy always seems to be conserved.

## 1.2. Forms of Energy and Energy Transformations

There are many forms of energy.

And each form of energy has its own formula or formulae as mentioned above in § 1.

It's tricky and tedious to try to list all the forms because there are a lot of them and the forms actually overlap. One reason the list of forms goes on and on is that often energy of one form is given different names in different contexts: e.g., horse energy and human energy are both animal energy and animal energy can be decomposed into other kinds of energy forms.

Here we will just run through a short list of important energy forms without giving formulae:

1. Kinetic energy: This is the energy of motion. Actually, there is subtlety here. An object in motion has kinetic energy, but in its own frame of reference it does not have kinetic energy since it is not moving. Energy is actually reference frame dependent. It's conserved in every reference frame though. There are rules for energy transformation between reference frames. Is there some frame of reference in which one calculates what can be called the true energy? Well maybe in that continuum of inertial reference frames that participate in the mean expansion of the universe—but I don't know what the great minds think on this fine point.

2. Potential energy: This is the energy of position in a field of force. Different fields of force can have their own potential energies. The gravitational field, electric field, and magnetic field can have potential energies. Actually, whether or not the energy of a field can be described as potential energy depends not only on the nature of the field, but on its structure (i.e., the distribution of field values in space and time). The electric field is often said to have a potential energy, but it doesn't in all contexts. The magnetic field is often said not to have a potential energy, but in some contexts it does. Note that fields of force always have energies it seems, it's just that those energies may not always be describable as potential energies.

3. Mechanical energy: This is the sum of kinetic and potential energy. This an example

of the overlap of forms of energy.

4. Electromagnetic field energy: The energy of the electromagnetic field. The electromagnetic field is the electric and magnetic fields considered jointly. That's all we can say here.

5. Chemical energy: The energy stored in chemical bonds of atoms and molecules. This is actually a form electromagnetic field energy.

6. Internal energy: This energy is also commonly, but improperly, called heat energy. Yours truly has given up being proper and will usually just write heat or heat energy.

   But what is internal energy or heat. It's the sum of the microscopic forms of all other energies. But what is a microscopic form of energy?

   Well microscopic kinetic energy is the energy of the uncorrelated motion of the atomic and subatomic particles. Uncorrelated motion means that those motions don't add up to any macroscopic motion.

   Chemical energy is also microscopic energy.

   The potential energy of atomic and subatomic particles in fields of force of microscopic extent or structure is microscopic energy.

   There are other microscopic energies in internal energy too.

7. Electromagnetic radiation energy: This is the energy of electromagnetic radiation or light. Electromagnetic radiation is a self-propagating electromagnetic field. Both electric and magnetic fields are needed for propagation. Electromagnetic radiation energy is a sub-form of electromagnetic field energy.

The list of energy forms goes on and on. It's quasi-endless.

In this lecture, we introduce kinetic, potential, and mechanical energy.

We do have to allude to chemical, heat, and electromagnetic radiation energy.

The first comes up in our examples as a source for mechanical energy.

The latter two come up as sinks for mechanical energy.

In general, of course, chemical, heat, and electromagnetic radiation energy can be either sinks or sources since all energy forms are transformable in to all other forms.

Frequently when mechanical energy has disappeared from our systems, it ends up as heat energy or as we often say it is dissipated to waste heat.

As mentioned in § 1, changes in a system can usually be discussed schematically in terms of energy transformations alone.

Now that we have some forms of energy we can do this. For concreteness, let's just do a specific example.

You metabolize food you've eaten changing one kind of chemical energy into another kind of chemical energy. That chemical energy gets changed both into heat energy in your body and into kinetic energy of motion. The kinetic energy of motion allows you to move around. Much of it ends up as waste heat. If you lift objects or yourself, you change some kinetic energy into gravitational potential energy.

What causes energy transformations?

Forces do.

To speak loosely: energy is the structure and forces change the structure.

For example, a net force changes some energy form into the kinetic energy. We'll see how this happens quantitatively in § 3 making use of Newton's 2nd law of motion.

We can now pretty easily see that how vague energy as used in everyday life connects with well defined energy.

If you have energy you have some kind of quantified structure. You can change that into other structures using forces.

So actually I think vague use of the term energy is quite correct.

## 1.3. Energy and Newtonian Physics

Part of the concept of energy was known in the 17th century to Galileo (1564–1642), Christian Huygens (1629–1695), and Gottfried Leibniz (1646–1716) as discussed in § 1.5.

Newton (1643–1727) never used energy insofar as it was known then though I guess he must have known of it (see § 1.5).

But energy is implicit in Newton's Newtonian physics in that work, kinetic energy, potential energy, mechanical energy and the conservation of mechanical energy can be developed consistently from it. Personally, I'd say that work, potential energy, and mechanical energy are defined and kinetic energy and the conservation of mechanical energy are derived. But the epistemic status of these concepts may be debatable. We develop the aforesaid concepts in this lecture.

Energy turns out be very useful in intro physics and in advanced Newtonian physics even more so.

Energy, however, is much broader than Newtonian physics.

It's a vital ingredient in all physics nowadays.

So it's very important to introduce in intro physics.

At advanced levels, energy methods offer a superior approach to doing straightforward

Newtonian physics.

At our level, energy methods or energy analysis (if you prefer) offer an alternative approach to the non-energy Newtonian methods we've seen in the lectures *NEWTONIAN I* and *NEWTONIAN II*. Sometimes the energy methods are just alternatives and not superior. But sometimes they are superior.

In particular, energy methods often give you partial information easily that the non-energy methods do not give you easily. By energy methods giving partial information easily, one usually means in intro physics that conservation of mechanical energy (§ 5.1) gives partial information easily.

## 1.4. $E = mc^2$

If there is one physics equation everyone knows, it's

$$E = mc^2 , \tag{3}$$

where $E$ is energy, $m$ is mass, and $c = 2.99792458 \times 10^8$ m/s is the vacuum light speed.

Usually people just name this equation by rattling it off. It is also called the Einstein equation or the mass-energy equivalence.

But what does $E = mc^2$ mean?

First of all, it means that all energy forms have mass: resistance to acceleration and gravitational charge. This is the concrete way of thinking about energy we alluded to at the end of § 1.

Now this may seem odd because there are negative amounts of energy and mass is never negative. The resolution of this paradox is that if you add up all the energy forms of a system, the result is always positive and it is the result that is always detectable as mass.

Second of all, $E = mc^2$ means there must be what is called rest mass energy of basic particles. The most important of the basic matter particles of are those of ordinary matter: protons, neutrons, and electrons. Rest mass energy is just the energy of existence of these particles in a frame of rest. The mass of the particles in a frame of rest is their rest mass. Some particles like photons don't have rest mass or rest mass energy—but this is OK because they are never observed at rest, but always moving at the vacuum speed of light. Massless (i.e., rest massless) particles are usually not counted as matter in modern jargon.

For example of rest mass energy, consider a proton which has (rest) mass $m = 1.672621637(83)\times 10^{-27}\,\text{kg}$. Its rest mass energy is

$$E = mc^2 \approx 1.503 \times 10^{-10}\,\text{J} \;, \tag{4}$$

where we see that the units work out to be joules.

Since all energy forms are transformable to all other forms of energy, the rest mass energy of basic matter particles can be transformed to all other forms of energy and vice versa. The transformations into rest mass energy are creations of the particles and the transformation out of rest mass energy are the annihilations of the particles. Creations and annihilations happen all the time, but in the ordinary terrestrial environments and most astrophysical environments at a very low rate. So low that it wasn't noticeable before the 20th century.

In particular, the ordinary matter particles are created or annihilated a very low rates in ordinary terrestrial environments and most astrophysical environments—we say they are nearly stable in these environments.

Since ordinary matter particles (protons, neutrons, and electrons) make up most of the luminous mass in the universe, the ordinary matter is nearly stable in most environments, in particular, the terrestrial environment.

And since ordinary matter is mostly stable, it seemed before circa 1900 that the mass of systems was conserved no matter what flows of kinetic, thermal, or chemical energy went into or out of the system. Only when obvious matter flowed into or out of a system was the mass changed as far the old-timers could tell then. That led to the classical conservation of mass principle that mass/matter was not created or destroyed. What the old-timers thought of as matter was various. Some believed in atoms as the unbreakable constituents of matter and some thought of matter as a continuum of stuff that probably didn't include electromagnetic fields and some sat an the fence. A consequence of the classical conservation of mass is that mass that is unchanged by energy flows and only by matter flows.

Nowadays, we use the term matter for particles with rest mass as mentioned above.

In fact, mass does change for energy flows without matter flows, but the changes were so small that people couldn't detect them before some time well into the 20th century. If a system gains kinetic energy, its mass increases. If you heat up system or increase its chemical energy somehow using energy from outside the system, the system mass increases. For example, say an object in motion has 10 J of kinetic energy—and we'll see what that means about its motion soon—but it's a human-scale amount of kinetic energy for human-scale objects moving at human-scale speeds. The object has zero kinetic energy when at rest. The difference between its rest mass and its mass when moving is

$$\Delta m = \frac{E}{c^2} \approx 10^{-16}\,\mathrm{kg}\ . \tag{5}$$

Such mass changes for ordinary macroscopic objects were unmeasurably small until I don't when.

On the other hand, the stability of the ordinary matter in the terrestrial environment, means that for most purposes the sum of energies excluding the rest mass energy of the ordinary matter particles is conserved. This limited energy conservation is the classical conservation of energy.

With $E = mc^2$, it is now seen that the true conservation of energy principle is also the conservation of mass principle. In fact, in special relativity jargon, one often speaks of mass-energy which explicitly shows energy and mass are in a sense the same thing. Personally, I tend to think of mass as an attribute of energy (i.e., an energy form's resistance to acceleration and gravitational charge), but this is just a way of thinking.

In intro physics, we essentially live in the 19th century and use the separate classical principles of conservation of mass and energy. But we won't reiterate this much.

Just a bit of history.

Einstein discovered $E = mc^2$ in 1905 by deriving it from special relativity which he also discovered that year. The derivation is a physicsy derivation with reasonable assumptions injected when needed, not a rigorous math derivation from clearly stated axioms.

He did not immediately jump to the idea that he had discovered a new source of energy. Rather, he knew that energy changes in radioactive decay processes—only known since 1896—were very large compared to chemical energy changes. So he thought correctly that $E = mc^2$ could be tested by measuring mass changes in radioactive samples.

By the way, $E = mc^2$ is **NOT** the singular key ingredient the development of nuclear energy. It's just one of the ingredients. But it's also fundamental cornerstone of all modern physics. And all modern physics is inextricably interconnected. If one main principle (e.g., $E = mc^2$) is wrong, somehow it's all wrong. Like a house of cards, it would all fall down. But it's much more stable than a house of cards—it doesn't fall down.

In human-made nuclear energy generation, one does not for most part release heat energy by destroying ordinary matter particles. What happens is that the nuclear bonding of nuclei changes and protons turn into neutrons or vice versa. This is somewhat analogous to changing chemical bonds to release or absorb energy. The key difference for energy generation is that the energy scale of nuclear bonds is of order $10^6$ larger than that of chemical bonds.

So one can get a factor of a million more heat energy out nuclear fuel than out of the same mass of chemical fuel.

That factor of million has mesmerized people since the early 20th century. So much energy from so little fuel.

Of course, there are only certain ways that we can do it—and it's beyond our scope to discuss them now.

Why can't one just convert ordinary matter into other forms of energy leaving no ordinary matter at all?

Well, in principle, you can. Processes go on in nature and in the laboratory all the time that convert ordinary matter into electromagnetic radiation and often into heat energy at the microscopic level. For example, some radioactive material is always around and a common radioactive decay particle is the positron, the antiparticle of the electron. There are a few in this room. When a positron meets up with an electron, the two particles mutually annihilate creating gamma-rays which are very short wavelength electromagnetic radiation.

So there is a way to convert ordinary matter into other forms of energy leaving no ordinary matter at all.

But neither we nor nature have a way to assemble macroscopic amounts of antimatter. It's keeps annihilating before either of us (we and nature) can accumulate much. The observable universe is completely dominated by matter—antimatter particles when they arise are annihilated when they interact with matter particles—which may take a long time in some space environments, but not long enough for the creation of macroscopic antimatter samples it seems.

In general, we have no practical way of converting ordinary matter samples entirely into other forms of energy on the macroscopic scale.

But say we could convert 1 kg of matter into some other form of energy such heat energy which in a small region would create explosively high pressures. How much energy would we get? Well

$$E = mc^2 \approx 9 \times 10^{16}\,\mathrm{J} \approx 20 \text{ megatons} \ , \tag{6}$$

where a megaton is the heat energy released by a megaton ($10^9$ kg) of TNT.

It is actually a good thing in our bombing age that this can't be done practically speaking.

## 1.5. The History of Energy

The term energy (*energiea*) was introduced by Aristotle (384–322 BCE) in the 4th century BCE as a philosophical term for which I've found no meaningful definition. I mean the definitions given convey to me no clear idea of what old Aristotle used energy for. *Energiea* is actually a compound word meaning "in work", but that is not what it meant (e.g., Smil 2006, p. 1). One suggested meaning is "actuality, identified with motion" (e.g., Smil 2006, p. 1)—but that doesn't help much. Here's an Aristotle quote "The energy of the mind is the essence of life." That doesn't help much either—he may have been using energy only metaphorically in this case.

The term energy kicked around in philosophical discourse for 2000 years because Aristotle used it.

But when did energy come into physics?

Well Galileo (1564–1642) had a quantitative of idea of work and power, and therefore of energy (e.g., Caldwell 1994, p. 87). But he didn't use the word energy. Galileo's concepts were very useful in quantitatively understanding machine performance and were utilized and extended by engineer-scientists in the 17th and 18th centuries somewhat independently of Newton's Newtonian physics it seems (e.g., Caldwell 1994, p. 89).

Christian Huygens (1629–1695) and Gottfried Leibniz (1646–1716) made use the quantity $mv^2$ which is object mass times object speed squared (e.g., Caldwell 1994, p. 96). Leibniz used the quantity from 1676 on. Leibniz called it *vis viva* (living force) (Wikipedia: *Vis viva*). Huygens and Leibniz thought this quantity was the true measure of an object's motion. It measures the degree to which resistance is overcome. For example, a bullet's velocity is doubled it penetrates four times as far into wood as the *vis viva* formula suggests. Also the height that an object launched straight upward can be determined from it's initial *vis viva* which is nowadays recognized as a simple consequence of conservation of mechanical energy (see § 5.1). Leibniz also noted that total *vis viva* is conserved in some systems. This result is the conservation of kinetic energy alone. We consider this conservation rule from a modern perspective in Appendix A.

I would guess that Newton (1643–1727) must have been aware of the energy concepts of his great predecessors and contemporaries, but he doesn't seem to have made any use of them.

The word energy was not yet in use in any of these physics conceptions yet.

In 1807, Thomas Young (1773–1829) renamed *vis viva* energy.

In the course of the 19th century, it was figured out that a factor of 1/2 should multiplied to *vis-viva* energy $mv^2$—this 1/2 was needed precisely to make *vis-viva* energy change equal to net work done in equation (25) (see § 3) without any klutzy explicit factors of 1/2.

When other forms of energy were introduced in the 19th century, $(1/2)mv^2$ was given the special name kinetic energy. “Kinetic” is derived from the Greek word for motion *kinesis* (Wikipedia: kinetic energy). The world cinema was also derived from *kinesis*—but cinematic energy is something Steven Spielberg has. As we will see, kinetic energy arises from the work concept used with Newton’s 2nd law. Thus, kinetic energy can be said to be implicit in Newton’s Newtonian physics. In our Newtonian physics, energy is, of course, quite explicitly used.

In certain kinds of isolated systems, kinetic energy is conserved: i.e., stays constant as the system evolves as mentioned above (see also Appendix A).

It was found in the course of the 19th century that one could extend the principle of conservation of energy by inventing new forms of energy.

It came to be believed that the sum of all energy forms is conserved: i.e., never created or destroyed. Thus, the principle of conservation of energy was discovered.

In the 19th century, I think people began to take the principle of conservation of energy as a postulate of physics.

I’m not sure of the modern view. As mentioned in § 1.1, Noether’s theorem, discovered in 1915, (Wikipedia: Conservation of energy; Noether’s theorem) proves conservation of energy from other principles. But the status of Noether’s theorem is beyond yours truly’s expertise.

Remember though energy conservation is just for the sum of all energy forms in general. Energy can be transformed from one form to another. In fact, any form of energy can be transformed into any other form. This is a good reason for saying all forms of energy are energy.

You may ask is energy conservation just an accounting trick—it is conserved because we have invented forms of energy to make it conserved.

The answer I think anyone would give is no.

One reason is Noether's theorem discussed above with a confession of authorial ignorance.

A second reason is because we've always been able to find new forms of energy whenever energy seemed to have been not conserved and this suggests that those forms were waiting for us to discover. Also, the new forms have always been useful in calculations.

A third reason for viewing energy as a real thing is the Einstein equation:

$$E = mc^2 \tag{7}$$

which we discussed in § 1.4.

Since all energy has mass, I'm prepared to say that energy is a real thing.

## 2. WORK

To get to energy in Newtonian physics, we first define work.

What is work in everyday life?

Well it's doing something.

You exert forces and move things if only electrons in your brain while thinking.

You also expend energy (in the vague everyday sense of the word) doing work: let's call this energy vague-energy.

This vague-energy is transformed by work: e.g., from the food energy of your body to the energy of motion of your body.

In physics, it was found useful to find a quantitative formula for work done on a system by a force. The formula incorporates the idea that force and movement come into work.

It turns out in our developments that work done (i.e., the work quantity) is energy transformed (which in some cases is just energy transferred) by the work process. As mentioned in § 1.2, forces transform energy and work is the process whereby they do it. But what the energy is transformed from and what it is transformed to is **NOT** explicit in the work formula. Formulae for energy forms must be developed and related to work. We do that in succeeding sections of this lecture.

Note the work formula is a definition that was found to be useful in developing the energy concept. We just say this formula defines what work is.

The basic differential work formula for work done on a system is really very simple:

$$dW = \vec{F} \cdot d\vec{s} \tag{8}$$

where $dW$ is differential work done, $\vec{F}$ is the force acting on the system and doing the work, $d\vec{s}$ is the differential displacement of the system center of mass while the work is being done, and the binary operation is the dot or scalar product. Note we are discussing work done on a system thinking of the system as whole. This is why $d\vec{s}$ is the center of mass displacement. If the system is or is treated as a particle, $d\vec{s}$ is the particle displacement. If you are considering work done on a subsystem of a system, the force must act on the subsystem and $d\vec{s}$ is the displacement of the subsystem center of mass.

In our developments, we usually only consider external forces doing work on the system. Internal forces can do work too, but usually we won't get into that.

Note that work has the dimensions $(\mathrm{ML/T^2})\mathrm{L} = \mathrm{ML^2/T^2}$ which as discussed in § 1 are the dimensions of energy. This must be so for work to be energy transformed. Evidently, the unit of work is

$$\mathrm{unit}[W] = \mathrm{unit}[Fs] = \mathrm{N\,m} = \mathrm{kg\,m^2/s^2} = \mathrm{J} \tag{9}$$

where unit[ ] is my idiosyncratic unit function. So the MKS unit of work is the joule, of course.

Work is a scalar quantity since it comes from the dot product of two vectors. One can write out the dot product quantities explicitly:

$$dW = \vec{F} \cdot d\vec{s} = F\,ds\cos\theta \ , \tag{10}$$

where $F$ is the magnitude of the force $\vec{F}$, $ds$ is the magnitude of the differential displacement force $d\vec{s}$, and $\theta$ is the angle between $\vec{F}$ and $d\vec{s}$ with their tails at one point.

Note the displacement is not in general caused by the force $\vec{F}$ and $\vec{F}$ is not in general the net force on the system. The force $\vec{F}$ is just the force for which the work is being calculated. The energy (and structure) of system may not be changing in the system if the force is being exactly canceled by another force. But somewhere in the chain of interactions energy is being transformed or transferred.

There are special case results for the differential work formula for $\theta = 0$, $\theta = \pi/2$, and $\theta = \pi$.

What are they? You have 30 seconds working individually or in groups.

Behold:

$$dW = \begin{cases} \vec{F} \cdot d\vec{s} & \text{in general;} \\ F\,ds & \text{for } \theta = 0; \\ 0 & \text{for } \theta = \pi/2; \\ -F\,ds & \text{for } \theta = \pi. \end{cases} \tag{11}$$

These special cases illustrate that there is positive and negative work and that if the force is perpendicular to the displacement there is zero work done.

There is also zero work done if the displacement is zero. No movement, no work no matter how strong the force. If I push on a wall, no matter how hard I try, it does not move macroscopically and I've done no macroscopic work on the wall. Now at the microscopic level, the wall has flexed and so microscopic work has been done, but that does not count in a macroscopic calculation. To anticipate at bit, the wall doesn't acquire macroscopic kinetic energy or potential energy. The microscopic flow of energy in the microscopic work ends up as waste heat pretty quickly. If one idealizes the wall as perfectly rigid as we often do, then no work is done at all.

Now positive work done on a system is energy transferred to the system and negative work done on the system is energy transferred from the system.

These interpretations of positive and negative work follow from the work-kinetic-energy theorem which we'll get to soon (see § 3).

The work-kinetic-energy theorem, in fact, gives the first validation that the work formula is useful in physics.

For finite work done and finite displacements, one in general has a three-dimensional integral work formula:

$$W = \int \vec{F} \cdot d\vec{s} \tag{12}$$

where the integral is along the path traced out by the system (i.e., its center of mass) as it moves. Such integrals along paths are called path or line integrals. We don't need to specify how the integration is to be done here. There are ways as we'll see. We just have to acknowledge that the integral exists and is the sum of the infinitely many differential work elements $\vec{F} \cdot d\vec{s}$ along the path made up of differential displacements $d\vec{s}$.

Note we don't specify endpoints in equation (12) since any labels for these are arbitrary. In the author's opinion, it is best to leave general formulae unadorned by subscript, superscript, and limit value symbols that are arbitrary choices and that obscure the simple meaning of the formulae. Of course, in particular cases one must use adornments to show the particularities of the particular cases.

If the force is constant, then the force factor can be taken out of the integration and the finite work formula becomes

$$W = \int \vec{F} \cdot d\vec{s} = \vec{F} \cdot \int d\vec{s} = \vec{F} \cdot \vec{s} \tag{13}$$

where $\vec{s}$ is the net displacement from the initial point of motion to the final point of motion. Note the path did **NOT** have to be straight, but the vector sum of the path is, of course, a straight line displacement vector.

Equations (8), (12), and (13) can all be referred to as the work formula.

### 2.1. Example Work Calculation

Say a man pulls a vacuum cleaner 3.0 m along the floor using a straight rope at an angle of 30° to the horizontal. Since a straight rope can only extert a tension force parallel to the rope itself, the rope force on the vacuum cleaner is at 30° to the horizontal. The rope tension is 50 N.

In this case, the force is a constant and we can use equation (13).

Solve tor the work done.

You have 30 seconds working individually. Go.

We find that the work done on the vacuum cleaner is

$$W = \vec{F} \cdot \vec{s} = 50 \times 3.0 \times \cos(30^\circ) = 130\,\mathrm{J} \tag{14}$$

to 3 digit accuracy.

Actually before one derives kinetic energy and the work-kinetic-energy theorem (which we do just below in § 3), there arn't many interesting example calculations that one can do.

One can just calculate work done by forces, but in itself that tells little of the system behavior. Even after we get kinetic energy and the work-kinetic-energy theorem really interesting examples have to wait for the derivation of the work-energy theorem (§ 5) which is **NOT** the same as the work-kinetic-energy theorem.

So don't expect to see numbers much until we get to the work-energy theorem.

## 3. THE WORK-KINETIC-ENERGY THEOREM

The work-kinetic-energy theorem is a really beautiful result.

The derivation shows how the work formula in combination with the 2nd law of Newtonian physics gives rise to the dynamical variable kinetic energy. So it finally puts energy into Newtonian physics.

The theorem itself shows how to calculate changes in kinetic energy using forces. Without the theorem, the work formula would be an arbitrary definition without any use.

Kinetic energy alone is physically useful because we can calculate speed using it.

Kinetic energy is also useful because we know that in certain systems of interacting particles or subsystems, it is conserved just by itself with certain qualifications to the word "conserved". We will briefly consider those cases in Appendix A.

But much more interesting uses of kinetic energy follow from the work-energy theorem (§ 5).

As Federigo da Montefeltro would say, now to business: the derivation of the work-kinetic-energy theorem.

Say you have a system that moves through three-dimensional space from point A to point B. Implicitly, we mean that the system center of mass moves from point A to point B.

What is the work done on the system by the **NET** force? Note **NET** force, not just the work done by any force.

Well

$$W_{\rm AB} = \int_{\rm A}^{\rm B} \vec{F}_{\rm net} \cdot d\vec{s} \ , \tag{15}$$

where $d\vec{s}$ is a differential path displacement vector for the center of mass. Now since we are considering the net force on the system, we can make use of Newton's 2nd law $F = ma$. The 2nd law applied to the system (which has mass $m$) is

$$\vec{F}_{\rm net} = m\vec{a} \ , \tag{16}$$

where $\vec{a}$ is the system's acceleration (i.e., the center-of-mass acceleration of the system). We assume the mass is constant. Non-constant mass cases can be handled too—but we won't get into all that.

Now substituting for net force $\vec{F}_{\rm net}$ with $m\vec{a}$ in the work integral, we get

$$W_{\rm AB} = \int_{\rm A}^{\rm B} m\vec{a} \cdot d\vec{s} \ . \tag{17}$$

Fig. 1.— System moving from point A to point B through three-dimensional space.

Now for some calculus trickery. Note

$$\vec{a} = \frac{d\vec{v}}{dt} \qquad \text{and} \qquad d\vec{s} = \frac{d\vec{s}}{dt}\,dt = \vec{v}\,dt \ . \tag{18}$$

Thus,

$$W_{\rm AB} = \int_{\rm A}^{\rm B} m\vec{a}\cdot d\vec{s} = \int_{t_{\rm A}}^{t_{\rm B}} m\frac{d\vec{v}}{dt}\cdot\vec{v}\,dt \ , \tag{19}$$

where we have introduced as the explicit path parameter time $t$ which is a variable of integration. With the explicit path parameter, the endpoints of the integration are $t_{\rm A}$, the time when the system is at point A, and $t_{\rm B}$, the time when it is at point B.

Now note that

$$\frac{1}{2}\frac{dv^2}{dt} = \frac{1}{2}\frac{d(\vec{v}\cdot\vec{v})}{dt} = \vec{v}\cdot\frac{d\vec{v}}{dt} \ . \tag{20}$$

where we have just used the product rule for the dot product. The ordinary product rule generalizes to the dot product rather obviously. For a proof, consider general vectors $\vec{P}$ and $\vec{Q}$. Now making use of the component formula for the dot product, we see that

$$\frac{d(\vec{P}\cdot\vec{Q})}{dt} = \frac{d\left(\sum_i P_i Q_i\right)}{dt} = \sum_i \frac{d(P_i Q_i)}{dt} = \sum_i\left(\frac{dP_i}{dt}Q_i + P_i\frac{dQ_i}{dt}\right) = \frac{d\vec{P}}{dt}\cdot\vec{Q} + \vec{P}\cdot\frac{d\vec{Q}}{dt} \ , \tag{21}$$

where the sum is over all the components of the vectors. Equation (21) is the proof.

We now substitute equation (20) into equation (19) to obtain

$$W_{\rm AB} = \int_{t_{\rm A}}^{t_{\rm B}} \frac{1}{2}m\frac{dv^2}{dt}\,dt = \frac{1}{2}mv_{\rm B}^2 - \frac{1}{2}mv_{\rm A}^2 \ , \tag{22}$$

where $v_{\rm A}$ is the speed at point A and $v_{\rm A}$ is the speed at point B.

We now define the center-of-mass or translational kinetic energy $KE$ for a system to be given by the formula

$$KE = \frac{1}{2}mv^2 \ , \tag{23}$$

where $m$ is the system's mass and $v$ is the system's speed (i.e., its center-of-mass speed).

The dimensions of kinetic energy are $\mathrm{ML^2/T^2}$ which are the foretold dimensions of energy. The units of $KE$ are energy units are joules, of course. To be explicit note that

$$\mathrm{unit}[KE] = \mathrm{unit}\left[\frac{1}{2}mv^2\right] = \mathrm{kg\,m^2/s^2} = \mathrm{J} \ , \tag{24}$$

where again unit[ ] is my idiosyncratic unit function.

Note that the system kinetic energy defined by equation (23) is he kinetic energy associated with the center-of-mass motion of the system. We can call it the center-of-mass kinetic energy. It is also commonly called the translational kinetic energy of the system or object. The center-of-mass kinetic energy does not include the kinetic energy of the subsystems the system which can be considerable. One could add up the kinetic energies of any set of subsystems of the system and that would be the total kinetic energy of the set of subsystems. The total kinetic energy of sets of subsystems is always greater than or equal to the center-of-mass kinetic energy. We take up the subject of total kinetic energy in § 3.3.

What are the subsystems? The subsystems could be non-interacting particles flying around with their own subsystem center-of-mass kinetic energies. They could parts of the system that vibrate or rotate and kinetic energy associated with those motions. The whole system could rotate and have rotational kinetic energy. But none of those kinetic energies are counted in the center-of-mass kinetic energy. In the lecture *ROTATIONAL DYNAMICS*, we consider rotational kinetic energy.

With the definition for kinetic energy equation (23), we can rewrite equation (22) to a form we call the **WORK-KINETIC-ENERGY THEOREM** for a system:

$$\Delta KE = W \ , \tag{25}$$

where $\Delta KE$ is the change in center-of-mass kinetic energy and $W$ is the work done on the system by the net force on the system (which is also the net external force). We have suppressed the subscript labels in equation (25) since they were just arbitrary choices useful in the derivation. For a general theorem or rule, we don't want any adornments that just reflect arbitrary names and are unmemorable. One simply understands from context that in equation (25), $W$ is the work done on a system by the **NET** force that act on it and $\Delta KE$ is the change in kinetic energy due to that work.

We have already said that work is an energy transfer process: it transforms energy from one form to another. The **WORK-KINETIC-ENERGY THEOREM** gives us one of the forms of energy the transfer process is between: kinetic energy. Work done on a system can change energy from some form (e.g., potential energy as we will discuss later in § 5) into kinetic energy of a system if $W > 0$ or the reverse if $W < 0$.

There is a key point to note: kinetic energy is always non-negative since the factors in its formula can only be non-negative: i.e.,

$$KE = \frac{1}{2}mv^2 \geq 0 \qquad \text{always.} \tag{26}$$

The kinetic energy bank (as one can call it) can be emptied.

Another key point is that the kinetic energy is frame dependent. Energy is conserved in all frames, but it's value can depend on the frame you observe it in. Recall we mentioned this in § 1.2.

For example, an object at rest in a moving car has no kinetic energy relative to the car, but it does relative to the ground.

A third key point is that the work-kinetic-energy theorem is referenced to inertial frames since derived it using Newton 2nd law which is referenced to inertial frames.

One can still use the work-kinetic-energy theorem in non-inertial frames if one uses the non-inertial frame generalization of the 2nd law introduced in the lecture *NEWTONIAN PHYSICS II*. One must then introduce the fictitious inertial forces in evaluating the work done. Recall the inertial forces allow one to reference Newton's laws to non-inertial frames. Using inertial forces is a useful trick because it reduces an unsolved case (which laws to apply in a non-inertial frame?) to a solved one (use Newton's laws).

### 3.1. Individual Forces and the Work-Kinetic-Energy Theorem

The work done by the net force changes a system's kinetic energy according to the work-kinetic-energy theorem.

Or another way to put it, the net force transfers energy into or out of the system's kinetic energy bank.

But what do individual forces do that make up the net force?

Acting alone they would each be the net force.

But what do they do individually when they act together.

One perspective, which may not be the only one, is that there are various inflows and outflows from the system's kinetic energy bank which can be continuous.

The individual forces doing positive work contribute inflows and the individual forces doing negative work contribute outflows.

The bank balance in the system's kinetic energy bank is the system's kinetic energy itself.

If the system's kinetic energy stays constant, no net force acts, but there can still be energy transfers via the system. So energy elsewhere changes and structure elsewhere changes.

Where to the inflows come from and where do the outflows go to?

Well these are other stories.

We will consider qualitatively a couple of illustrative cases. Say you push on a block and move it at a constant velocity. The friction force cancels your push force. Here chemical energy from your body becomes kinetic energy of your body and flows into the block's motion, but friction extracts it as quickly as it flows in and turns it into waste heat.

Another case outside of the realm of mechanics is that of a constant electrical current in direct current (DC) circuit. Energy from some source creates an electromagnetic field in and around the circuit and that field drives the electrons in the wires. Energy from the source continually flows into the circuit system, but the energy of the electromagnetic field stays constant and the electron kinetic energy stays constant. An energy outflow occurs that keeps these energy pools constant. The outflow could be macroscopic kinetic energy in an electrical motor or heat energy in a resistor.

But we will quantitatively treat the flows energy between kinetic and potential energy in § 5 on the work-energy theorem.

In simple kinetic energy examples, we often just name some energy source or sink without going into the details.

### 3.2. Example Work-Kinetic-Energy Theorem Calculation

Given the work-kinetic-energy theorem

$$\Delta KE = W = F\Delta x \ , \tag{27}$$

what is an formula for final speed $v$ in terms of initial speed $v_0$, mass $m$, and work done $W$?

You have 1 minute working in groups or individually. Go.

Behold:

$$\begin{aligned}
\Delta KE &= KE - KE_0 \\
KE &= \Delta KE + KE_0 \\
v &= \sqrt{\frac{2KE}{m}} = \sqrt{\frac{2(\Delta KE + KE_0)}{m}} \\
v &= \sqrt{\frac{2\Delta KE}{m} + v_0^2} \\
v &= \sqrt{\frac{2W}{m} + v_0^2} \ .
\end{aligned} \tag{28}$$

Now for a number problem.

We pull a block of mass $m = 6\,\mathrm{kg}$ with a net force $F = 12\,\mathrm{N}$ in the horizontal direction for a displacement $\Delta x = 3\,\mathrm{m}$. The block is on a frictionless horizontal surface. All other forces (gravity and the normal force) are perpendicular to the direction of motion, and so do **NO** work.

From the **WORK-KINETIC-ENERGY THEOREM**, the change in the block's kinetic energy is what?

You have 10 seconds. Go.

Behold:

$$\Delta KE = W = F\Delta x = 36\,\mathrm{J} \ . \tag{29}$$

What is the block's final speed $v$ given that its initial speed is $v_0 = 0$?

You have 30 seconds working individually or in groups. Go.

Behold:

$$v = \sqrt{\frac{2W}{m}} = \sqrt{\frac{2 \times 36}{6}} = \sqrt{12} \approx 3.5\,\mathrm{m/s} \ . \tag{30}$$

Of course, the above calculation could have been done using Newton's 2nd law to find a constant acceleration and then using the appropriate constant-acceleration kinematic equation (i.e., the timeless equation) to the find the speed. Recall, the timeless equation is

$$v^2 = v_0^2 + 2a\Delta x \ .$$

In fact, we could have gotten more information about the motion from using the 2nd-law approach. We would have could have found the whole kinematic evolution of the block. So the energy method gives less information in this case and is just an alternative method for finding the speed. But there will be cases as we'll see where energy gives us partial information very easily and we'd have to work very hard to get any information by 2nd-law approach.

## 3.3. Total Kinetic Energy

Note equation (23) $KE = (1/2)mv^2$ defined the center-of-mass kinetic energy (or commonly the translational kinetic energy) since $\vec{v}$ is the center-of-mass velocity. But this is not the total kinetic energy of the system, but just the kinetic energy associated with the center-of-mass velocity. One could add up all the kinetic energies of the parts of the system and that would be the total kinetic energy of the system.

The total kinetic energy is always greater than or equal to the center-of-mass kinetic energy.

To prove the last statement, let's decompose the system into arbitrary subsystems. The subsystems $i$ have masses $m_i$ and (center-of-mass) velocities $\vec{v}_i$. Let the subsystem velocities relative to the center-of-mass velocity $\vec{v}$ be $\vec{v}\,'_i$. Now

$$\vec{v}_i = \vec{v} + \vec{v}\,'_i \ , \tag{31}$$

of course. The total kinetic energy of the set of subsystems is

$$\begin{aligned} KE_{\rm tot} &= \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\vec{v} + \vec{v}\,'_i) \cdot (\vec{v} + \vec{v}\,'_i) \\ &= \sum_i \frac{1}{2} m_i (v^2 + 2\vec{v} \cdot \vec{v}\,'_i + v_i'^2) \\ &= \sum_i \frac{1}{2} m_i v^2 + \vec{v} \cdot \sum_i m_i \vec{v}\,'_i + \frac{1}{2} \sum_i m_i v_i'^2 \\ &= KE + \frac{1}{2} \sum_i m_i v_i'^2 \ , \end{aligned} \tag{32}$$

where the first term is the center-of-mass kinetic energy, the cross term vanished since a factor in it is the center-of-mass velocity of the whole system in the center-of-mass frame of reference of the whole system, and the third term is always greater than or equal to zero. The third term is only zero when all the relative velocities $\vec{v}\,'_i = 0$ which means that the subsystem center-of mass are moving in formation.

The proof shows that the total energy of any set of subsystems is always greater than center-of-mass kinetic energy of the system.

How small can we make the subsystems?

Well we could go right down the mythical classical point particles that often use as the starting point for physics derivations.

Such particles could be regarded as actual basic particles that are assigned their classical average behavior. Their quantum mechanical behavior is then eliminated.

In fact, the random kinetic energy of microscopic particles is part of the internal energy (AKA heat energy) and is never added to the macroscopic kinetic energy. So the particles in a macroscopic kinetic energy are just assigned the macroscopically non-random velocities—which is actually easy to do.

In actual calculations, the classical particles are treated in the continuum limit so that one can use integration.

## 4. POTENTIAL ENERGY

Potential energy is the energy of position of a particle in the region of space where there is a field force. For classical physics, the particle is a classical point particle. We will elaborate this definition below.

But why is potential energy called potential energy? Historical reasons I suppose. If one was doing the names all over again, one could more reasonably call it "position energy".

Remember a field force is a force that is defined at every point space or at least a region of space. We will just say space below for brevity.

Say that we have field force $\vec{F}$.

Say we have general points A and B in space.

Going from A to B along some path, the work done by the field force on a particle is

$$W_{\rm AB} = \int_{\rm A}^{\rm B} \vec{F} \cdot d\vec{s} \ , \tag{33}$$

where we note this is **NOT** the net work in general, but just the work done by the field force.

Now we assume that the work done by the field force is, in fact, path-independent for the particle moving from general point A to general point B. This means one gets the same work value for $W_{\rm AB}$ for any path.

Forces for which this assumption holds are called **CONSERVATIVE FORCES**. The term "conservative force" is probably used since only work done by conservative forces allow mechanical energy to be conserved (see § 5.1).

For a conservative force, the work done by the force going around any closed path is zero. The proof is simple. Say we did work $W_{\rm AB}$ going from A to B along a particular path.

Remember A and B are general points. The work going from point B to point A by along the same path in the reverse direction is

$$W_{\rm BA} = \int_{\rm B}^{\rm A} \vec{F} \cdot d\vec{s} = -\int_{\rm A}^{\rm B} \vec{F} \cdot d\vec{s} = -W_{\rm AB} \ , \tag{34}$$

where the differentials $d\vec{s}$ in the 2nd integral are implicitly the inverses of the differentials $d\vec{s}$ in the 1st integral. Clearly, $W_{\rm AB}$ and $W_{\rm BA}$ sum to zero. But since the work done going from A to B and the work going from B to A are both path independent, the sum $W_{\rm AB}$ and $W_{\rm BA}$ is zero for any path from A to B for $W_{\rm AB}$ and any path from B to A for $W_{\rm BA}$. Thus $W_{\rm AB}$ and $W_{\rm BA}$ sum to zero for any pair of paths. Thus work is also the work done going around any closed path from general point A to general point A is zero. This completes the proof.

One can also show that if the work done by a field force around any closed path is zero, then the force is conservative. Say one went from A to B along a path and came back along a fixed path and the work done was zero always. Then the work done going from A to B by any path must be the same since it must be canceled by the work coming back along the fixed path. So if one assumes that the work done around any closed path is zero, the work is done between any two points is path-independent. In fact, the two properties are equivalent in that one implies the other. So a conservative force has both properties for the work it does.

We now define **POTENTIAL ENERGY** as being the energy source for the conservative force. If the conservative force does positive work, the particle's potential energy is decreased. If the conservative force does negative work, the particle's potential energy is increased. The general formula for potential energy changes is

$$\Delta PE = -W \ , \tag{35}$$

where we have dropped arbitrary subscript labels, $W$ is the work done by the conservative force, and $\Delta PE$ is the potential energy change.

Yours truly likes to use the symbol $PE$ for potential energy since it is mnemonic, but some texts use $U$ and in the context of quantum mechanics and advanced classical mechanics one uses $V$ (e.g. Goldstein et al. 2002, p. 4). Potential energy is sometimes called potential, particularly in quantum mechanics and advanced classical mechanics. But in electromagnetism, the word potential is used for something else (the thing commonly called voltage), and we will stick to saying potential energy for potential energy.

Consistent with our discussion above, $W > 0$ reduces $PE$ and $W < 0$ increases $PE$.

Since the work done by the conservative force between any two points in space is path-independent, we see that the potential energy change between any two points in space is path-independent. Therefore a definite potential energy can be assigned to each point in space for the particle aside from an arbitrary additive constant. The arbitrary additive constant exists because general potential energy formula equation (35) gives us no absolute zero point of the potential energy. It only gives changes in potential energy between points in space. In fact, the zero point of potential energy is arbitrary. In any calculation, only differences in $PE$ affect the results, except for those results like the value of $PE$ itself which have arbitrary additive constants.

So as we'll see in the examples, the zero point is chosen either for convenience of a particular example or by a convenient convention that is widely used.

Equation (35) is the general potential energy formula. Particular conservative forces will each have their own particular potential energy formulae which will depend explicitly on position. These potential energy formulae are derived from the general potential energy formula for those particular conservative force or a postulated and the conservative force is derived from the potential energy formula. We will discuss the latter procedure in §§ 4.7 and 4.8.

Given how potential energy is defined, we see that it makes sense to call it the energy of position in a field of force. Note we usually just say potential energy is a function of position in space—and don't explicitly mention "for a particle" usually.

## 4.1. Potential Energy of Systems

We have defined potential energy for particles.

What can be done for systems?

Well the potential energy for systems can be calculated by summing the potential energy for the particles that make up the systems.

These particles don't have to be classical point particles which are tricky to handle in actual calculations. The particles are the bits that are small enough that one position fully specifies them.

If the system is a continuum, one can integrate up the potential energy contribution of the differential bits that make up the system.

Now in general doing the summing process might be difficult.

Can anything simpler be done?

One can approximate the system potential energy as being the potential energy for some representative point for the system. The center of mass seems a natural choice since that is consistent with using the center-of-mass kinetic energy—which is center-of-mass kinetic energy in the sense that the velocity used in evaluating it is the center-of-mass velocity.

The representative point approximation may be poor sometimes.

But if the potential energy per particle varies little over the extent of the system the approximation may be very good. In such cases, the potential energies of the parts of the system can be evaluated using just the representative point: i.e.,

$$PE = \sum_i PE_i(\vec{r}_i) \approx \sum_i PE_i(\vec{r}_{\rm rep}) \ , \tag{36}$$

where the parts are labeled by $i$, $\vec{r}_i$ is the position of part $i$, $PE_i(\vec{r}_i)$ is the potential of part $i$ evaluated at $\vec{r}_i$, $\vec{r}_{\rm rep}$ is the representative point position, and $PE_i(\vec{r}_{\rm rep})$ is the potential of part $i$ evaluated at $\vec{r}_{\rm rep}$.

Sometimes it is possible to find a representative point that gives exactly the potential energy of the whole system.

Fortunately, this is the case for the examples we consider: the potential for gravity near the Earth's surface (where the representative point is the center of mass) and the potential for the one-dimensional simple harmonic oscillator (where the representative point can be any point of the system if it is rigid—but the center of mass seems a good choice).

### 4.2. What is Potential Energy?

What is potential energy?

And where is it?

The question means can potential energy be understood in terms of other physical concepts.

In fact, usually yes.

For real physical conservative forces, potential energy is the energy associated with the field structures of those forces. The field is a real physical thing in modern physics, not just a mathematical auxiliary. The field is the cause of a field force recall. The energy of the structure of fields is mostly beyond our scope though it comes up in the second semester when the electric and magnetic fields are introduced.

Can you calculate the potential energy from the fields? Yes. One integrates a field energy density over space. So the energy of the field is spread out, in general unevenly, through the region of the field extends over. Calculations of field energy can be done for electric and magnetic fields easily enough. For gravity fields, it can be done too, but yours truly is essentially ignorant of the finer details of gravitational fields.

Actually, the field energy of fields is more fundamental/intrinsic to the fields than potential energy. As discussed in § 1.2 whether a potential energy can be used to describe the energy of a field (and one can describe the force as conservative) depends not only on the nature of the field, but on its structure (i.e., the distribution of field values in space and time). We discuss examples of forces and when they are conservative forces in § 4.4.

But whether a potential energy can be defined or not, one can calculate the field energy by an integration of the field energy density.

Potential energy can be described as shortcut way of calculating changes in the energy of the field structure. Sometimes the shortcut is available, sometimes not. When it is available, it can be very useful and allows one to avoid what may be awkward calculations. The downside of using potential energy is that rather hides the structure of the field energy and makes this energy a bit mysterious and not obviously locatable.

Can one find non-arbitrary zero points for potential energy by a field energy calculation?

Well in sense. But first note that having a zero field in a region does not mean that the natural choice of potential energy for that region is zero. The field energy density of that region is zero, and thus that region contributes nothing to an integral of the field energy. But the potential energy at a point (for a point particle) depends on the structure of the field as whole not just on its value at any particular point.

To give an example, the gravitational field inside an isolated hollow sphere (which has mass) is zero, but one usually doesn't assign zero potential energy to objects placed in that hollow—it isn't a natural assignment it turns out. The natural assignment is to set the zero potential at infinity (i.e., infinitely far from the sphere). We may consider gravitating hollow spheres the lecture *GRAVITY*—or maybe not.

It also seems natural to say the gravitational potential energy of two point particles is zero when they are at infinity relative to each other. Similarly, the electric potential energy of two point electric charges is naturally said to be zero when they are at infinity relative to each other. But there are ambiguities in classical physics. The electric field energy (and therefore the potential energy) of a classical point electric charge is infinite. Classical point particles don't actually exist which is a relief in this context. Quantum mechanical point particles may exist (e.g., the electron seems to be a quantum mechanical point particle) and their energy (i.e., there rest mass energy) is not infinite. But there are still infinities in quantum mechanical treatments that have been worked around by a theory called renormalization and not solved to everyone's satisfaction.

For ideal conservative forces that one just invents for heuristic purposes (e.g., the ideal linear force), the potential energy is a just-so: it can't be further explained.

## 4.3. What Field Forces are Conservative Forces?

As mentioned in §§ 1.2 and 4.2, whether a potential energy can be used to describe the energy of a field depends not only on the nature of the field, but on its structure (i.e., the distribution of field values in space and time).

For example, the Coulomb or electric force (which is caused by the electric field) is often said to be a conservative force, but it isn't in all contexts. For example, if the charges that cause the electric field are moving in the inertial frame of interest, then no potential energy can be defined over finite times—you can define potential energy instant by instant. One can define potential energy instant by instant and that may or may not be useful. Another example where no potential energy can be defined is when the electric field is caused by Faraday's law of induction which is a topic in the 2nd semester of intro physics.

The magnetic field is often said **NOT** give rise to a conservative force, but there are contexts in which it does. For example, a magnetic dipole in a magnetic field has a potential energy associated with its orientation with respect to the magnetic field direction. This is also a topic in the 2nd semester of intro physics.

Gravity is often called a conservative force, but like the Coulomb force, one can only define a potential energy if the sources of the forces (i.e., masses) are not moving in the inertial frame of interest. If the sources move, a potential energy is not strictly definable for finite time intervals. One can again define potential energy instant by instant and that may or may not be useful. If the source motion is slow enough, potential energy may be a valid approximation.

If the sources of the electric or gravity force are in some sort of steady stream, then a potential energy can be defined.

By the way, the potential energy of the Coulomb force is not a stranger to you. The potential energy per unit charge is called electrical potential and, more commonly, voltage. A 10-volt battery means that for every coulomb of charge that flows the battery, 10 J of electrical potential energy is transformed into some other form of energy. A coulomb is the standard SI unit of charge. It is a macroscopic unit of charge. A coulomb of charge flow per second is the familiar ampere or amp.

The one-dimensional linear force is conservative as well as we'll see below in § 4.6. But the linear force has only a one-dimensional field.

The linear forces often arise from the elastic forces for of materials. One usually thinks of such forces as the forces of a body that resist compression, extension, and shearing. Elastic forces can be regarded as field forces in the coordinate system defined by the body. Perfect elastic forces don't lose energy to waste heat through internal resistive forces and can have associated potential energies. At the microscopic level, elastic forces are manifestations of the electromagnetic force.

In the case of the ideal linear force of examples, there is no field energy other than what is described by the potential energy. This potential energy has no defined location in space. I suppose you could invent such field energy and locating in space, but there's no advantage I can think of in doing this. Now real linear forces arise from manifestations of fundamental forces as we'll discuss in § 4.6.

## 4.4. What Forces are Non-Conservative?

Well usually, one can say the ideal normal force, the ideal tension force, and friction are not conservative forces and do not have potential energies.

They are non-conservative forces.

You can use the first two to push or pull an object through space, but the work done is path-dependent. They also are not field forces. But one if one considers the normal force as a perfect elastic force of a non-rigid surface, then a potential energy can be defined. As you compress the surface potential energy is stored and as it is decompressed the potential energy is released. So context is important in deciding if a normal force has a potential energy.

The kinetic friction force always does negative work on an object relative to the surface over which the sliding occurs since the friction force always points opposite to the direction of motion on that surface. To put this in a formula form, the work done by kinetic friction is

$$dW_{\rm ki} = \vec{F}_{\rm ki} \cdot d\vec{s} = -F_{\rm ki}\, ds \ , \tag{37}$$

where $\vec{F}_{\rm ki}$ is the kinetic friction force and $d\vec{s}$ is a differential displacement between the surfaces. The work done clearly depends on the path and no potential energy can be defined.

The kinetic friction work is an outflow from the object's macroscopic kinetic energy bank. Where does that energy outflow go to?

Well to delve for a moment into the microscopic level, chemical bonds form between the sliding surfaces. This bond formation is actually a transformation of microscopic electric potential energy between the atoms of the surfaces into heat energy. So the heat energy that heats the surfaces comes from the bond formation. To break the bonds, the electrical potential energy between the atoms must be restored. The energy to do the restoration comes from the macroscopic kinetic energy bank of the sliding object. From another point of view the bonds exert a force on the sliding object that does negative work on the object, and thus removes energy from the object's kinetic energy bank.

Of course, both static and kinetic friction can do positive work on an object in certain very common contexts. Just think of block in the back of the pickup truck. As the truck accelerates, the static friction force does work to accelerate the block if the block doesn't slide. If the block slides back in the truck, kinetic friction force does negative work relative to the truck. But as long as the block still goes forward relative the ground, kinetic friction is doing positive work relative to ground.

### 4.5. Gravitational Potential Energy

Here we are concerned only with gravity near the Earth's surface where the force law for gravity on a classical point particle is

$$\vec{F} = mg(-\hat{y}) \ , \tag{38}$$

where $m$ is the particle mass, $g$ is gravitational field magnitude (acceleration due to gravity) idealized as **CONSTANT** (with fiducial value 9.8 N/kg), and $\hat{y}$ is a unit vector pointing in the upward vertical direction.

Going from A to B along some path, the work done by gravity on the particle is

$$W_{\rm AB} = \int_{\rm A}^{\rm B} \vec{F}\cdot d\vec{s} = \int_{\rm A}^{\rm B} mg(-\hat{y})\cdot d\vec{s} = -\int_{y_{\rm A}}^{y_{\rm B}} mg\,dy = -mg\Delta y_{\rm AB} \ , \tag{39}$$

where $\Delta y_{\rm AB} = (y_{\rm B} - y_{\rm A})$ is the difference in $y$ coordinates between the endpoints. The work done is, in fact, independent of the path as the explicit evaluation of the integral shows. In fact, it's spatial dependence is only on the $y$ coordinates of the endpoints.

Motion in the horizontal direction contributes no work. Any path loops cancel out.

Since the work is path independent, a potential energy can be defined for gravity.

Using the potential energy formula equation (35) $PE = -W$ and dropping the arbitrary subscripts gives

$$\Delta PE = mg\Delta y \tag{40}$$

for gravitational potential energy changes. There is no conventional zero point of gravity near the Earth's surface, and so the zero point is chosen for convenience. Often the ground level is the convenient choice.

We see that gravitational potential energy increases with height.

What about a finite system?

Well say the system is made of classical point particles $i$ and has total mass $m$.

The masses of the particles are $m_i$ and in some displacement of the system the particles undergo height displacements $\Delta y_i$. The total change in potential energy of the system is

$$\Delta PE = \sum_i m_i g \Delta y_i = mg \left( \frac{\sum_i m_i y_i}{m_i} \right) = mg\Delta y \ , \tag{41}$$

where $\Delta y$ is the center of mass displacement.

So calculating potential energy for gravity near the Earth's surface is simple. One just needs to keep track of the center-of-mass position.

The potential energy for a system is the same as for a point particle with the meaning of the factors changed appropriately: i.e.,

$$\Delta PE = mg\Delta y \ , \tag{42}$$

where $m$ is total mass and $y$ is center of mass height.

### *4.5.1. Example Gravitational Potential Energy Calculation*

Say you raise 100 kg object through 1 m.

This could be for example a large man climbing about 3 steps.

The gravitational potential energy change is what?

You have 10 seconds. Go.

Behold:

$$\Delta PE = mg\Delta y \approx 100 \times 10 \times 1 = 1000\,\text{J} \ . \tag{43}$$

## 4.6. Linear Force Potential Energy

As you should recall from the lecture *NEWTONIAN PHYSICS I*, the linear force is a force of many names: linear force, linear restoring force, spring force, Hooke's law force, and simple-harmonic-oscillator force.

For one dimension, it is

$$F = -kx \ , \tag{44}$$

where $x$ is the displacement of an object from the equilibrium position (where the linear force is zero) along an $x$ axis and $k$ is the force constant. The linear force is one-dimensional field force since it is defined everywhere in a one-dimensional space.

What point in the object should $x$ be?

If we assume a rigid object where all parts are constrained to move in formation, any point will do.

That point will have an equilibrium position and the force varies varies linear with the point's displacement from equilibrium.

You could make the point the center of mass or for an ideal spring or the point where the object is attached to the spring. The center of mass is the probably best choice since that is consistent with all of our developments.

The linear force is extremely important since the physical parts of stable static structures (i.e., those that resist distortion) are usually acted on by the linear force (for one or more dimensions) for relatively small displacements from their stable equilibrium positions (i.e., the positions where all forces cancel). We discussed this point in the lecture *NEWTONIAN PHYSICS II* and well discuss it again in § 7.1

To find the potential energy of the linear force, we consider the work done on an object moved from a zero-point $x_0$ to general $x$ along some path in one-dimension which may include loops. This work is

$$\begin{aligned} W &= \int_{x_0}^{x} \vec{F}\cdot d\vec{s} = \int_{x_0}^{x} (-kx'\hat{x})\cdot(dx\hat{x}) \\ &= \int_{x_0}^{x} (-kx')\,dx' = -\frac{1}{2}kx'^2\Big|_{x_0}^{x} \\ &= -\frac{1}{2}k(x^2 - x_0^2) \ . \end{aligned} \tag{45}$$

The work done is, in fact, independent of the path as the explicit evaluation of the integral shows. Any one-dimensional loops in the get canceled out since again $\vec{F}\cdot d\vec{s}+\vec{F}\cdot(-d\vec{s}) = 0$. The work depends only on the $x$ coordinates of the endpoints.

Using the general potential energy formula equation (35), we find

$$\Delta PE = -W = \frac{1}{2}k(x^2 - x_0^2) \ . \tag{46}$$

There is an obvious conventional zero point for the linear force: i.e., the equilibrium point which can often be chosen to be $x = 0$. Thus, usually one writes the absolute linear force potential energy as

$$PE = \frac{1}{2}kx^2 \ . \tag{47}$$

The linear force potential energy is parabolic.

Really we have to wait until we have the work-energy theorem (§ 5) before we can do any interesting calculations with the linear force potential energy.

In the case of the ideal linear force of examples, there is, as mentioned in § 4.3, no field energy other than what is described by the potential energy. This potential energy has no defined location in space. I suppose you could invent such field energy and locating in space, but there's no advantage I can think of in doing this.

Now real linear forces arise from manifestations of fundamental forces. For example in real springs, it is the electromagnetic force between the atoms that is the ultimate source of the linear force. So, in fact, the spring potential energy is actually electromagnetic field energy. The linear force can arise as manifestations of other fields too: gravity and the strong nuclear force. I don't know about the weak nuclear force. An example of a linear force from gravity is the case of the simple pendulum discussed in the *NEWTONIAN PHYSICS II*.

### 4.7. Deriving Force from Potential Energy

Obviously, we have the differential result

$$dPE = -dW = -\vec{F} \cdot d\vec{s} \ , \tag{48}$$

where $\vec{F}$ is the force that has the potential energy the equation defines.

In one dimension, equation (48) becomes

$$dPE = -dW = -F dx \ , \tag{49}$$

where the **POSITIVE DIRECTION** for the force is the positive $x$ direction and the **NEGATIVE DIRECTION** for the force is the negative $x$ direction.

We find immediately, that

$$F = -\frac{dPE}{dx} \ . \tag{50}$$

So given a functional form for $PE$, the force can be obtained by differentiation.

The minus sign in equation (50) seems formally annoying, but it indicates an important fact.

The force points in the direction that the potential energy decreases in.

If the $PE$ is increasing with $x$, the force points in the negative direction.

If the $PE$ is decreasing with $x$, the force points in the positive direction.

As an example, let's derive the gravitational force from the gravitational potential energy. We get

$$F = -\frac{dPE}{dx} = -\frac{d(mgy)}{y} = -mg \tag{51}$$

which is what expected.

In multi dimensions, it turns out that the force points in the direction of fastest spatial decrease in potential energy.

This gives a useful mental picture. As we'll discuss in § 6, one can view the potential energy as forming a landscape with high potential energy regions forming hills and low potential energy regions forming valleys or—as they are usually called **WELLS**.

The force derivable from the potential energy always points down the hills and to the bottoms of the wells.

This mental image is a useful mnemonic since it corresponds to our usual way of thinking with real hills and valleys with the combined force of gravity and the surface normal force being a force that points down the hills. This combined force is actually a conservative force since moving along the land surface the normal force does no work and the gravity component all the surface is conservative. Real hills and valleys (which can be called wells in this context) with gravity are actually a special case of a potential energy landscape.

## 4.8. Deriving Force from Potential Energy in Three Dimensions: Reading Only

This is verging on being out of the scope of this course—but not quite.

Just as in § 4.7 (see eq. 48), we have the differential result

$$dPE = -dW = -\vec{F} \cdot d\vec{s} \;, \tag{52}$$

where $\vec{F}$ is the force that has the potential energy the equation defines.

We now expand $d\vec{s}$ in a Cartesian coordinate system:

$$d\vec{s} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz \tag{53}$$

and then we find

$$dPE = -(F_x\,dx + F_y\,dy + F_z\,dz) \tag{54}$$

from which it follows that

$$\vec{F} = -\left(\frac{\partial PE}{\partial x}\hat{x} + \frac{\partial PE}{\partial y}\hat{y} + \frac{\partial PE}{\partial z}\hat{z}\right) \ , \tag{55}$$

where the $\partial$ is the partial derivative derivative. A partial derivative is a derivative with respect to only one variable when the function depends on multiple variables.

To see non-rigorously that equation (54) is correct consider equation (55) with only one differential at a time non-zero.

The gradient operator of vector calculus for Cartesian coordinates is defined

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \ . \tag{56}$$

This definition can be generalized to all orthogonal coordinates (e.g., spherical polar and cylindrical).

The gradient is in some respects the three-dimensional analog of the derivative in one dimension.

Thus, we get the general result

$$\vec{F} = -\nabla PE \ : \tag{57}$$

the force is the minus the gradient of the potential energy.

The gradient of a function actually points in the direction of greatest spatial increase of a function which we will not prove—but it's not very hard to do that—hold on until vector calculus catches up with you.

The negative sign means that force points in the direction of fastest potential energy spatial decrease.

The mental picture of the potential energy landscape that goes with this understanding of the force derivable from a potential energy was discussed above in § 4.7.

## 5. THE WORK-ENERGY THEOREM

This section is the biggy for this lecture—well one of the biggies.

Recall the **WORK-KINETIC-ENERGY THEOREM** equation (25),

$$\Delta KE = W \ , \tag{58}$$

where $W$ is the net work (i.e., the net work done by all forces) during displacement of a system center of mass and $\Delta KE$ is the change in center-of-mass kinetic energy of the system during that displacement. The work is by the net force on the system and for a displacement of the center of mass.

Nothing forbids us from decomposing $W$ into the work done by conservative forces and the work done by non-conservative forces. We do this thusly:

$$W = W_{\rm con} + W_{\rm non} \ , \tag{59}$$

where $W_{\rm con}$ is the work done by conservative forces and $W_{\rm non}$ is the work done by non-conservative forces.

Actually, we could decompose the net work $W$ in any way we like among the work contributions by various forces. Sometimes that turns out to be a useful trick. For example, it sometimes useful to include only some the conservative forces in $W_{\rm con}$ and put the others in $W_{\rm non}$ treating them as non-conservative even though they are conservative—nothing forbids us from doing this.

Recalling the general potential energy formula (eq. (35) in § 4), we can find the potential energy change due to the conservative forces: i.e.,

$$\Delta PE = -W_{\rm con} \ , \tag{60}$$

where $PE$ is the potential energy for the conservative forces that do work $W_{\rm con}$.

A tricky point now. If the potential energy for a system can be evaluated accurately enough using the center-of-mass position as a representative point for the system, then making use of potential energy to replace $-W_{\rm con}$ is straightforward. If not, not then not. The displacement of parts of the system may need to be treated in accurately evaluating the changes in potential energy.

Now we substitute equation (60) into equation (59)—with the assumption that all conservative potential energy forms have been included in equation (60) or at least those we wanted to consider conservative—and then substitute that into work-kinetic-energy theorem equation (25) to get

$$\Delta KE = -\Delta PE + W_{\rm non} \ . \tag{61}$$

Now that $\Delta PE$ is just begging to be moved to the other side of the equal sign.

We do this and get the **WORK-ENERGY THEOREM**

$$\Delta KE + \Delta PE = W_{\rm non} \ . \tag{62}$$

It is now convenient to define **MECHANICAL ENERGY** as the sum of kinetic and potential energy. Thus,

$$E = KE + PE \ , \tag{63}$$

where one can often distinguishes $E$ with a subscript to make it explicit that mechanical energy is meant, not just any old energy: e.g., $E_{\rm mech}$. But for general formulae, I prefer a clean look with context giving the meaning. Mechanical energy is an example of a form of energy that subsumes two other forms (i.e., kinetic energy and potential energy), and thus is illustrates the difficulties encountered in drawing up a simple list of energy forms.

With the definition of **MECHANICAL ENERGY**, we can now write the **WORK-ENERGY THEOREM** as

$$\Delta E = W_{\rm non} \ . \tag{64}$$

Either equation (62) or equation (64) is fine to remember as the **WORK-ENERGY THEOREM**: equation (64) is more compact, but one often has to decompact it to equation (62) anyway in solving problems.

To emphasize, equation (64) is the **WORK-ENERGY THEOREM** for a system as whole. The mechanical energy is the sum of center-of-mass potential energy and the total potential energy of the external forces. We are not counting the internal kinetic energy of the system due to rotations, vibration, or other motions of the parts. We are not counting the potential energy of internal forces.

A tricky point redux. If the potential energy for a system can be evaluated accurately enough using the center-of-mass position as a representative point in the system, then making use mechanical energy and the **WORK-ENERGY THEOREM** is straightforward. If not, not then not. The displacement of parts of the system may need to be treated in accurately evaluating the changes in potential energy.

The **WORK-ENERGY THEOREM** can be generalized to include internal kinetic energies in some cases. The only case I really know of is that of a rigid-body roller (e.g., a sphere or cylinder) that is rolling without slipping on some surface. Here one can include the rotational kinetic energy in the **WORK-ENERGY THEOREM**. Because of the noslip condition, the roller center-of-mass velocity $v$ (i.e., the translational velocity) and the angular velocity $\omega$ are related by $v = r\omega$, where is $r$ radius from the roller axis to the surface on which the roller is rolling.

### 5.1. Conservation of Mechanical Energy

Now overall energy is always conserved for a closed system. This is consequence of the principle of conservation of energy and an empirical fact as far as we know.

But it—energy—can be transformed among all forms in principle. In macroscopic systems, it often transforms into heat energy and often heat energy is not transformed back into any macroscopic form of the energy—in which case we say the system has dissipated energy to waste heat.

But if non-conservative forces do **NO** net work, then that special form of energy **MECHANICAL ENERGY** is conserved just by itself—then we have conservation of **MECHANICAL ENERGY**.

From the **WORK-ENERGY THEOREM** with $W_{\rm non} = 0$, we have

$$\Delta E = \Delta KE + \Delta PE = 0 \ . \qquad (65)$$

The conservation of mechanical energy can often be used to obtain answers to problems as an alternative to simply employing Newton's laws.

Often the alternative is no better, but often it is.

In particular, it often turns out that using conservation of **MECHANICAL ENERGY** is useful in obtaining partial information about the motion of a system easily when full information about the motion is very hard to obtain.

These cases often arise when there are non-conservative forces around acting, but **NOT** doing net work. The non-conservative forces are often guiding the motion in such cases. In physics jargon, they are **CONSTRAINT FORCES**. When they do no work, they are **WORKLESS CONSTRAINT FORCES**.

The conservation of mechanical energy equation (65) can be rewritten

$$\Delta KE = -\Delta PE \ . \tag{66}$$

Say one defines $\Delta PE_{\text{inverse}} = -\Delta PE$, then one gets

$$\Delta KE = \Delta PE_{\text{inverse}} \ , \tag{67}$$

where $\Delta PE_{\text{inverse}}$ means the additive inverse of $\Delta PE$. This formula is often useful in simple calculations. Say a ball drops from rest by $|\Delta y|$ where $\Delta y < 0$ is the actual displacement. It's final kinetic energy is

$$KE = \Delta KE = -\Delta PE = -mg\Delta y = mg|\Delta y| = \Delta PE_{\text{inv}} \ . \tag{68}$$

Unfortunately, students unclear on concept often use equation (67) without understanding what the variables mean—frequently on tests. They often just say any amount of potential energy equals any amount of kinetic energy and vice versa.

Do **NOT** do that.

Equation (65) is the basic conservation of mechanical energy equation. It should be the formula memorized and used in general. Only equation (67) when one is clearly understands what the variables mean.

### *5.1.1. Example 1: A Falling Ball*

There is ball of mass $m$ that we drop from an initial height $y_0$ to $y$. We take upward as the positive direction, and so $y - y_0 < 0$. The ball is in free fall, and thus gravity is the only external force.

There are no **CONSTRAINT FORCES**.

Since gravity is a conservative force, **MECHANICAL ENERGY** is conserved.

Say the initial height of the ball is $y_0$ and the initial speed of the ball is $v_0$.

What is the speed $v$ of the ball at any other height $y$ as a function of $y$, $y_0$, $v_0$, $g$ and $m$?

The height and speed are center-of-mass values, of course.

You have 1 minute working in individually or groups? Go.

We can apply, the conservation of mechanical energy equation (65):

$$\Delta E = 0 \tag{69}$$

which implies

$$E = KE + PE = KE_0 + PE_0 = E_0 \ , \tag{70}$$

where the unadorned symbols are for any time and the 0 subscripted symbols apply at time zero.

Solving for the general speed $v$ gives

$$\begin{aligned} KE &= KE_0 - (PE - PE_0) \\ v &= \sqrt{\frac{2KE_0}{m} - 2g(y - y_0)} \\ v &= \sqrt{v_0^2 + 2g(y_0 - y)} \ , \end{aligned} \tag{71}$$

where we have used the fact that gravitational potential energy is given by

$$PE = mgy \ , \tag{72}$$

where $g$ is gravitation field (or acceleration due to gravity) which we assume is constant since the problem is implicitly a near-Earth-surface problem. The zero point of the potential energy is the arbitrary zero of our height scale. Recall that the zero point for potential energy can always chosen for convenience or convention.

Equation (71) is mass independent.

Should we have expected this?

Yes.

All the terms in the conservation of mechanical energy are homogeneously linear in mass. So mass cancels out. So canceling mass is to be expected with the only potential energy in a system conserving mechanical energy is the potential energy of gravity.

In this case, we know the ball was moving down, and so knew that the solution formula for velocity is

$$v = -\sqrt{v_0^2 + 2g(y_0 - y)} \ , \tag{73}$$

where $v$ means velocity now and not speed.

We could have been obtained equation (73) using Newton's laws (which would have given a constant acceleration $a = -g$) and then using the timeless equation of the constant-acceleration kinematic equations.

Equation (71) is exactly the result the timeless equation would have given in fact.

In this case, the solution by either mechanical energy conservation or Newton's laws takes about the same amount of work.

But the Newton's laws approach would have given us more. We could have found the whole time evolution of position and velocity from that approach: i.e., full information.

Now the energy method can be made to give full information—but that takes formalism well beyond the scope of this course. But as we'll see for other systems, getting partial information from energy approach is often useful.

Say $v_0 = 0$ and $y_0 - y = 10\,\mathrm{m}$.

What is $v$?

You have 10 seconds. Go.

Behold:

$$v = \sqrt{2g(y_0 - y)} \approx \sqrt{2 \times 10 \times 10} = \sqrt{200} \approx 14\,\mathrm{m/s}\ . \tag{74}$$

### *5.1.2. Example 2: The Bead-on-Wire System*

Now for a much more interesting example that illustrates the power of the conservation of mechanical energy method.

There is bead of mass $m$ on a frictionless, fixed wire—which is in the near-Earth-surface environment—which is always the case pretty well unless we say otherwise.

The wire curves around in three-dimensional space. The bead freely slides along it.

The wire has no cusps.

What forces act on the bead?

Gravity—a conservative force.

The normal force of the wire. It's not a conservative force—no potential energy can be associated with it.

But the normal force of the wire does **NO** work on the bead:

$$dW_{\mathrm{normal}} = \vec{F}_{\mathrm{normal}} \cdot d\vec{s} = 0 \tag{75}$$

always since the bead path vector $d\vec{s}$ is always perpendicular to the normal force.

The normal force in this case is a **WORKLESS CONSTRAINT FORCE**: it does no work on the bead, but it does guide the bead's path through space.

The friction force? It's non-conservative, but since it's zero, it does no work.

Since the non-conservative forces do **NO** net work—although the normal force provides a **WORKLESS CONSTRAINT FORCE**—we can apply the conservation of **MECHANICAL ENERGY**.

We want to solve for the speed $v$ for any height $y$ given initial speed $v_0$ and height $y_0$.

You have 1 minute working individually or in groups. Go.

By steps absolutely identical to those in the falling ball example (§ 5.1.1), we arrive again at

$$v = \sqrt{v_0^2 + 2g(y_0 - y)} \ . \tag{76}$$

Note we only get the speed, not the velocity. We don't know which way on the wire the bead is moving just from conservation of mechanical energy alone.

Fig. 2.— Bead on a frictionless wire.

Solving for $y$ given $v$, $v_0$ and height $y_0$, one obtains

$$y = y_0 + \frac{(v_0^2 - v^2)}{2g} \ . \tag{77}$$

Note that again mass has canceled out of equations (76) and (77) as it does when gravity is the only force in a case of conservation of mechanical energy.

Now for a really important point.

In falling ball example, $g$ was the actual constant acceleration magnitude of the falling ball as well the gravitational field magnitude and parameter in the gravitational force law and gravitational potential energy formula.

In that context, equation (76) was also a special case of the timeless equation of the constant-acceleration kinematic equations as well as being a constant mechanical energy result.

But in the present example, the acceleration is **NOT** constant.

The acceleration magnitude is not $g$.

**Question:** What is the acceleration along the wire of the bead at any point?

a) $\pm g$ everywhere.

b) $g$ downward everywhere.

c) $g \sin\theta$ downward everywhere, where $\theta$ is angle of the wire from the horizontal at the current position.

Yes, it's (c).

Our conservation of **MECHANICAL ENERGY** analysis has allowed us to get partial information (i.e., eq. (76) and (77)) about the bead's motion very easily. We can get even more partial information easily about the bead-on-wire system using energy methods as we'll see in § 6.1.

It is one of the great benefits of conservation of mechanical energy analysis in general that one can get partial information easily. Actually, it is generally true that conservation laws (e.g., conservation of mechanical energy, conservation of total energy, conservation of momentum, conservation of angular momentum) give partial information easily.

We cannot get complete information from conservation of mechanical energy: we won't get position, velocity, and acceleration as functions of time.

In fact, to get such information we would have to specify the whole path of the wire and the initial conditions of the bead fully (i.e., initial velocity and position). And then we'd have to do a lot of work.

There is in general no analytic solution for full information to the general bead-on-wire system. In general, one would need a numerical solution. We won't go there.

Of course, special cases can be done. Say the wire was straight. In this case, $\theta$ is constant and the problem becomes a constant acceleration problem.

Here's another question for the class.

**Question:** What is the maximum height $y$ of the bead?

a) $y_{\rm max} = y_0 + v_0^2/(2g)$.

b) $y_{\rm max} = y_0$.

c) 0.

Yes, it's (a). This is the maximum value of $y$ according to equation (77). Also this value for $y$ causes the expression under the radical sign to be zero in equation (76): any larger value $y$ would give an imaginary speed and kinetic energy. Obtaining the upper limit on $y$ is another example of the partial information that energy analysis allows.

At the maximum height, all the bead's energy is potential energy and none is kinetic energy.

Here's still another question for the class.

**Question:** When the bead is at a specified height $y$, its speed:

a) is 0.

b) depends on acceleration.

c) is the same no matter what the bead's location is in other coordinates.

Yes, it's (c).

We'll return to the bead-on-wire system in § 6.

## 5.2. Non-Conservation of Mechanical Energy

If non-conservative forces do work, then mechanical energy energy is **NOT** conserved.

Let's consider kinetic friction which is a prime example.

Recall kinetic friction is caused by the formation of chemical bonds as discussed in § 4.4.

The bond formation causes the transformation of microscopic electrical potential energy into heat energy when the bond forms.

The heat energy is randomized microscopic energy as seen from the macroscopic scale and cannot easily nor entirely be converted back into macroscopic energy.

In many ordinary human situations of friction, the heat energy created by bond formation is not recovered in any direct sense at the macroscopic level.

A sliding object on a surface breaks the chemical bonds between object and surface almost immediately upon formation. The bond forces point opposite the direction of motion relative of object and surface, and so do negative work on the object and reduce it's kinetic energy.

### *5.2.1. Example of Mechanical Energy Loss due to Kinetic Friction*

Say we have an object of mass $m$ sliding on a horizontal surface.

We can set the surface to zero gravitational potential energy for convenience.

It's initial kinetic energy is $KE_0$.

Only the kinetic friction force does work on the object. The coefficient of kinetic friction $\mu_{\rm ki}$.

The kinetic friction force always points opposite to the direction of motion.

Thus, the work done by friction over displacement $\Delta\vec{s}$ of object relative to surface is

$$W = \vec{F}_{\rm ki} \cdot \Delta\vec{s} = -\mu_{\rm ki} mg\Delta s \ , \tag{78}$$

where $mg$ is the magnitude of the normal force in this case and $\Delta s$ is just a magnitude.

By the work-energy-theorem with no change in potential energy, we find that

$$\Delta KE = W_{\rm non} = -\mu_{\rm ki} mg\Delta s \tag{79}$$

Given initial kinetic energy is $KE_0$, how far does the object travel before it comes to a stop where it has zero kinetic energy?

You have 30 seconds working individually. Go.

Behold:

$$\begin{aligned} \Delta KE &= 0 - KE_0 = -\mu_{\rm ki} mg\Delta s \\ \Delta s &= \frac{KE_0}{\mu_{\rm ki} mg} \ . \end{aligned} \tag{80}$$

If we were given the initial speed $v_0$, then

$$\Delta s = \frac{v_0^2}{2\mu_{\rm ki} g} \tag{81}$$

which is independent of mass.

The independence of mass is, of course, because the kinetic frictional force is homogeneously linearly dependent on mass in this context.

Say $v_0 = 10\,{\rm m/s}$ and $\mu_{\rm ki} = 0.3$, what is $\Delta s$?

You have 30 seconds working individually. Go.

Behold:

$$\Delta s = \frac{v_0^2}{2\mu_{\rm ki} g} \approx \frac{100}{2 \times 0.3 \times 10} \approx 16.7\,{\rm m} \ . \tag{82}$$

We can complicate things in finding the stopping distance for the object by making the surface an incline at angle $\theta$ from the horizontal. To be definite, we say $\theta \geq 0$ if the surface rises in the direction of motion and $\theta < 0$ if the surface decreases in the direction of motion.

We also consider $\Delta s$ to be a one-dimensional displacement (rather than a magnitude of displacement) that is positive in the direction of motion which is opposite to the direction of kinetic friction force. We do this to use $\Delta s$ in describing vertical displacement. We assume the direction of motion does not change.

Now from the work-energy theorem, we find that

$$\Delta E \ = \ 0 - KE_0 + mg\Delta y = -KE_0 + mg\Delta s \sin\theta = -\mu_{\rm ki} mg \cos\theta \Delta s$$

$$
\begin{aligned}
KE_0 &= \Delta s m g(\sin\theta + \mu_{\rm ki}\cos\theta) \\
\Delta s &= \frac{KE_0}{mg(\sin\theta + \mu_{\rm ki}\cos\theta)} \\
\Delta s &= \frac{v_0^2}{2g(\sin\theta + \mu_{\rm ki}\cos\theta)} \ .
\end{aligned} \tag{83}
$$

The last equation reduces to our horizontal surface equation if $\theta = 0$.

**Question:** What does it mean if $\sin\theta + \mu_{\rm ki}\cos\theta = 0$ which can happen for $\theta < 0$?

a) The analysis fails.

b) The net force on the object is zero and it continues moving at a constant velocity forever: i.e., $\Delta s = \infty$. It can't have zero velocity since we assumed kinetic friction.

c) The object explodes.

Yes, it's (b). The gravitational force and the kinetic friction force cancel each other.

What is the net force along the slope?

Well

$$
F_{\rm net} = -mg\sin\theta - \mu_{\rm ki} mg\cos\theta = -mg(\sin\theta + \mu_{\rm ki} mg\cos\theta) \ , \tag{84}
$$

where we note that the gravitational term is a downhill force in all cases and the kinetic friction term is a force opposing the direction of motion (which we assume does not change). From the net force, we can see explicitly that the velocity will be constant and stopping distance will be infinite if $\sin\theta + \mu_{\rm ki}\cos\theta = 0$.

If $\Delta s$ becomes negative (which requires $\theta < 0$) in equation (83), then that just means that the object was at rest in the past when the object was higher up the slope. In this case, the gravity force down the slope is greater than the kinetic friction force up the slope and the object never comes to rest after it starts to slide and accelerates downhill forever.

Note that mechanical energy keeps decreasing in the case of $\Delta s < 0$ because of the loss to waste heat by friction. There is a net lost in energy. But the loss is all from the potential energy form. The kinetic energy keeps increasing.

We can show this explicitly using the the work-kinetic-energy theorem (eq. (25) in § 3). The kinetic energy for $|\Delta s|$ (i.e., the magnitude of the stopping distance for a stopping in the past which requires$\theta < 0$ $(\sin\theta + \mu_{\rm ki}\cos\theta) < 0$) is given by

$$\Delta KE = -mg|\Delta s|\sin\theta - \mu_{\rm ki}mg\cos\theta|\Delta s| = -mg(\sin\theta + \mu_{\rm ki}\cos\theta)|\Delta s| \ . \tag{85}$$

We can see that $\Delta KE$ grows as $|\Delta s|$ increases.

Sometimes the equations take some thought to understand.

## 6. POTENTIAL ENERGY LANDSCAPES AND ENERGY DIAGRAMS

Can we learn anything more about the bead-on-wire system from § 5.1.2 without a detailed analysis?

Yes.

Say we observed the bead at initial height $y_0$ and speed $v_0$, but we also observed its initial direction of motion.

To be definite let's define the initial direction as positive.

The bead will travel along slowing down as it rises hills, speeding up as it descends valleys, and staying constant as it travels on the level plains.

The literal hills, valleys, and plains of the wire are also potential energy hills, valleys, and plains in the case of the bead-on-wire system since

$$PE = mgy : \tag{86}$$

i.e., potential energy is proportional to height.

Thus, there is a **POTENTIAL ENERGY LANDSCAPE** that directly corresponds to the actual elevation landscape of the bead-on-wire system.

The concept of a potential energy landscape can be generalized to all potential energy cases.

In this generalized potential energy landscape, hills, valleys, and plains are just features in potential energy as a function of space and not literal features in general.

In fact, the expression "potential energy landscape" is not all that common, but the concept is under whatever name a person chooses. We use "potential energy landscape" for convenience in our discussions.

One common name is **WELL** which is a potential energy valley. People talk of potential energy **WELLS** all the time.

A picture of a **POTENTIAL ENERGY LANDSCAPE** is an **ENERGY DIAGRAM**.

For two-dimensional or three-dimensional cases, it's hard to draw a fully representative **ENERGY DIAGRAM**, but partially representative ones or schematic ones are often easily drawn and are useful in contemplating the potential energy landscape.

For the bead-on-the-wire system the **ENERGY DIAGRAM** is comparatively simple since the potential energy is proportional to height, and so in two dimensional cases ($x$ and $y$) can be plotted on a two-dimensional diagram of $PE$ versus $x$.

To quantitatively accurate for two-dimensional or three-dimensional cases, one must resort to complex visualization techniques. But qualitative accuracy can be obtained by a simply schematically visualizing the potential energy as varying with one coordinate as we can do with the bead-on-wire system.

As well as potential energy, one can put other system energies on **ENERGY DIAGRAMS**.

A common situation is to plot the mechanical energy for mechanical energy conserving systems. In these cases, the mechanical energy just has a constant value. For one-dimensional cases, the mechanical energy is just a horizontal line.

But conventionally one only plots the mechanical energy for only for locations that the system can actually reach. The system cannot reach locations where the kinetic energy would have to be negative to conserve mechanical energy since there is no negative mechanical energy. The difference between the mechanical energy and the potential energy values at any point is the kinetic energy, of course.

## 6.1. The Bead-on-Wire System Redux

To return to the bead-on-wire system again.

Now the bead will continue traveling along the wire in the positive direction until the speed goes to zero.

**Question:** When would the speed go to zero?

a) When the bead returned to $y_0$.

b) Never.

c) When the bead reached $y_{\rm max}$.

Yes, it's (c), but also maybe (b).

In order or the direction of motion to change, the velocity in the direction of motion must go to zero, and so the speed must go to zero.

This can only happen at $y_{\text{max}}$.

Of course, the bead may never encounter $y_{\text{max}}$ and may just travel on forever in the positive direction.

Speed zero happens when the wire is rising in the direction the bead is moving, and so its acceleration points opposite to the direction of motion (i.e., in our defined negative direction).

This means that the bead will change directions at that point of zero speed since the acceleration is negative.

The point of changing direction is called a **TURNING POINT**.

After the turning point the bead will head off in the negative direction until its speed goes to zero again—unless there are no other $y_{\text{max}}$ points in the negative direction in which cases it just keeps going in the negative direction forever.

The new zero-speed point is another **TURNING POINT**, but with a positive acceleration since the wire is now rising in the negative direction at the zero-speed point. After this **TURNING POINT**, the bead heads back in the positive direction.

As time passes, the bead will just oscillate back and forth between the two **TURNING POINTS**—unless in one direction there is no $y_{\text{max}}$ in which case, as already understood, the bead will just head off in that direction forever.

In the first case, the system is **BOUND**.

In the second case, the system is **UNBOUND**.

In the bound case, the oscillation is actually exactly periodic since at every point, the bead has exactly the same velocity as when it was at that point before going in the same direction. Therefore integral of the velocity with respect to time must necessarily result in exactly periodic motion. But finding the period of the motion requires an exact specification of the wire a detailed calculation with Newton's 2nd law which in many cases must be done numerically.

The only possible deviation from the above scenario for the bead-on-wire system is if the bead speed goes to zero exactly at local maximum of the potential.

At a potential maximum, the net force on the bead goes to zero.

So the bead will come to rest and stay at the local maximum.

To come to such an ideal resting place would take ideal fine-tuning of the bead's initial conditions.

The slightest deviation from rest at the potential maximum and the bead will slide down from it one way or the other.

If one plots the mechanical energy on the bead-on-wire **ENERGY DIAGRAM** it is just a line segment between the turning point potential energy values. The difference between the mechanical energy line and the potential energy curve is the kinetic energy, of course. At the turning points the kinetic energy is zero.

The picture of a system moving about in a potential energy landscape generalizes to any system with a significant potential energy. If the system can escape to infinity, it is unbound and otherwise it is bound. In one dimensional cases, there will be a periodic oscillation between turning points. In multi-dimensional cases, the time evolution of a bound system can be quite complex and may not ever repeat exactly in finite time I think.

The situation of a system with only conservative forces is an idealization in the macroscopic realm. There are other forces besides the conservative force giving rise to the potential energy. We take up the question of other forces in § 7, but only for an important special case. The idea of potential energy landscapes is still very useful when non-conservative forces are present.

Examples of systems moving in potential energy landscapes come up in celestial mechanics and microscopic physics. In celestial mechanics, planets are bound to stars. They are in gravitational potential energy wells and don't have enough energy to escape. If an object has enough energy to escape from the star's potential energy well, then the object has a speed at or exceeding the escape speed and is unbound. In microscopic systems, there is a similar situation with electrons bound to atoms: they are in the atom's potential energy well and can't escape usually. If the electron is given an injection of energy, it may have enough energy to become unbound.

Both cases are more complicated than our bead-on-wire case. For one thing they are three-dimensional cases in general. Also the potential energy zero point is at infinity. Angular momentum, in fact, puts extra restrictions are where the objects can move in these wells: i.e., restrictions beyond those of conservation of mechanical energy. Discussions of angular momentum in this context are beyond our scope of class.

## 7. EQUILIBRIA

In mechanics, equilibria are situations of zero net force on an object.

If the object is not moving, then the equilibrium is a static equilibrium.

Of course, the staticness is reference frame dependent: whether an equilibrium is static or not depends on the frame of reference you are in.

Let's consider an inertial reference frame in which there is a defined potential energy. The equilibria for the force of this potential energy acting alone or in combination with workless constraint forces are all static equilibria. They occur where the potential energy has zero derivative in one-dimensional systems and zero gradient in higher dimensional systems. Recall from § 4.8 that gradient is in some respects the three-dimensional analog of the derivative in one dimension.

Let's consider the our bead-on-wire system again as a concrete example a system with static equilibria due to a potential energy field.

The potential energy of a bead of mass $m$ is

$$PE = mgy \ . \tag{87}$$

But we also have a workless force of constraint, the normal force of the wire. This normal force causes $y$ to be a function of a path length parameter $s$. So a useful point of view $PE$ is a function of $s$ and is a one-dimensional potential energy. Using results from § 4.7, the one-dimensional force along the wire path derived from $PE$ consistent with the constraint force is

$$F = -\frac{dPE}{ds} = -mg\frac{dy}{ds} = -mg\sin\theta \ , \tag{88}$$

where $\theta$ is the angle of wire from the horizontal direction is aligned with the positive $s$ direction.

The stationary points or equilibria of the potential energy occur when $\theta = 0$.

Can we make our bead-on-wire system a bit more realistic?

Let's say that there is kinetic friction, but no static friction—which isn't realistic for the bead-on-wire, but is for many other kinds of systems that are somewhat analogous to the bead-on-wire system.

In this case, mechanical energy is **NOT** conserved.

The kinetic friction force always opposes the direction of motion, and so always reduces speed and mechanical energy.

So the bead perpetually loses mechanical energy until it comes to permanent rest.

The bead can only come to permanent rest where the net force on it is zero.

Since at rest only gravity and the normal force act on the bead—remember there is no static friction—it can must come to rest at potential energy equilibria.

Now let's consider a more general kind of system.

Let's consider a object (which could be just a particle) governed by a conservative force and maybe workless constraint forces a **DAMPING FORCE** that like kinetic friction always reduces the kinetic energy.

Since the conservative force is conservative, there is a potential energy with peaks, wells, and plains in general.

We assume there are no cusps in potential energy which would cause discontinuities in force. Remember a force in one dimension is minus the derivative of the potential energy (§ 4.7). If the potential energy has a cusp, it has two slopes at one point and there a discontinuity in the derivative and in the force. In three dimensions, similarly cusps lead to discontinuities in force. Real discontinuities probably do not exist when one studies a system on a small enough scale.

For systems like that described or even approximately like it, there is conventional classification of equilibria into four kinds:

1. **STABLE EQUILIBRIUM:** This is a potential energy minimum located in a potential energy well of relatively large depth. It is not in general a global minimum of the potential. Any small perturbation of the object from the exact minimum will lead to

a small scale oscillation of the object between nearby turning points similar to that discussed in for the bead-on-wire system (§ 6.1). Recall a conservative force points in the direction of fastest decrease of potential energy (§ 4.7). Thus, the conservative force tries to restore the object to the exact minimum. The damping force (if present) will damp out the oscillation by reducing the kinetic energy to zero and causing the object to come to rest at the minimum again. Except in very rare cases, the potential energy in the vicinity of the minimum is parabolic and the restoring force it gives rise to is linear. We discuss why this is so in § 7.1 below.

2. **UNSTABLE EQUILIBRIUM:** This is a potential energy maximum. Any perturbation no matter how small starts the object accelerating away from the maximum. Recall a conservative force points in the direction of fastest decrease of potential energy (§ 4.7). The damping force (if present) will continue to reduce the mechanical energy of the object and the object will never have enough mechanical energy to climb to the maximum again unless there is some new injection of energy.

   Inflexion points of potential energy are also unstable equilibria. Recall an inflection point is one where the derivative has gone to zero, but the function is monotonically increasing or decreasing at the point. Usually any perturbation will cause the system to move off in the direction that the potential energy is decreasing (i.e., in the direction the force points). One can image cases where the perturbation damps out by a damping force, but such cases are probably rather rare.

   In multi-dimensional cases, it is also possible to have equilibrium points that are stable in some directions, but unstable in others. Overall such points probably have to be judged as unstable since general perturbations will lead to a un-returning motion away from the equilibria. The obvious example is a saddle point in a two-dimensional space. In one direction the point is a maximum of potential energy (and so stable) and in an orthogonal direction it is a minimum (and so unstable).

3. **NEUTRAL EQUILIBRIUM:** This is a potential energy plain. The object can come to rest on the plain. A small perturbation of added kinetic energy will cause the object to move. But the object will come to rest again a short distance away when if there is a damping force to remove all the kinetic energy, unless the object moves off the plain. If the object moves off the plain, then the object position will evolve according to the forces it encounters in the potential energy landscape.

4. **METASTABLE EQUILIBRIUM:** This is potential minimum, but one that is at the bottom of a small potential energy well in the potential energy landscape. The equilibrium is stable against sufficiently small perturbations, but not larger ones. A sufficiently small one with a damping force present just leads to damped oscillations and rest again at the minimum. A large one will cause the object to escape the well and lead to further evolution. In reality, all stable equilibria are metastable. There can always be a sufficiently large perturbation to make the object move away from a minimum permanently if there are no other effects not included in the conservative or damping force to bring it back. But metastable category is often used only for equilibria that are considered delicate (i.e., potential energy wells that are shallow) in the context of system.

If there was no damping force in the stable and metastable equilibria, then perpetual oscillations or bound-state motions would occur as discussed in § 6.1. This can actually happen in quantum mechanical systems which do not obey Newton's laws of course. For macroscopic systems, the absence of all damping forces is an ideal limit that cannot be reached in reality.

The discussion of equilibria becomes more complex if our object is subject to something like static friction as a real bead-on-wire system is.

Nevertheless, the equilibrium categories we have introduced above have general utility.

For example, a building should be in overall stable equilibrium.

This means all its parts should be in stable equilibrium.

A little damped oscillation is necessarily allowed for in building design, but not irreversible motions that can lead to collapse.

On the other hand, a balance scale should be designed to be at a nearly unstable metastable equilibrium when balanced.

This allows a fine mass determination.

Any stabilizing resistive force in the pivot will cause an uncertainty in a mass determination.

For a very fine balance scale, one wants the stabilizing resistance force as small as possible while still allowing a balance.

## 7.1. The Nature of Stable Equilibria

In equilibrium, the net force on an object (which may be just a particle or may be a part of a larger object) is zero.

To be a stable equilibrium, there has to be a restoring force that will try to push the object back to the equilibrium position after a perturbation has displaced it as we discussed above in § 7.

If the restoring force varies continuously with displacement, then the restoring force will be linear in the displacement for a sufficiently small displacement from equilibrium, except in very rare cases.

Nature usually abhors discontinuities at least if you look on a fine enough scale. So we will not bother with cases of discontinuities in the dependence of the restoring force on displacement.

In some cases, the linear regime for the restoring force may be very tiny.

A more mathematical argument can be given for the linearity of the restoring force about an stable equilibrium for the one-dimensional case.

Consider a potential energy $PE$ for the system.

At an equilibrium point $x_0$ for the force of the potential energy, the force is zero, and thus $PE$ has a stationary point.

We can can Taylor expand $PE$ around the equilibrium point to get

$$PE(x) = PE_0 + \sum_{n=2}^{\infty} \frac{(x-x_0)^n}{n!} \frac{d^n PE}{dx^n}\bigg|_{x=x_0} , \tag{89}$$

where the linear term in the expansion is zero since $PE$ is stationary at $x_0$.

For the equilibrium to be stable, there must be at least a small region about $x_0$ where there is a restoring force due to the potential: i.e., where

$$F(x > x_0) < 0 \qquad \text{and} \qquad F(x < x_0) > 0 \tag{90}$$

so that the force tends to push the system back to equilibrium when there are small perturbations.

Since

$$F(x) = -\frac{dPE}{dx} \tag{91}$$

(eq. (50) in § 4.7), we demand for stability that the lowest non-constant term in equation (89) have an even order $n$ and a positive coefficient $d^n PE/dx^n|_{x=x_0}$. This lowest non-constant term is the stabilizing term. Say the stabilizing term is the $n$th order term, then for some small region about $x_0$,

$$F(x) = -\frac{(x - x_0)^{n-1}}{(n-1)!}\frac{d^n PE}{dx^n}\Big|_{x=x_0} \ . \tag{92}$$

We see explicitly that the last equation gives the behavior of equation (90).

In fact, almost always in nature and technology, the stabilizing term is the 2nd order term. There is no general reason for this term to vanish at stationary points and it doesn't usually. Thus, for some small region about $x_0$,

$$F(x) = -(x - x_0)\frac{d^2 PE}{dx^2}\Big|_{x=x_0} \tag{93}$$

which is just the linear force with $d^2PE/dx^2|_{x=x_0}$ acting as the linear force constant $k$.

One can, of course, create higher order stabilizing terms for specially constructed systems, but I know of no practical reason for doing so.

A system which may turn up sometimes is where the potential energy at the bottom of a well goes flat for a small region. Technically, the flat region is a neutral equilibrium, but if this region is very small it might act effectively as a stable equilibrium giving rise to oscillations for small perturbations of an object in the that are not simple harmonic motion. If such oscillations damp out, then the object will come to rest nearly where it started from. The system is not far removed from the case where one has a linear restoring force, but is not exactly that case.

The above argument can be generalized to three dimensions and again the stabilizing term is almost always 2nd order in displacement from stable equilibrium.

Since the 2nd order term is almost always the stabilizing term, the linear force has immense importance both in nature and in technology. It is almost always the restoring force for small displacements from the stable equilibria of a potential energy's force.

## 8. THE SIMPLE HARMONIC OSCILLATOR

Once more unto the linear farce—er force.

Recall for one dimension, it is

$$F = -kx \ , \tag{94}$$

where $k$ is a constant and $x$ is measured from zero-point location for the force.

The equilibrium point for the linear force is $x = 0$. It's actually a stable equilibrium as we discussed in §§ 4.6 and 7.1 since the force tries to push the particle it acts on back toward the $x = 0$ location after any displacement.

Say we have one-dimensional system in which a particle of mass $m$ is subject only to the linear force or the linear force is the only unbalanced force.

The equation of motion follows from $F = ma$:

$$ma = -kx \ , \tag{95}$$

where the linear force is the net force in this case.[1]

Since $a = d^2x/dt^2$, the equation of motion is

$$m\frac{d^2x}{dt^2} = -kx \tag{96}$$

which is called the simple harmonic oscillator equation since the system it describes is called the simple harmonic oscillator. It is a differential equation recall.

We have already solved the simple harmonic oscillator equation in the lecture *NEWTONIAN PHYSICS II*.

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[1]Equation of motion is a term used in several ways. It can mean $F = ma$ itself or a particular application of $F = ma$. The latter is the usage I usually use. It can also mean the solution to $F = ma$. This is usage I avoid.

That solution gave us full information: position as a function of time and thus all orders of derivatives of position with time. Recall

$$x = A\cos(\omega t) + B\sin(\omega t) \ , \tag{97}$$

$$v = -A\omega\sin(\omega t) + B\omega\cos(\omega t) \ , \tag{98}$$

$$a = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) \ , \tag{99}$$

where $\omega = \sqrt{k/m}$ is the angular frequency and $A$ and $B$ are constants of integration determined by the initial conditions. Position $x$ and its derivatives are all sinusoidal. The behavior of the simple harmonic oscillator is called simple harmonic motion.

The usual initial conditions specified for initial time $t = 0$ are the initial position $x_0$ and the initial velocity $v_0$. Obviously at this point in the course,

$$A = x_0 \qquad \text{and} \qquad B = \frac{v_0}{\omega} \ . \tag{100}$$

An example of a simple harmonic oscillator is an ideal spring with an object with mass attached.

But since the linear force is nearly always the stabilizing force for stable equilibria for small displacements from equilibrium points (see § 7.1), approximate simple harmonic oscillators are ubiquitous in nature and technology. Thus, the simple harmonic oscillator is one of the most important of all ideal systems.

Actually, at the macroscopic level, an ideal simple harmonic oscillator is virtually impossible. There are always damping forces. However, the damping forces can be made very small in some cases. Also one can always add a driver force to counteract the damping forces and maintain a driven harmonic oscillator quasi-perpetually.

Although we have the full solution for the simple harmonic oscillator, energy methods give us some information without having to know the solution.

The linear force is a conservative force as we showed in § 4.6 where we also found the linear force potential energy to be given by

$$PE = \frac{1}{2}kx^2 \ , \tag{101}$$

where the $PE$ zero point is set to $x = 0$ (the equilibrium point) by a convention that is obviously completely reasonable.

Since the only force that does work is conservative, mechanical energy is conserved.

Thus

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2 = E_0 \ , \tag{102}$$

where the subscripts 0 indicate initial values. The initial conditions $x_0$ and $v_0$ allow us to calculate $E_0$.

We can't learn everything about the system from the energy method. We can't learn the time evolution. But we can find speed $|v|$ as a function of $|x|$ and vice versa:

$$|v| = \sqrt{\frac{2E_0}{m} - \frac{k}{m}x^2} \ , \tag{103}$$

$$|x| = \sqrt{\frac{2E_0}{k} - \frac{m}{k}v^2} \ . \tag{104}$$

From these equation it follows that $|v|_{\rm max}$ occurs for $x = 0$ and $|x|_{\rm max}$ for $v = 0$:

$$|v|_{\rm max} = \sqrt{\frac{2E_0}{m}} \ , \tag{105}$$

$$|x|_{\rm max} = \sqrt{\frac{2E_0}{k}} \ . \tag{106}$$

We could have found these values from the full solution simply by differentiating the expressions for $x$ and $v$ by time and setting the expression to zero and solving for the time of the stationary points which then get plugged back into the $x$ and $v$ expressions for the stationary points which we then take the absolute values of. The energy method is just alternative which may simpler to use in some cases. But in any case, shows how $|x|_{\rm max}$ and $|v|_{\rm max}$ follow from specifying the mechanical energy of the simple harmonic oscillator.

The quantity $|x|_{\rm max}$ is called the amplitude of the oscillation of the particle.

And there is an oscillation as we know from the full solution. But even without that solution, we can know there is an oscillation.

First, we can see that the potential energy landscape is parabola with a single stationary point at $x = 0$ which is a minimum.

From our discussion in § 6.1, we know that the particle will oscillate back and forth between **TURNING POINTS** which are, in fact, at $x = \pm|x|_{\rm max}$.

The total mechanical energy on an **ENERGY DIAGRAM** is a straight line between the turning point potential energy values. At the turning points, the kinetic energy is zero.

So the energy method has taught us a lot.

But not the whole time evolution of the simple harmonic oscillator.

For that we need the full solution.

But note that as we discussed energy methods often give information easily when the full solution is not available and not easily obtainable: e.g., for the bead-on-wire system (§§ 5.1.2 and 6.1).

## 9. POWER

In science and technology, power is the rate of energy transfer or transformation (which is a kind of transfer).

The instantaneous power is given by

$$P = \frac{dE}{dt} \ , \tag{107}$$

where $E$ is any kind of energy being transformed into any other kind.

The basic SI unit of power is the familiar watt (W) which is a joule per second:

$$\text{unit}[P] = \text{unit}[E/t] = \text{J/s} = \text{W} \ , \tag{108}$$

where where unit[ ] is my idiosyncratic unit function.

Watts are familiar power units in the context of electricity.

Electric companies bill in kilowatt-hours which are actually energy units:

$$1\,\text{kW-h} = 1\,\text{kW-h} \times \left(\frac{1000\,\text{W}}{1\,\text{kW}}\right) \times \left(\frac{3600\,\text{s}}{1\,\text{h}}\right) = 3.6 \times 10^6\,\text{J} = 3.6\,\text{MJ} \ . \tag{109}$$

So electric companies could bill you in megajoules, but nooOOOooo they have to use an obscure unit that confuses all energy discussions.

The watt is named for James Watt (1736–1819) who vastly improved the steam engine and invented an obsolete unit of power, the horsepower. He used horsepower (meaning the rated power of his engines in horsepower) in marketing his engines.

There are various definitions of horsepower (which differ by smidgens), but the electrical horsepower is defined exactly by the following equation:

$$1\,\text{hp} = 746\,\text{W} \ . \tag{110}$$

Actually, only very strong horses can deliver a horsepower of power to external objects for very long (Wikipedia: Horsepower). Maybe Shire horses, where the stallions have typically more than 900 kg of mass, can do it easily.

Horse metabolic rate (which the power to just live and move their bodies as well as external objects) probably exceeds a horsepower often.

We are familiar with watts from electrical devices. A 100-watt incandescent light bulb uses 100 W of power. In fact, only a few percent of this power comes out as visible light. The rest is mostly in the form of infrared light (which quickly becomes waste heat) or waste heat directly. This remarkable inefficiency is why the venerable incandescent light bulb will probably be phased out in a few years.

In the context of macroscopic work on an object, we can find a power-work formula. Since basic differential work formula for work done on an object by a force is

$$dW = \vec{F} \cdot d\vec{s} \ , \tag{111}$$

it follows that power produced by the force is

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{s}}{dt} = \vec{F} \cdot \vec{v} \ . \tag{112}$$

Positive power puts energy into an object's kinetic energy bank and negative work takes it out.

### 9.1. Example: Power by Elevator Motor

There is an elevator with mass of 1800 kg including cargo. It moves upward with constant velocity and is acted on by a tension force of a cable and a kinetic friction force of 4000 N.

Note that since the velocity is constant as much energy flows out of the kinetic energy bank as flows in.

The tension force puts energy in. This energy comes from the motor.

Energy flows out into gravitational potential energy and into waste heat due to the kinetic friction.

How much power must be generated by the motor to raise the elevator at a constant $v = 3.00\,\mathrm{m/s}$?

You have 2 minutes working individually or in groups. Go.

We need to find the tension force to find the motor power.

Since the elevator is unaccelerated, Newton's 2nd law tells us that

$$T - F_{\rm ki} - mg = 0 \ , \tag{113}$$

where $T$ is the tension force magnitude, $F_{\rm ki}$ is the friction force magnitude, and $mg$ is the gravitational force magnitude.

Clearly,

$$T = F_{\rm ki} + mg \ . \tag{114}$$

Using equation (112), we find

$$P = \vec{F} \cdot \vec{v} = Tv = (F_{\rm ki} + mg)v \approx (4000 + 1800 \times 10) \times 3.00 = 66000\,\mathrm{W} = 66\,\mathrm{kW} \ . \tag{115}$$

## 10. WEIRD ENERGY UNITS: READING ONLY

One of my pet peeves is that in special contexts, traditional special energy units are used.

Now if one had a concrete sense of the size of an energy unit, then context-dependent units might make some sense.

But in all contexts, energy is a pretty abstract quantity.

Using special—and weird—energy units just makes it difficult to compare energy usage in different contexts.

Table 1 below shows energy unit conversions.

We won't describe it all.

But the food calories are really egregious. A food calorie is 4.1868 kJ.

You need thousands of food calories per day.

Why not just put megajoules on the food containers?

The human food needs are typically in the range 8–12 MJ per day.

Then there is the kilowatt-hour which is unit of energy not power: it's a kilowatt of power times one hour. It's 3.6 MJ.

I admit the kilowatt-hour is a somewhat convenient unit for domestic energy usage. But in the modern age with calculators and energy measuring meters, its easy enough to use joules, kilojoules, and megajoules.

There is no sense, however, in using kilowatt-hours for large scale energy reports and calculations—at least I think not.

I believe that all discussions of energy usage in society would be more rational and effective if people simply knew by using standard SI energy units the relative sizes of the energy involved.

But no one ever listens to me.

Table 1. Energy Unit Conversions

| Energy Unit | SI equivalent | Comment |
|---|---|---|
| 1 food calorie | 4.1868 kJ | Typical human food needs are in the range 2000–3000 food calories per day. |
| 1000 food calories | 4.1868 MJ | That turns into 8–12 MJ. So the megajoule is a perfectly convenient unit for food energy. It's better than food calories |
| 1 calorie | 4.1868 J | A food calorie is really a kilocalorie. The real calorie is the amount of energy needed to raise the temperature of one gram of water by 1° Celsius. Various versions exist because the amount of energy needed varies with conditions. The shown one is the International Steam calorie (See Wikipedia: Calorie). |
| 1 kilowatt-hour | 3.6 MJ | The kilowatt-hour is hybrid unit that is (kilojoule/second)×hour. The MJ is good-sized replacement. |
| 1 Btu | 1.0545 kJ | British thermal units of slightly different size still linger around. Kilojoules can obviously replace them. |
| 1 kg of gasoline | 44–45 MJ | About 5.5 times daily human food needs. You could live on a about 0.2 kg of gasoline per day. |
| 1 kg of oil | 41.868 MJ | This is standard definition since the chemical energy content of oil varies. It looks like the calorie digits. |
| tonne oil equivalent (toe) | 41.868 GJ | A tonne is a metric ton (1000 kilograms). It really ought to be called a megagram (Mg). |

Table 1—Continued

| Energy Unit | SI equivalent | Comment |
|---|---|---|
| barrel (bl) of oil equivalent | 6.12 GJ | This is approximate. The oil industry insists are reporting oil in barrels—though no one has put oil in barrels in a jillion years (to be precise). Why not just report oil quantities in energy equivalent since energy content is the key issue. |
| 1 Mbl of oil | 6.12 PJ | World daily consumption is often given in mbls. |
| 1 Gbl of oil | 6.12 EJ | World yearly consumption is often given in Gbls. |
| tonne coal equivalent | 29.3076 GJ | This must be a standard definition since the chemical energy content of coal varies. You can see one the reasons why people prefer oil to coal. Oil has a higher energy density typically. (See tonne oil equivalent just above.) |

Note. — The values are from Wikipedia's article *Conversion of Units*, except the gasoline value is from Smil (2006, p. 16).

## 11. HUMAN BODY ENERGY: READING ONLY

How much energy does a human transform from food energy to other forms?

The food energy is chemical energy.

The other forms are mainly kinetic energy, gravitational potential energy, and waste heat.

Mostly the kinetic energy and the gravitational potential ends up as waste heat.

However, putting books up onto bookshelves for eternity is certainly making quasi-eternal contributions to gravitational potential energy.

We need some terminology now.

Human body power—human body energy output per unit time—is called the metabolic rate.

The lowest power a healthy human can have is the basal metabolic rate (BMR) measured when a person is doing absolutely nothing not even digesting food. It's measured some hours after last eating and in some temperature controlled setting.

The resting metabolic rate (RMR) is the metabolic rate of ordinary resting. Actually, the difference between BMR and RMR is not large. They can be taken to be almost the same, except when being exact.

The sustained metabolic rate (SMR) is the daily average rate or at least that is the definition we will use.

There are also high metabolic rates that can be sustained for limited times.

Table 2 below shows typical metabolic rates.

The values are not carefully defined in the sources and must be treated as typical (or representative) and not as exact or exact averages.

Table 2. Typical Metabolic Rates

| Specification | Metabolic Rate (W) | Metabolic Rate (MJ/day) |
|---|---|---|
| BMR adult female of 50–80 kg | 55–80 | 4.7–6.9 |
| BMR adult female of 50–80 kg | 60–90 | 5.2–7.8 |
| SMR female scientist | 100 | 8.5 |
| SMR male scientist | 140 | 12 |
| SMR male miner | 180 | 16 |
| SMR male soldier | 200 | 17 |
| SMR male antarctic explorer | 230 | 20 |
| SMR male Tour de France cyclist | 380 | 33 |
| elite male endurance athlete for some hours | 1750 | |
| maximum MR for a few seconds | $\sim 8000$ | ... |

Note. — The values are from Smil (2006, p. 59–61) and Peterson et al. (1990). These are not the best or most up-to-date sources. They are just the ones I can lay my hands on. BMR is basal metabolic rate and SMR is sustained metabolic rate. It's pretty easy to find scientists to measure, but I wonder where one lays hands on an antarctic explorer—yes, I know there are lots of outdoor scientists in Antarctica to study while they are studying something else—the observer observed.

If you are sedentary, your power is like that of bright light bulb and you may need only 8 to 12 megajoules of food energy per day.

If you are high performance athlete, your SMR can be over 400 W and you need over 35 MJ of food energy per day. This, of course, is when you are training or performing for hours per day for weeks on end.

In fact, somewhere in the range 400–500 W may be the upper limit for SMR for a human since even elite athletes can't seem to do better. There may be a limit to how much a human can eat and metabolize.

The sustained metabolic scope is the ratio of SMR to BMR. The human limit sustained metabolic scope may be about 6. In fact, a sustained metabolic scope of 7 may be the limit for all animal life (Peterson et al. 1990).

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## A. Conservation of Kinetic Energy

Is kinetic energy ever just conserved by itself alone?

A single particle alone in force-free space has conserved kinetic energy obviously, by the work-kinetic-energy theorem.

No forces, no net force, no net work done on the particle, no change in kinetic energy.

But this is a pretty trivial example.

Say you imagine a system of non-interacting particles in force-free space, then each particle has it's own constant kinetic energy.

Then each particle's kinetic energy is conserved and the total kinetic energy of the system is conserved.

But this is a pretty trivial example too.

Say the space is force free, except of the forces the particles exert on each other when they collide.

If these forces are perfect elastic forces, then the kinetic energy lost in compression of the particles in a collision is stored as a potential energy of compression which is then entirely converted back to kinetic energy by the end of the collision. But the kinetic energy of the individual particles can be changed in a collision.

Nevertheless, the total system kinetic energy between collisions is a constant.

But whoa now. How can a particle be compressed?

We have to de-assume that our objects are "particles".

They are objects with extension. But between the collisions, let's assume they are **RIGID**.

But there is another qualification.

Since the objects have extension, they can rotate and have kinetic energy associated with rotation.

So the total kinetic energy that is conserved is the sum of what we later call the translational (or more exactly center-of-mass kinetic) energies plus the rotational kinetic energies.

Well this subsection is getting into deeper waters than I thought, and so that's why it's optional.

Historically, such a system of interacting particles is probably important.

Leibniz's concept of *vis viva* (his name for the quantity $mv^2$ for a particle) partially had its basis in that he recognized the concept of conservation of kinetic energy in some contexts.

Wikipedia is unenlightening on which contexts. But I imagine Leibniz as studying billiard balls (if they had them in the 17th century) colliding on level surfaces.

If the balls are **IDENTICAL** and roll without slipping either on the surface or in collision events, then the center-of-mass and rotational kinetic energies will be conserved separately in the ideal case as we'll now prove. The ideal case means no dissipation to waste heat by any kind of dissipative force (friction, rolling friction, air drag, internal friction inside the balls, etc.) and the collisions are instantaneous. I think it is true that total kinetic energy is conserved in such cases. With no dissipation to waste heat, total kinetic energy is conserved, except during the instants of collision if you count them.

Now consider an equation that anticipates later results:

$$KE = \sum_i \frac{1}{2}\left(m + \frac{I}{r^2}\right) v_i^2 = \left(\frac{1}{2}m + \frac{1}{2}\frac{I}{r^2}\right)\sum_i v_i^2 \ , \tag{A1}$$

where $KE$ is the total kinetic energy (i.e., the sum of center-of-mass and rotational kinetic energies) and the sum is over all balls which have identical mass $m$, radius $r$, and rotational inertia $I$. The quantity $(1/2)m\sum_i v_i^2$ is the total center-of-mass kinetic energy, and the quantity $(1/2)(I/r^2)\sum_i v_i^2$ is the total rotational kinetic energy. If $KE$ is constant, then $\sum_i v_i^2$ is constant Then so are total center-of-mass and rotational kinetic energies. This is what we started out to prove.

Leibniz may have recognized the conservation of the total center-of-mass kinetic energy. I don't think in his time, it could have treated rotational kinetic energy.

From total center-of-mass kinetic energy conservation in this case, he might have inferred that point particles (which have no rotational kinetic energy) would have conservation of total kinetic energy even when the particle mass differs.

But all this is speculation. Historians of science must know the real story.

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