ONE-DIMENSIONAL KINEMATICS

David J. Jeffery[1]

2008 January 1

ABSTRACT

Lecture notes on what the title says and what the keywords say.

Subject headings: keywords — calculus — differentiation — integration — antidifferentiation — position — displacement — velocity — acceleration — jerk — distance — speed — constant acceleration — constant-acceleration kinematic equations — relative motion — gravitational field — free fall — gravitational field magnitude g — acceleration due to gravity g

1. INTRODUCTION

KINEMATICS is the study of motion without causes—and in the context of Newtonian physics the causes are **FORCES**.

So aside from occasional allusions, no **FORCES** in this lecture. We dig into forces in the lectures *NEWTONIAN PHYSICS I* and *NEWTONIAN PHYSICS II*.

KINEMATICS relies on those well known known kinematic quantities position, displacement, velocity, and acceleration all of which are used to describe motion.

In **ONE** dimension, you only have motion along a line—real exciting stuff.

[1] Department of Physics, University of Idaho, PO Box 440903, Moscow, Idaho 83844-0903, U.S.A. Lecture posted as a pdf file at http://www.nhn.ou.edu/~jeffery/course/c_intro/introl/001_intro.pdf .

In this lecture by line we usually mean a straight line.

But curved lines (i.e., curves) can also be treated using the same formalism we develop. In the case of curves, one simply measures displacement along a curve through space.

We are usually thinking of Cartesian axes for the description, and so the straight line of our developments is usually the x axis when we are considering horizontal motion and the y-axis when we are considering vertical motion.

But remember "A rose by any other name would smell as sweet." The x and y are just the conventional variables to measure perpendicular directions with.

We will only be considering the motions of **PARTICLES**.

But **PARTICLE** has a special meaning in this context—it does not mean small necessarily. It means that the internal structure of the object can be neglected.

It may seem odd that planes and trains and automobiles can be considered particles, but indeed they can be.

Treating them as particles is justified by the concept of center of mass that discuss in detail in the lecture *NEWTONIAN PHYSICS I*.

Here it is enough to say that the center of an object or a system (to use more physicsy jargon) is the mass-weighted mean position of the object.

A position is just a point in space, and so is an extensionless particle.

So whatever object we treat in our developments and treat as a particle, it is usually to be understood that we are treating its center of mass.

The object's velocity is then its center-of-mass velocity and its acceleration is its center-of-mass acceleration.

2. CALCULUS

Calculus-based intro physics courses assume that calculus is being taken as a corequisite by students.

This is necessary. While many students may have completed a calculus course or two, many are taking it concurrently.

So calculus results are introduced gradually in intro physics.

But it is inevitable that intro physics that we need calculus results before they are taught in intro calculus.

What is calculus?

The branch of math that deals with limits, derivatives, differentiation (which is the process of creating derivatives), integrals, and integration (which is the process of creating integrals). Lots of other things like functions and series come up too.

Now while the calculus course is going on and on about limit theorems, we already need differentiation and integration in intro physics.

So we now introduce some calculus formulae.

Just accept them and use them as formulae for the time being. The proofs will catch up to you sooner or later.

2.1. Differentiation and Derivatives

Differentiation is a calculus operation on functions with respect to variables that the function depends on. The result is the derivative of the function.

Say we have general variable x and $f(x)$ is a function of x.

As x varies $f(x)$ will vary in general.

And also to be mathematical, there is one-to-many correspondence from $f(x)$ to x: i.e., a single value of $f(x)$ corresponds in general to many values of x although in many special cases it may correspond to one. On the other hand, to be a function as we ordinarily understand the term, there is only one value of $f(x)$ for every x.

One can plot $f(x)$ as a function of x using 2-dimensional Cartesian axes. In this case, x is plotted along the horizontal axis which is conventionally the x axis and the $f(x)$ is plotted along the vertical axis which is conventionally the y axis.

To get to the derivative formula, first note that the Greek capital delta Δ is the common symbol for finite change in a quantity. For example, a change in the value of x is commonly written Δx and vocalized as Delta X.

The average slope of $y = f(x)$ over some x interval Δx is **DEFINED** to be

$$\frac{\Delta f}{\Delta x} , \tag{1}$$

where Δf is the change in $f(x)$ over the interval Δx.

The average slope can also be called the average rate of change of $f(x)$ with x over Δx. This makes sense since

$$\frac{\Delta f}{\Delta x} \times \Delta x = \Delta f , \tag{2}$$

is the total change in $f(x)$ over interval Δx.

Derivative of $f(x)$ with respect to x is the slope of $f(x)$ or the rate of change $f(x)$ with x at a single point x. The formula definition—but not the explicit definition which you'll get in intro calculus—is

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} . \tag{3}$$

Note that as Δx goes to zero, Δf goes to zero. But the limiting ratio is not in general zero over zero.

It's a finite value.

We will not do proofs—we leave that to your intro calculus course.

But a graphically, one can illustrate $\Delta f/\Delta x$ changing as Δx goes to zero and settling down to some finite value.

This process is easy to see for a straight line function or any smooth function.

In fact, there are points where a derivative does **NOT** exist.

For example, where the function has a singularity like a cusp, a discontinuity, or an infinity.

For a derivative to exist at point or, in other words, for the function to be differentiable there, the function has to be sufficiently smooth there.

We will usually deal with differentiable functions and leave worrying about other cases to math classes. We will briefly discuss the cusp, discontinuity, and infinity singularities in § 4.

There an alternative form for writing the derivative of $f(x)$. This is $f'(x)$. The prime symbol $'$ means the function is differentiated. Thus

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \ . \tag{4}$$

But note that primes are often conventionally used to mean things other than differentiation. Yours truly uses those other uses when convenient.

There are orders of derivatives for a function.

When you just say derivative without qualification, you mean 1st order derivative.

The zeroth order derivative of a function is just the function itself.

Higher order derivatives are naturally generated by multiple differentiations.

Naturally, the nth order derivative of $f(x)$ is derivative of the $(n-1)$th order derivative of $f(x)$. In symbols, the nth order derivative is

$$\frac{d^n f}{dx^n} , \tag{5}$$

where in this case n is **NOT** a power, but is the derivative order—and yes the derivative order in the "numerator" is on the "d". In symbols, the nth order derivative is obtained from

$$\frac{d^n f}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1} f}{dx^{n-1}}\right) . \tag{6}$$

The term "nth order derivative" is often abbreviated to "nth derivative".

Using the prime notation, multiple differentiations are denoted using multiple primes.

Thus, $f''(x)$ is the 2nd order derivative and $f'''(x)$ is the 3rd order derivative.

But this multiple prime formalism gets klutzy for higher orders. For higher orders, one expresses the order by a bracketed order number. For the n order derivative, one writes $f^n(x)$. Actually, some folks use the bracketed order number from 2nd order on.

2.2. Differentials

Now what the devil are *df* and *dx* that turn up in the derivative notation?

In the context of calculus, they are **NOT** products of d with f and x.

They are called **DIFFERENTIALS**: *df* is the differential of $f(x)$ and *dx* is the differential of x.

In calculus formalism, any quantity symbol with a "d" in front of it means the differential of that quantity.

A **DIFFERENTIAL** is an infinitesimal change in a quantity. Recall Δ with a quantity means finite change in the quantity.

But what does infinitesimal change mean?

It meaning is that there is a limiting process acting on the quantity.

It is being differentiated or it is the variable of differentiation or it is being integrated (which we'll go into in § 2.4).

If a **DIFFERENTIAL** is standing alone, the limiting process is implicit or hasn't been carried out.

For example, df is the differential of f. So it could be that f will be differentiated with respect to some variable, but we havn't specified the variable.

We have left that variable general.

What if you see

$$\frac{d}{dx} \tag{7}$$

standing alone? In math jargon d/dx is an operator. An operator is an entity that transforms one function into another. In the case of d/dx, it is an operator that transforms a function into its derivative with respect to x.

DIFFERENTIALS will be used a lot in our developments in this course.

It's helpful to think of them as itty, bitty, changes in a quantity. In fact, one often makes the approximation

$$dx \approx \Delta x \ . \tag{8}$$

But this is **NOT** an ordinary approximate equality.

It means we are approximating a limiting process with dx by simply multiplying or dividing something by Δx, where Δx is sufficiently small in some sense. For example, from the exact result

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \ , \tag{9}$$

one obtains the approximate result

$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x} \ . \tag{10}$$

When is equation (10) an adequate approximation?

Well that depends on the case?

But for crude purposes, a crude graph of function $f(x)$ will tell if the slope over some region approximates the derivative at a point in that region.

There are special cases where equation (10) is obviously adequate. For example, when $f(x)$ is a linear function: its derivative is a constant and equal to the slope over any region. In this special case, $df/dx = \Delta f/\Delta x$ exactly for any region Δx.

Taylor expansions give a general way to test the adequacy of equation (10). We'll introduce in the lecture *NEWTONIAN PHYSICS II*), but those of you will some calculus background may know all about Taylor expansions already.

2.3. Power Law Function Derivative

For our purposes in this lecture, the only explicit derivative formula we need is for power law functions.

We will **NOT** prove this derivative law formula—that's for your calculus class whenever it gets around to it.

A general power law function is

$$f(x) = Ax^p \ , \tag{11}$$

where A is a general constant and p is a general power.

Sans proof, the general derivative formula and some special cases there of are given by

$$\frac{df}{dx} = A \times \begin{cases} px^{p-1} & \text{for } p \neq 0 \text{ and } p = 0 \text{ with } x \neq 0; \\ 0 & \text{for } p = 0; \\ 1 & \text{for } p = 1; \\ 2x & \text{for } p = 2; \\ 3x^2 & \text{for } p = 3. \end{cases} \tag{12}$$

Question for the class: what is the derivative of x^5?

You have 5 seconds. Go.

Of course, $5x^4$.

Question for the class: what is the derivative of $x^{5/2}$?

Of course, $(5/2)x^{3/2}$.

Question for the class: what is the derivative of $x^{-5/2}$?

Of course, $(-5/2)x^{-7/2}$.

2.4. Integration

Integration is the process of finding integrals.

An integral for functions of one variable is the area under the curved line of the function: the curve above the variable axis gives a positive contribution and the curve below the variable axis gives a negative contribution.

The symbolic notation for a definite integral of function $f(x)$ between $x = a$ and $x = b$ is

$$\int_a^b f(x)\,dx \ , \tag{13}$$

where $\int$ is the integration symbol and dx is the differential of the variable x.

The integration symbol $\int$ was introduced by Gottfried Leibniz (1646–1716) who also introduced our other most basic calculus notations. Leibniz and Isaac Newton (1643–1727) independently invented calculus at about the same time circa 1670–1680. The integral symbol is, in fact, a stylized S and meant summation.

In fact an integral is a limit of a summation. The definition is

$$\int_a^b f(x)\,dx \equiv \lim_{\Delta x_i} \sum_i f(x_i)\Delta x_i \ , \tag{14}$$

where i is an index labeling small x intervals Δ_i, the Δx_i are the small intervals which tile the region from a to b, x_i is a point in interval Δx_i, the summation is over all intervals, and the limit is the limit of the intervals going to zero while their number goes to infinity.

Now evaluating an integral from the limit definition directly is a pain.

Fortunately, the fundamental theorem of calculus—which we absolutely positively will not prove here—saves us.

The fundamental theorem of calculus tell us that

$$\int_a^b f(x)\,dx = F(b) - F(a) \ , \tag{15}$$

where $F(x)$ is the antiderivative of $f(x)$.

What's an antiderivative?

Just what you'd think.

It's what you get when you do the opposite of differentiation to $f(x)$.

This means that

$$\frac{dF}{dx} = f(x) \ . \tag{16}$$

Some people think of antiderivatives as being unspecified to within an additive constant since derivative of a constant is zero. But yours truly along with most people (I think) use antiderivative to mean only those functions without any constant terms. Antiderivatives in this sense are completely specified by the antidifferentiation operation.

Question for the class: what is the derivative of x^p?

Of course,

$$\frac{x^{p+1}}{p+1} \ . \tag{17}$$

But what if $p = -1$?

This is a special case. The antiderivative of x^{-1} is $\ln(x)$: i.e, the natural logarithm of x. We won't need this special case much for awhile anyway and will never prove it.

An indefinite integral is

$$\int f(x)\,dx = F(x) + C \ , \tag{18}$$

where there are no integration limits, C is an unspecified additive constant, and $F(x)$ is antiderivative with **NO** unspecified constant term.

If by antiderivative one means antiderivative with an unspecified constant term, then equation (18) is written without C and indefinite integral and antiderivative become synonyms. But as aforesaid, yours truly along with most people (I think) use antiderivative to mean only those antiderivatives without additive constants.

So to us good-thinkers, indefinite integral and antiderivative are not synonyms.

What determines C?

The boundary or initial conditions of the system you are considering.

To see this explicitly, we can write the indefinite integral as a definite integral in terms of a variable upper limit:

$$\int_{x_0}^{x} f(x')\,dx' = F(x) - F(x_0)\ , \tag{19}$$

where x_0 is the initial value for the integral, the variable x is the upper limit, and x' is a dummy variable.

Dummy variables are variables that are integrated over. Note that the prime symbol here does **NOT** mean differentiate, it is just used to distinguish x' from x.

By the way, the zero symbol as a subscript of a variable is often vocalized "nought" (which is actually a synonym for zero) when vocalizing the variable. For example, x_0 is vocalized as X-nought. The vocalization of subscript zero as "nought" is a common convention in physics and probably other mathematical sciences.

Get used to it.

The zero subscript is frequently used to indicate that the variable is an initial value in time or an initial value in some other sense.

2.5. More Calculus Results

More calculus results such as the chain rule, the product rule and Taylor expansions will be introduced as needed in the course of the course.

One hopes that the calculus course will have caught up to the physics course by then and you'll already know all the other calculus results.

3. KINEMATIC QUANTITIES: POSITION, DISPLACEMENT, VELOCITY, ACCELERATION

In this section, we go into those kinematic quantities position, displacement, velocity, and acceleration.

Time is also a kinematic quantity. We discussed time in the lecture *INTRODUCTION TO INTRODUCTORY PHYSICS*.

We'll just use time here.

Speed will also be discussed as an adjunct to velocity.

3.1. Position and Displacement

Position and displacement are nearly synonyms and can be exactly so depending on how you think of them.

Usually people think of position as being the location of a point in space relative to some absolute physical system: e.g., a point relative to the Earth.

Displacement on the other hand is a vector: i.e., a quantity with magnitude and direction.

Displacement specifies position relative to some coordinate system origin.

In two dimensions, one usually thinks first of the two-dimensional Cartesian coordinates of the x-y plane.

In three dimensions, one usually thinks first of the three-dimensional Cartesian coordinates of the x-y-z space.

Two and three dimensional Cartesian coordinates with a displacement vector are show in Figure 1

A displacement vector specifies position by giving the straight-line distance to the position and the direction.

In one dimensional cases, everything is simple.

The positions are along an axis which we will usually call the x axis.

The x value associated with any position is the displacement.

The absolute value or the magnitude of $|x|$ is the distance from the origin.

The sign of x gives the direction from the origin: there are only two possible directions or, in math jargon, two senses.

In math jargon, "sense" means one of two opposite directions in which a vector can point (e.g., Barnhart 1960, p. 1102).

In one-dimensional cases, there is no need for explicit vector symbols since the direction is specified by sign.

In multi-dimensional cases, one puts an arrow over a vector quantity symbol or in cursive by a squiggle under a vector quantity symbol.

Fig. 1.— Two and three dimensional Cartesian coordinates with a displacement vector.

3.2. Velocity and Speed

Velocity is the time rate of change of displacement.

The usual velocity symbol is v.

Given our discussion in § 2.1 on differentiation and derivatives, clearly

$$v = \frac{dx}{dt} \quad : \tag{20}$$

i.e., velocity is the derivative of displacement.

Velocity is a vector. In one-dimensional cases, this means it can be positive or negative. Note x can be positive and v negative and vice versa.

Velocity defined by differentiation can be called instantaneous velocity since it a quantity with a value at an instant in time.

But usually one only says "instantaneous velocity" to clearly distinguish velocity from average velocity.

The average velocity $v_{\rm avg}$ over a time interval Δt is given by

$$v_{\rm avg} = \frac{\Delta x}{\Delta t} \ , \tag{21}$$

where Δx is the change in displacement in Δt.

There are different ways of defining average quantities. Our definition for average velocity seems most reasonable since

$$v_{\rm avg}\Delta t = \Delta x \quad : \tag{22}$$

i.e., the average velocity as we've defined it for time interval Δt gives the displacement change in Δt by a simple multiplication operation.

The magnitude of velocity $|v|$ is the speed in physics jargon.

Actually, many people often use velocity as a synonym for speed when talking though not when writing. Context must decide if velocity means velocity or means speed.

There is no special symbol for speed. One can use $|v|$.

But there is an awkwardness.

In multi-dimensional cases, one denotes a vector with an arrow over the quantity symbol. For general vector one has $\vec{A}$ say. The magnitude of $\vec{A}$ is often written A or if one wants to be more explicit $|\vec{A}|$, but the explicit form is too tedious for yours truly and looks awkward.

So in multi-dimensional cases, velocity is $\vec{v}$ and its magnitude (i.e., its speed) is v.

It's not possible for the one-dimensional case to be consistent with the multi-dimensional case, since in one-dimensional cases v is velocity.

So one can use $|v|$ or v for speed as long as knows what one means.

3.2.1. Example: Average Velocity and Average Speed

Say I went 10 km north in 2 hours and then went 5 km south in 3 hours.

Question for the class: what is my average velocity?

You have 15 seconds. Go.

Behold:

$$v_{\rm avg} = \frac{10-5}{5} = 1\,{\rm km/h}\ . \tag{23}$$

Question for the class: what is my average speed?

You have 15 seconds. Go.

Behold:

$$v_{\text{avg,speed}} = \frac{10+5}{5} = 3\,\text{km/h} \ . \tag{24}$$

Note there is an oddity. Speed is $|v|$ where v is velocity, but $v_{\text{avg,speed}}$ is not $|v_{\text{avg}}|$ at least in the way people usually interpret questions about average speed. Average speed should be the total distance divided by time traveled so that average speed times time traveled gives the total distance traveled.

3.2.2. Constant Velocity

Say velocity v is a constant.

What is displacement x as function of time?

You have 15 seconds working individually **OR** 1 minute working in groups. Go.

You integrate to get

$$x = \int v\,dt = vt + x_0 \ , \tag{25}$$

where x_0 is the initial displacement at time zero.

Recall from § 2.4 that the zero subscript is vocalized as "nought" when vocalizing the variable. So x_0 is vocalized as X-nought.

3.3. Acceleration

Acceleration is the rate of change of velocity.

The usual acceleration symbol is a.

Given our discussion in § 2.1 on differentiation and derivatives, what is the formula for a in terms of v and in terms of x?

You have 15 seconds. Go.

Behold:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \ : \tag{26}$$

i.e., acceleration is the derivative of velocity and the 2nd derivative of displacement.

Acceleration is a vector. In one-dimensional cases, this means it can be positive or negative.

You can have a negative acceleration with a positive velocity and vice versa.

A negative acceleration with a positive velocity means the object is slowing down in the direction it is moving.

A positive acceleration with a negative velocity means the same thing.

In both situations, the acceleration can be called a **DECELERATION** which means a decrease in speed or an acceleration opposite to direction of motion. Deceleration is a useful descriptive term, but not essential in kinematics.

Acceleration defined by differentiation can be called instantaneous acceleration since it a quantity with a value at an instant in time.

But usually one only says instantaneous acceleration to clearly distinguish velocity from average acceleration.

The definition of the average acceleration $a_{\rm avg}$ over a time interval Δt is analogous to that of average velocity. Thus, we have

$$a_{\rm avg} = \frac{\Delta v}{\Delta t} \ , \tag{27}$$

where Δv is the change in displacement in Δt.

There are different ways of defining average quantities. Our definition for average acceleration seems most reasonable since

$$a_{\rm avg}\Delta t = \Delta v \ \ : \tag{28}$$

i.e., the average acceleration as we've defined it for time interval Δt gives the velocity change in Δt by a simple multiplication operation. The reasoning for reasonableness of the definitions of average velocity and and average acceleration are the same.

The magnitude of acceleration has not special name of its own in physics jargon. It's just called acceleration too. Context must decide what is meant.

The symbol for magnitude of acceleration is $|a|$ or just a. Context must decide what is meant whether a is acceleration or magnitude of acceleration. There is the analogous ambiguity in the symbols for speed in one-dimensional cases (see § 3.2).

3.3.1. Constant Acceleration

Say acceleration a is a constant.

What are velocity v and displacement x as functions of time?

You have 15 seconds working individually **OR** 1 minute working in groups. Go.

You integrate to get

$$v = at + v_0 \ , \tag{29}$$

$$x = \frac{1}{2}at^2 + v_0 t + x_0 \ , \tag{30}$$

where v_0 is initial velocity at time zero and x_0 is the initial displacement at time zero.

We note that velocity is linear with time and displacement is quadratic with time when acceleration is constant.

The constant acceleration case is a very special case of motion that does **NOT** occur all that often in nature or technology.

But it is easy to analyze constant acceleration cases, and so they are educationally useful.

So will consider the constant acceleration case in succeeding sections.

The constant velocity case is even simpler to analyze, but not so interesting.

There is one very important case of constant acceleration in nature. This is the case of acceleration due to gravity in a constant gravitational field.

The gravitational field of the Earth near the Earth's surface is constant to good approximation. So the case of acceleration due to gravity in a constant gravitational field is very important for all systems near the Earth's surface.

Motion under the force of gravity alone is called **FREE FALL**—in the first meaning of the term free fall. We consider free fall in § 6.

3.4. Jerk

Using differentiation, we go from displacement to velocity to acceleration.

Can we go further? Should we go further?

Yes. No.

The derivative of acceleration is called jerk which according to Wikipedia—the supreme authority—has symbol j. Thus

$$j = \frac{da}{dt} = \frac{d^2v}{dt^2} = \frac{d^3x}{dt^3} . \tag{31}$$

So jerk can be defined and it has its uses particularly in some technologies. But in Newtonian physics, jerk is conceptually much less important than the kinematic variables displacement, velocity, and acceleration. You need those three to understand motion.

Jerk is around, of course, whenever acceleration changes, but you don't have to use the name and, in fact, it is seldom used.

More important no basic physical laws depend on jerk alone to my knowledge. There are basic laws that on displacement, velocity, and acceleration. This means displacement, velocity, and acceleration occur in basic formulae, but jerk doesn't.

So at our level, jerk seldom comes up.

It might turn up in some problems.

4. MOTION DIAGRAMS

One can plot acceleration, velocity, or displacement versus time on Cartesian plot to create motion diagrams.

In Figure 2, we illustrate a constant-acceleration case on acceleration, velocity, or displacement motion diagrams.

As we showed in § 3.3.1, a constant acceleration leads to velocity linear with time and displacement quadratic with time. This is behavior is seen in Figure 2.

Smooth curve motion diagrams of any sort can be imagined.

In fact, for educational reasons, we sometimes want to consider motion with **SINGULARITIES**. Singularities are points where a function or some derivative order of it is not defined as mentioned in § 2.1. A singularity in a diagram is loosely speaking an unsmoothness. At our level, the notable singularities are cusps, discontinuities, and infinities.

A cusp is a point of a function where the function is continuous, but the derivative is not. It's a kink or a corner in a function plot. A discontinuity is a point of a function where the function is discontinuous (i.e., has two values speaking loosely). An infinity is a point of a function where the function diverges to positive or negative infinity.

Figure 3 illustrates the singularities cusps, discontinuities, and infinities.

Cusps, discontinuities, and infinities in motion diagrams have to regarded as ideal limits that can't actually happen in nature.

For example, a cusp in a displacement motion plot means that there are two different velocities at one point in time. This can't actually happen. But one can imagine a change in velocity that is very rapid in time and the ideal limit as the time of change goes to zero is a cusp.

Since a cusp in a displacement motion plot means two velocities at one instant in time, the cusp corresponds to a discontinuity in the corresponding velocity plot. A discontinuity in a velocity plot means there is a change in velocity at one instant in time and thus an infinity in acceleration at that time. We illustrate the situation in in Figure 4.

Fig. 2.— Acceleration, velocity, and displacement diagrams for a constant-acceleration case.

Fig. 3.— The singularities cusps, discontinuities, and infinities.

Fig. 4.— Cusp, discontinuity, and infinity in, respectively, displacement, velocity, and acceleration motion diagrams. Displacement has two linear regions: one where it is growing linearly and one where it is constant. There is a cusp between the two regions. Velocity has two constant regions: one with a positive value corresponding to increasing displacement and one with a zero value corresponding to constant displacement. There is a discontinuity between the two regions where displacement has a cusp. Velocity has two zero regions both corresponding to constant velocity regions. There is an infinity between the two regions where displacement has a cusp and the velocity a discontinuity.

The cusp-discontinuity-infinity situation arises for any function differentiated with respect to variable. A cusp in the function causes a discontinuity in the derivative and an infinity in the 2nd derivative.

Discontinuities in acceleration (which are cusps in velocity) are common in intro physics problems.

The reason for this is as follows. It is relatively easy to analyze constant-acceleration cases as we'll see in § CONSTANT-ACCELERATION KINEMATIC EQUATIONS. So if we want tractable problems with multiple constant accelerations, we have to change the acceleration discontinuously at points in time. One can calls such problems multi-phase problems since they have multiple phases of constant acceleration.

Any function which is continuous between isolated points where it is discontinuous is called **PIECEWISE** continuous. Piecewise something means the something holds except at isolated points in math jargon. So acceleration that is continuous except at isolated points is piecewise continuous.

Figure 5 illustrates case of a piecewise-continuous acceleration with constant-acceleration phases.

Fig. 5.— A case of a piecewise-continuous acceleration with constant-acceleration phases.

5. CONSTANT-ACCELERATION KINEMATIC EQUATIONS

As mentioned in § 3.3.1, constant-acceleration cases are simple to analyze, but are also relatively interesting—unlike constant velocity cases which are relatively uninteresting to analyze.

We got two kinematic equations for constant-acceleration cases from integration in § 3.3.1:

$$v = at + v_0 , \tag{32}$$

$$\Delta x = \frac{1}{2}at^2 + v_0 t , \tag{33}$$

where $\Delta x = x - x_0$. Note again that a constant acceleration leads to velocity linear with time and displacement quadratic with time.

The reason for using Δx rather than x and x_0 is that in formulating and solving constant-acceleration problems, it is usually possible and easier to deal with a single position variable.

Solving intro physics problems often comes down to doing algebra to solve for unknowns in terms of knowns after having identified the correct equations relating the knowns and unknowns.

Equations (32) and (33) are the basic equations for solving constant-acceleration kinematic problems. They are algebraically independent: you can't create one from the other by algebra. You can create one from the other using calculus since that is how we got equation (33) from equation (32).

How many distinct variables are there in equations (32) and (33)?

Equations (32) and (33) have 5 variables: a, v, v_0, Δx, and t.

How many unknowns can we solve for from equations (32) and (33)?

Right/wrong.

You can solve for 2 unknowns at most in general.

In general, you can solve for only as many unknowns as you have algebraically independent equations. We won't do a proof—which yours truly doesn't know in general nor in all its special cases anyway. In special cases, you can solve for some unknowns even if you don't have enough knowns to solve all unknowns. Such special cases do turn up in this course.

Since we have 5 variables altogether in our 2 basic equations, we need at least 3 knowns to solve a constant-acceleration problem.

Be warned, the 3 knowns may **NOT** be given explicitly in problems. Frequently, one or more must be inferred from the problem. Problems frequently have a lot of camelflaj—which is the art of hiding camels in flaj.

Now all constant-acceleration problems can be solved from equations (32) and (33), but sometimes rather tediously since one sometimes has to go through a procedure of eliminating one unknown to get one unknown in one equation.

The process is speeded up by algebraically creating 3 more constant-acceleration kinematic equations from equations (32) and (33).

First, from equations (32) and (33) find an equation without the time t.

The start is to get an explicit expression for time from equation (32) substitute it into equation (33).

You have 30 seconds working individually **OR** you have 2 minutes working in groups. Don't look at any notes. Go.

Well $t = (v - v_0)/a$ from equation (32). Substitute this into equation (33) to find

$$\begin{aligned} \Delta x &= \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 + v_0\left(\frac{v - v_0}{a}\right) \\ 2a\Delta x &= v^2 + v_0^2 - 2vv_0 + 2v_0 v - 2v_0^2 \\ 2a\Delta x &= v^2 - v_0^2 \\ v^2 &= v_0^2 + 2a\Delta x \ . \end{aligned} \tag{34}$$

This equation yours truly calls the **TIMELESS EQUATION** since it has no time in it. But this is not a conventional name—just my idiosyncratic mnemonic name.

The timeless equation solved for either v or v_0 gives two possible solutions that are additive inverses of each other. Which solution is the desired one must be determined from other conditions of the problem.

Second, from equations (32) and (33) find an equation without the acceleration a.

You have 30 seconds working individually **OR** you have 2 minutes working in groups. Don't look at any notes. Go.

Well $a = (v - v_0)/t$ from equation (32). Substitute this into equation (33) to find

$$\begin{aligned} \Delta x &= \frac{1}{2}\left(\frac{v - v_0}{t}\right)t^2 + v_0 t \\ \Delta x &= \frac{1}{2}(v - v_0)t + v_0 t \\ \Delta x &= \frac{1}{2}(v_0 + v)t \ . \end{aligned} \tag{35}$$

We note that $(1/2)((v_0 + v)$ is the average of the initial and final velocities. More importantly it is the average velocity as we defined it in § 3.2 since $(1/2)(v_0 + v)$ times the time interval t gives the displacement.

Third, from equations (32) and (33) find an equation without the initial velocity v_0.

You have 30 seconds working individually **OR** you have 2 minutes working in groups. Don't look at any notes. Go.

Well $v_0 = v - at$ from equation (32). Substitute this into equation (33) to find

$$\begin{aligned} \Delta x &= \frac{1}{2}at^2 + (v - at)t \\ \Delta x &= -\frac{1}{2}at^2 + vt \; . \end{aligned} \tag{36}$$

This 5th kinematic equation is often omitted in textbooks. But it should be in the set of constant-acceleration kinematic equations to make the set complete. The complete set allows one to solve for any of the 5 variables given any 3 knowns by solving for one equation for one unknown. The trick is based on the fact that each of the 5 constant-acceleration kinematic equations lacks one of the 5 variables. We show how to use the complete set of constant-acceleration kinematic equations to solve problems below in § 5.1.

It is important to note that though there are 5 constant-acceleration kinematic equations, only 2 are algebraically independent.

You can still only solve for 2 unknowns.

What happens if you try to solve for more unknowns?

Typically you end up with the equation $0 = 0$.

For example, say you know t and v_0 only.

You now try to eliminate unknowns v and Δx from the timeless equation using equations (32) and (33) and try to solve for a in terms of t and v_0. What do you get? Well

$$\begin{aligned} v^2 &= v_0^2 + 2a\Delta x \\ (at + v_0)^2 &= v_0^2 + 2a\left(\frac{1}{2}at^2 + v_0 t\right) \\ a^2t^2 + v_0^2 + 2v_0 at &= v_0^2 + a^2t^2 + 2v_0 at \\ 0 &= 0 \; . \end{aligned} \tag{37}$$

A true equation, but not a solution.

5.1. Solving a Constant-Acceleration Problem

When confronted with a constant-acceleration problem, the procedure is as follows.

Identify the 3 knowns and 2 unknowns.

Find the constant-acceleration kinematic equation that does **NOT** contain the unknown that you are **NOT** solving for.

That equation allows you to solve for the unknown you do want as one-unknown-in-one-equation problem.

With a little practice, you hardly need to think of this procedure—unless the camelflaj is very good—you just jump to the right equation and solve the problem.

You see why the 5th constant-acceleration kinematic equation (i.e., eq. (36)) is necessary as we said in § 5. It allows you to follow the procedure in cases where the initial velocity v_0 is the unwanted unknown.

Say you have solved for the wanted unknown. You now have 4 knowns. You can now solve for the last unknown using any equation that contains that unknown since all the other variables in that equation are known. You can't solve for a unknown using an equation that does not contain it: e.g., you can't solve for Δx from the 1st constant-acceleration kinematic equation $v = at + v_0$.

Table 1 summarizes the constant-acceleration kinematic equations and gives the missing variable in each equation.

Table 1. The Constant-Acceleration Kinematic Equations

Equation Number	Equation	Missing Variable
1	$v = at + v_0$	Δx
2	$\Delta x = \frac{1}{2}at^2 + v_0 t$	v
3 (timeless equation)	$v^2 = v_0^2 + 2a\Delta x$	t
4	$\Delta x = \frac{1}{2}(v_0 + v)t$	a
5	$\Delta x = -\frac{1}{2}at^2 + vt$	v_0

Note. — There are variables a, v, v_0, Δx and t. One can only solve for two unknowns since only two equations in the set of 5 equations are independent. The easiest way is to solve for the wanted unknown using the equation where the other unknown is the missing variable. The solution is then a 1 unknown in 1 equation case. The first 3 equations get most of the use in practice in intro physics courses. There is no physical reason for this, it's just the way problems are written. The 5th equation is rarely used because problems are rarely written for it and many textbooks omit it. But logically it is needed to complete the set. Δx can be written $x - x_0$, but some other is needed to know x and x_0 separately.

The first 3 equations in Table 1 get most of the use in practice in intro physics courses. There is no physical reason for this, it's just the way problems are written. The 5th equation is rarely used because problems are rarely written for that require it and this is why many textbooks omit it. But there can be problems that to require it, and so it shouldn't be omitted. Even if there were no such problems, to complete our understanding of constant-acceleration cases we should specify it.

Now cover up all your notes.

Write down the 5 constant-acceleration kinematic equations from memory while yours truly does the same.

You have 30 seconds. Go.

How many got them all?

Enough yes/no.

If no, cover your notes and everyone try again.

5.2. Example: A Jet Lands on an Aircraft Carrier

Say a jet lands on an aircraft carrier.

It's initial speed on landing is 63 m/s.

It stops in 2 s.

Assuming it has a constant acceleration, what is that acceleration?

Identify the knowns, the unknowns, and appropriate constant-acceleration kinematic equation to use.

You have 30 seconds working individually or 2 minutes working in groups. Go.

First note that implicitly the final velocity $v = 0$ since the jest–er jet—stops.

So we have 3 knowns: $t = 2$, $v_0 = 63$, and $v = 0$.

The unknowns are a and Δx.

We don't want Δx.

The appropriate constant-acceleration kinematic equation is the 1st one.

Solving this equation for acceleration gives

$$a = \frac{v - v_0}{t} = \frac{0 - 63}{2} = -31.5\,\mathrm{m/s^2}\ . \tag{38}$$

If we wanted Δx, we could find in many ways. Using the 4th kinematic equation looks easiest. We get

$$\Delta x = \frac{1}{2}(v_0 + v)t = 63\,\mathrm{m}\ . \tag{39}$$

The Δx number is coincidentally the same as the initial velocity number.

5.3. Factoids About Constant-Acceleration Cases

There are some general factoids—in the sense of tiny little facts—about constant-acceleration cases that are worth knowing. Knowing them explains why certain results turn up many times.

Fig. 6.— The curve for a constant acceleration case on displacement motion diagram. Here $a > 0$ and so the curve opens upward. The object comes in from positive infinity along a line, reaches a turning point (which is its minimum displacement), and then goes back to positive infinity.

Recall that the 2nd constant-acceleration equation

$$\Delta x = \frac{1}{2}at^2 + v_0 t \tag{40}$$

is a quadratic function of t for x if $a \neq 0$.

If $a = 0$, the formula is actually a straight line function of t for Δx. We won't consider the $a = 0$ case below because it's trivial and complicates the discussion.

On a displacement motion diagram, the curve is a **PARABOLA** which opens upward if $a > 0$ and downward if $a < 0$. So an object with a constant acceleration will come in from positive/negative infinity and reach a **TURNING POINT** and then turn around and head back to positive/negative infinity. The positive case is for $a > 0$ and the negative for $a < 0$.

See Figure 6 for displacement motion diagram for a constant-acceleration case with $a > 0$.

The discussion shows that the object is twice at every point on one side of the turning point and never on the other side. It is at the turning point once.

Of course, the above discussion assumes that acceleration never changes. In many problems, the acceleration does change. But the constant-acceleration kinematic equations themselves don't incorporate that information. They behave as if they applied at all times. The person dealing with them has to turn them on or off as appropriate to account for different phases of motion.

We can solve for time as a function of position. The solutions are

$$t = \frac{-v_0 \pm \sqrt{v_0^2 + 2a\Delta x}}{a} \tag{41}$$

where we have used the quadratic formula which is the general solution of a quadratic equation.

If the **DISCRIMINANT** $v_0^2 + 2a\Delta x > 0$, there are two real solutions for t for a given Δx. This just confirms what we said above, the object is twice at every point on one side of the turning point.

If the discriminant $v_0^2 + 2a\Delta x < 0$, there are **NO** real solutions for t for a given Δx. This just confirms what we said above, the object is never at any point on the other side of the turning point.

If the discriminant $v_0^2 + 2a\Delta x = 0$, there is only one solution. This obviously when the object is at the turning point which it only is once as we said above.

What is the time interval from when the object is at Δx and the time of the turning point? Well given that the discriminant is zero at the turning point, we find

$$\Delta t = \frac{-v_0 \pm \sqrt{v_0^2 + 2a\Delta x}}{a} - \frac{-v_0}{a} = \pm \frac{\sqrt{v_0^2 + 2a\Delta x}}{a} \ . \tag{42}$$

There are two solutions since the object is at Δx twice. The solutions are equal in magnitude. Thus, it takes as long to go from Δx to the turning point as it does to return from the turning point to Δx.

The timeless equation allows us to immediately see another factoid.

Recall the timeless equation is

$$v^2 = v_0^2 + 2a\Delta x \ . \tag{43}$$

Whenever the object is at a specific Δx, there are only two possible velocities:

$$v = \pm\sqrt{v_0^2 + 2a\Delta x} \ , \tag{44}$$

The magnitudes of the velocities or the speeds of motion are the same whenever at Δx is the same.

One velocity corresponds to approaching the turning point and the other to going away from it.

What if $\Delta x = 0$: i.e., what if the object is at its initial position either because it is at initiation or because it is on its only return?

Then

$$v = \begin{cases} v_0 & \text{for initiation;} \\ -v_0 & \text{on return.} \end{cases} \tag{45}$$

So on return to its initial point, an object has the same speed, but is moving in the other direction. This is just a special case of result that the object has the same speed at a given Δx whether it is coming or going.

5.4. An Example Problem from the Homework

A two-object problem from the homework might be good example at this point.

5.5. Relative Motion

Motion is always relative to something.

Hitherto we have mostly left the something unspecified or have been implicitly assuming that motion is relative to the ground.

But there can be motion relative to anything including another object.

Say you have object 1 and object 2.

Let's say object 1 is taken to define the origin.

The displacement of object 2 relative to object 1 is

$$x_{12} = x_2 - x_1 \ . \tag{46}$$

We can differentiate to get the relative velocity and relative acceleration. The full set of relative motion variables is then

$$x_{12} = x_2 - x_1 \ , \tag{47}$$

$$v_{12} = v_2 - v_1 \ , \tag{48}$$

$$a_{12} = a_2 - v_1 \ . \tag{49}$$

What if a_{12} is constant?

Well all the same calculus and algebra we did for constant acceleration without specifying our motion as relative can be repeated exactly.

So the constant-acceleration kinematic equations apply to relative motion too.

Here's a good place for doing a relative motion problem from the homework as an example.

6. THE GRAVITATIONAL FIELD AND FREE FALL

In modern physics, the gravitational field is a real physical thing that emanates from mass and causes the gravitational force (i.e., causes gravity).

The gravitational field is a vector field, in fact. At every point in space it has a magnitude and direction. Yours truly like to picture this as little arrows attached to every point in space—there are a continuum of points and arrows, but one can only draw a finite discrete set. The vectors extend in their own abstract gravitational field space: they have no extent in space space. They do point in space space.

The symbol for the gravitational field is $\vec{g}$, where the arrow indicates its vector nature.

We'll go further into the gravitational field in the lectures *NEWTONIAN PHYSICS I* and *GRAVITY*

Now forces cause accelerations.

Forces and how they cause acceleration is part of dynamics which we get to in the lecture *NEWTONIAN PHYSICS I*.

So the gravitational field causes acceleration.

Here we will just say that if the only external force on an object is gravity, then the object will be in motion and that motion is called **FREE FALL**.

But like a lot of other terms, **FREE FALL** can mean something else depending on context. It can mean when gravity, drag (AKA fluid resistance), and the buoyancy force are the only external forces acting on an object. Context usually must decide what is meant. Except in § 7, we mean free fall in the first sense. We discuss drag qualitatively in §§ 6.2 and 7.

Now if the object is a point particle, the object's acceleration will be equal to the gravitational field at the point particle's position: i.e.,

$$\vec{a} = \vec{g} \ . \tag{50}$$

We give no proof of this result here. We prove it in the lecture *NEWTONIAN PHYSICS I*.

Note acceleration and gravitational field have the same dimensions even though they are different quantities. They can be thought of has having the same derived unit too (i.e., the m/s^2), but since they are different quantities people often use different unit symbol for gravitational field. We'll get to that different unit symbol in lecture *NEWTONIAN PHYSICS I*—we don't want to get ahead of ourselves.

It's easy to prove (and we will prove it in the lecture *NEWTONIAN PHYSICS I*) that if the gravitational field is constant over an object, then acceleration of the center of mass of the object (which is a point recall) in free fall is exactly equal to the gravitational field at its center of mass.

This result is **INDEPENDENT** of the peculiar nature of object which is a remarkable fact. Especially noteworthy is that there is **NO** mass dependence. The physical understanding of object-independence of gravitational center-of-mass acceleration is given in the lecture *Newtonian Physics I.*

Near the Earth's surface the gravitational field is nearly constant in space, in fact, and thus to excellent approximation the center-of-mass acceleration of any object in free fall near the Earth's surface is equal the gravitational field at the center of mass.

From here on, we'll just say object when we mean center of mass of object and displacement, velocity, and acceleration when we mean center-of-mass displacement, velocity, and acceleration. These are essential common abbreviations. One usually only bothers with saying center of mass or center-of-mass something when clarity demands that.

So let's consider free-fall motion near the Earth's surface.

Since in this lecture we only deal with one-dimensional motion, the motion can only be up or down (i.e., in vertical direction).

Since y is the usual symbol for vertical coordinate, we will use y instead of x for displacement whenever the motion is in the vertical direction.

Now do we take up as positive or down as positive.

It's our choice.

Yours truly's ordinary rule is to take up as positive if the motion is up or up and down and to take down as positive when the motion is only downward.

The first case conforms to our ordinary idea that up is positive, and so allows one to treat up and down motions together without confusion. The second is a simplification to get rid of negative signs everywhere. There is no confusion in the second case since all motion is downward.

The gravitational field near the Earth's surface points **DOWN** toward the Earth's center nearly exactly.

Thus, free-fall acceleration near the Earth's surface is

$$a = \begin{cases} -g & \text{for up positive;} \\ g & \text{for down positive,} \end{cases} \tag{51}$$

where we have dropped the vector arrows since we are considering one-dimensional motion only and g is the magnitude of the gravitational field.

Note absolutely, positively, always, g is a magnitude, and so is always a positive value.

6.1. Values of g for Near Earth's Surface

For near Earth's surface what is g's value?

The fiducial (i.e., reference value) for g near the Earth's surface in **THIS CLASS** is exactly $9.8\,\mathrm{m/s^2}$. Note the units of g are the units of acceleration, but g is only an acceleration for an object in free fall.

The quantity g for the Earth is often called the acceleration due to gravity—but absolutely, positively, its only the acceleration for an object in free fall. The quantity g for the Earth is also sometimes called little g to distinguish it from big G which the gravitational constant in Newton's law of universal gravitation (which we'll get to in the lecture *GRAVITY*).

Wikipedia—the supreme authority—says that g can also be called the Earth's gravity—but I don't like that term and won't use it.

In fact, the gravitational field near the Earth's surface is not exactly constant either in magnitude or direction. The variations are too small for human senses to note, but are quite easily measured.

The largest variations are actually due to altitude and latitude. For reasons that we'll go into in the lecture *GRAVITY*, the Earth's gravitational field decreases as distance from the Earth's center increases as long as you outside of the interior of Eath. Thus, as altitude increases, g decreases.

In the case of latitude, there is a small amount variation due the Earth's oblateness and much more variation due to the **CENTRIFUGAL FORCE** which we discuss below. Oblateness is a flattening at the poles and bulging at the equator. For the Earth, the equatorial radius 6378.1 km and then there is a monotonic decline in radius on average going north/south to reach a polar radius 6356.8 km. The decrease in radius with increasing latitude causes a small decrease in g with latitude.

Now for the **CENTRIFUGAL FORCE** affect on g. The Earth has a **CENTRIFUGAL FORCE** due to its rotation. The centrifugal force is an inertial force: we discuss inertial forces in the lecture *NEWTONIAN PHYSICS II*. They arn't real forces, but they sure seem as if they are. The centrifugal force is really a way of accounting for the fact that objects move in straight lines if not acted on by forces. So in a rotating frame, there seems to be a force trying to throw you outward: this seeming force is the centrifugal force.

Remarkably the centrifugal force acts exactly like a gravitational force pointing outward from the axis of rotation. So one can define a centrifugal force field. It is zero on the axis and grows linearly with distance from the axis for constant rotational period. For the Earth then the centrifugal force field is zero at the poles and grows with decreasing latitude to be $0.043\,\mathrm{m/s^2}$ outward from the axis at the equator.

Because the centrifugal force acts just like a gravity force, when you measure g directly, you actually measure the combined effect of gravity and the centrifugal force.

Consequently, by convention g values for the Earth are for the combined gravitational and centrifugal fields. These g values are the magnitude of the vector sum of the gravitational field pointing toward the Earth's center to high accuracy and the centrifugal force field pointing outward from the Earth's axis. The contribution of centrifugal force field is sufficiently small that gravity still seems to down to the Earth's center to human perception, but actually it points minutely off center. The off-centeredness can be measured. In fact, the direction of the gravitational field is often used to define local downward direction.

There are also small variations in g due to the local geology of the Earth. They are used extensively in geological analysis.

To summarize the g values relevant to Earth,

$$g = \begin{cases} 9.8\,\mathrm{m/s^2} & \text{a fiducial exact value for our course;} \\ 9.80665\,\mathrm{m/s^2} & \text{an official average value called standard gravity;} \\ 9.832\,\mathrm{m/s^2} & \text{polar sea-level value;} \\ 9.780\,\mathrm{m/s^2} & \text{equatorial sea-level value;} \\ 9.823\,\mathrm{m/s^2} & \text{equatorial sea-level value without the centrifugal force field;} \\ 9.749\,\mathrm{m/s^2} & \text{equatorial value at altitude 10 km in free air} \end{cases} \tag{52}$$

(Wikipedia: Earth's gravity).

As one can see, the variations in g are small. The latitude variation is only about $0.5\,\%$. The change from sea level to $10\,\mathrm{km}$ at the equator is only about $0.3\,\%$ decrease. But this a free air value. On land surfaces at above sea level, the gravitational effect of the elevated land mass prevents gravity from decreasing so quite so quickly with height. Nevertheless, on

Mount Everest at 8.850 km above sea level, g decreases from sea level value by about 0.3 %.

All the variations in g are below human perception, but are easily measurable.

6.2. More on Free-Fall Motion Near the Earth's Surface

For free-fall motion near the Earth's surface, g is nearly constant and consequently free-fall acceleration is nearly constant.

For educational reasons, we will assume g is exactly constant and has the fiducial $g = 9.8\,\mathrm{m/s}^2$.

Say we drop an object from **REST**, what is its velocity and displacement after time t?

You have 30 seconds. Go.

Using the 1st and 2nd constant-acceleration kinematic equation, we find

$$v = gt \, , \tag{53}$$

$$y = \frac{1}{2}gt^2 \, , \tag{54}$$

where we have taken downward as positive.

Note both results are independent of the nature of the object dropped and especially that they are independent of the mass of the object. This is just a consequence that the acceleration is object-independent as discussed in § 6.

We further see that two objects released from rest at the same height should hit the ground at the same time. The fall time is

$$t = \sqrt{\frac{2y}{g}} \, . \tag{55}$$

I will now do the Galileo-Leaning-Tower-of-Pisa demonstration with a book and sheet of paper.

You saw. They hit the ground at the same time right?

Yes/No.

Why not?

Air resistance or, for fluids in general, fluid resistance or drag. I'll use the term drag since Wikipedia—the supreme authority—prefers drag.

Free fall is the absence of other forces.

But in air there will always be drag.

The drag force always points opposite to the direction of motion and increases with velocity magnitude: its zero for zero velocity. By velocity, we mean velocity relative to the fluid that is causing the drag force.

For relatively dense objects and/or relatively short falls, drag can be negligible. We discuss drag briefly in § 7. We may or may not depending on available time go into the formalism of drag in the lecture *NEWTONIAN PHYSICS II*.

When Galileo (1564–1642) did his famous ball-dropping experiment, the balls didn't hit the ground exactly at the same time. He couldn't release them exactly at the same time and drag would affect their motions a bit differently. But they nearly hit the ground at the same time.

Galileo's point was that ideally they should have the same fall time. You had to imagine removing the complicating effects and imagine the ideal experiment. In fact, Galileo later did more careful experiments and showed that the acceleration due to gravity became closer to a constant, the closer the ideal case was approached. They weren't ball dropping experiments. They were ball rolling down the incline experiments in which the physics is a bit different, but the conclusion of constant acceleration due to gravity is the same.

Galileo's method of imagining the ideal case of an experiment has become an important conceptual tool in all the sciences. When you casually observe nature—which is the way many natural philosophers did before Galileo—then you see complex motions with all kinds effects going on and it is not at all obvious that physics obeys exact mathematical laws. You have to imagine the ideal cases and try to get as close to them as you can in actual experiment in order to get to those exact mathematical laws. Then you can add the complications back trying to understand how they to obey exact mathematical laws.

Actually, Galileo's ball-dropping experiment is sometimes claimed to never have happened—even the supreme authority Wikipedia inclines to this view. But Galileo's earliest biographer Vincenzo Viviani (1622–1703), who knew Galileo personally, tells us Galileo did it though without much detail. So Galileo probably did do it. But it may have been more of demonstration for the students than an experiment. If it did happen, it must have happened in the interval 1589–1592 when he was a professor (untenured) at the University of Pisa.

6.3. Free Fall with Negligible Drag

If I drop two fairly dense objects, they do approximately have the same fall time.

See.

Yours truly drops two dense objects.

Objects of different density and shape would have the same fall time in a vacuum.

Now for a **DEMONSTRATION** with a near vacuum.

Here is an low-pressure tube with a ball and a feather. The air density and hence air drag is much reduced inside the tube.

Now see the ball and feather do have the same fall time as far as one can tell.

It's hard to make them hit at the same time, because it's hard to start their falls at the same time.

6.4. Example: Ball Thrown Up

A ball is thrown straight up from a just off the side of a tall building.

The ball's initial speed is 20 m/s.

Neglect drag.

We pretty well always neglect drag, unless we say otherwise.

6.4.1. Maximum Height

Find the maximum height the ball reaches from the top of top the building.

You have 2 minutes working in groups. Go.

Take upward as positive. This means that $a = -g$.

We know $v_0 = 20$.

At the top of the trajectory, $v = 0$, but this is not given explicitly—it's been camelflajjed. One has to figure it out that $v = 0$ at the top of the trajectory

So we have three knowns and two unknowns: i.e., Δy and t.

We don't have and don't want time t.

So its a job for the timeless equation.

Rearranging the timeless equation, we get

$$\Delta y = \frac{v^2 - v_0^2}{2a} = \frac{0 - 20^2}{-2g} \approx 20\,\mathrm{m}\ . \tag{56}$$

6.4.2. Time to Maximum Height

What is the time to maximum height?

You have 30 seconds working individually. Go.

Since we have four knowns now, any equation containing the unknown will do.

Probably, the 1st constant-acceleration kinematic equation is easiest.

Rearranging that equation, we get

$$t = \frac{v - v_0}{a} = \frac{0 - 20}{-g} \approx 2\,\mathrm{s}\ . \tag{57}$$

6.4.3. Time to Return to the Initial Height

What is the time to return to the initial height?

You have 30 seconds. Go.

Well there are several ways of doing this.

Probably the easiest is just to calculate the time to free fall $|\Delta y|$.

We know

$$|\Delta y| = \frac{1}{2} g t^2\ , \tag{58}$$

and so

$$t = \sqrt{\frac{2|\Delta y|}{g}} \approx \sqrt{\frac{2 \times 20}{g}} \approx 2\,\mathrm{s}\ . \tag{59}$$

This is the time it takes to fall down from maximum height.

Therefore the total time is approximately 4 s.

Actually, the rise and fall times are equal.

Is there anyway that we should just have known this?

Well yes.

In § 5.3 on factoids about the constant-acceleration cases, we proved in general that the time interval it takes an object to go from some point to the turning point is equal to the return time from the turning point to that point.

In our case, the maximum height point is the turning point.

So we should have known immediately that the fall time equals the rise time.

6.4.4. Velocity at the Time of Return to the Initial Height

What is the velocity when the ball gets back to its initial height?

This is an easy one.

You have 30 seconds working individually. Go.

Well from the timeless equation, we know that

$$v = \pm\sqrt{v_0^2 + 2a\Delta y} = \pm|v_0| = \pm 20\,\mathrm{m/s}\ . \tag{60}$$

The upper case solution is just initial velocity which is positive.

The lower case solution is negative which is what velocity during a fall should be.

So the velocity on return to the initial height is $-20\,\mathrm{m/s}$.

Actually, we should have known this result right away from the results in § 5.3 on factoids about the constant-acceleration cases.

6.5. Example from the Homework

Here's a good point to do a two-phase constant-acceleration problem from the homework.

7. TERMINAL VELOCITY

In § 6.2, we briefly discussed drag.

In intro physics, we largely avoid drag because it is tricky to deal with. Without drag, projectile motion near the Earth's surface is relatively simple and can be treated by simple analytic means. Thus it is educationally good to avoid the complications of drag mostly. It is a simplifying assumption to neglect drag.

We also usually neglect the fluid buoyancy force. The effect of the buoyancy force is equivalent to a mass reduction of the object by a multiplicative factor that depends on ratio of fluid density to object density for total immersion in the fluid. For objects much denser than fluid, the buoyancy force is relatively small and is often negligible. The buoyancy force is discussed in fluid dynamics which may or may not turn up in our study of intro physics. We assume the buoyancy force is negligible to simplify our discussion below of terminal velocity.

Now we may or may not depending on available time go into the formalism of drag in the lecture *NEWTONIAN PHYSICS II*.

But we can say a few more words about drag here and its consequence terminal velocity.

As we mentioned in § 6.2, drag is the force of fluid resistance to motion. It always points opposite to an object's velocity, it is zero for zero velocity, and it generally increases in magnitude for increasing velocity. By velocity, we mean velocity relative to the fluid.

In falling near the Earth's surface, the force of gravity is (nearly) constant and this gives rise to (nearly) constant acceleration downward if gravity is the only force that acts on the object. This is free fall in its first meaning as we discussed in § 6.

If drag and the buoyancy force also acts on the object, then we have free fall in its second meaning as we discussed in § 6.

Usually near the Earth surface, there is air drag. It can be negligible for relatively short falls and/or high density objects or in artificial low-pressure environments.

But very commonly, drag is **NOT** negligible in reality—though in problems we often neglect it as simplifying assumption as aforesaid.

Qualitatively what happens when you drop an object from rest.

At rest there is no drag, there is only the gravity force (neglecting the buoyancy force).

So the object starts accelerating under gravity with acceleration g downward.

As the object speeds up, drag increases from zero and it opposes gravity.

The faster the object, the bigger the drag force.

At some point, the drag is equal to gravity in magnitude and opposite in direction.

This is net force zero situation.

As we'll discuss in the lecture *NEWTONIAN PHYSICS I*, when the net force is zero, there is no acceleration.

So once drag equals gravity in magnitude, the acceleration turns off and the object continues to fall at a constant velocity. This is still free fall in the second meaning.

The constant velocity is called **TERMINAL VELOCITY**. Some folks like to say terminal speed rather than terminal velocity. But since the direction of motion at terminal velocity is understood to be downward in gravitational field, direction is specified and so terminal velocity is a reasonable term. Anyway terminal velocity trips off the tongue.

Ideally **TERMINAL VELOCITY** is only reached at time infinity after the fall is initiated.

But often in finite time, the velocity is equal to **TERMINAL VELOCITY** to within measurement uncertainty or to within the size of perturbations in velocity due variations in air conditions.

So practically **TERMINAL VELOCITY** is reached and in many cases rather quickly.

Being in **TERMINAL VELOCITY** is a stable state. If a perturbation slows the fall a little from **TERMINAL VELOCITY**, gravity again exceeds drag in magnitude and the object speeds up to **TERMINAL VELOCITY** again. If a perturbation speeds the fall a little from **TERMINAL VELOCITY**, drag exceeds exceeds gravity in magnitude and the object slows to **TERMINAL VELOCITY** again.

Stable states are very common in nature. They are not states where the system has exactly certain state variable values at all times. They are where a stabilizing effect keeps driving variable values back to the state variable values after any perturbation of the system.

For some representative objects, Table 2 gives terminal velocities calculated from an approximate formulae (which is described in the lecture *NEWTONIAN PHYSICS II*) and the corresponding typical measured terminal velocities and their 95 % distances (the fall distance from rest to distance to reach 95 % of the terminal velocity). The measured values are typical since exact values require exact specifications of the objects and the air conditions they are in.

Table 2. Terminal Velocities for Typical Objects

Object	Calculated Terminal Velocity (m/s)	Measured Terminal Velocity (m/s)	Measured 95 % Distance (m)
Shot put shot	114	145	2500
human	127	60	430
baseball ball	24	42	210
tennis ball	17	20	115
raindrop (radius = 1.5 mm	6	7	6
cat tucked	64	27	...
cat spread	64	19	...

Note. — The measured values are from Halliday et al. (2001, p. 105). They are typical values not exact values. Exact values require exact specifications of the objects and the air conditions they are in.

The calculated values in Table 2 are not so bad considering that the formulae with crudely estimated input values was used to calculate them.

One can see that a human doesn't accelerate indefinitely going down.

So there is no problem skydiving without a parachute: aim a haystack or a soft snowdrift: it has been done: check out URL

http://www.greenharbor.com/fffolder/carkeet.html .

Then there are cats.

Cats have tendency to jump out of high windows—they're just so smart. This is called the high-rise syndrome (Wikipedia: High-rise syndrome).

But their survival rate is pretty good—and it increases with fall height.

What happens is a falling cat can orient itself right side up in the air—using the cat righting reflex—and spread out if it has enough time. Cats do this automatically—millions of years of tree-climbing and tree-off-falling evolution is at work here. In their spread-out form, they have a rather low terminal velocity because of a big cross sectional area for their mass. Of course, they must slow down to that low terminal velocity from pre-spread-out higher one—a fall of more than 6 stories seems to be enough. When the cats see the ground approaching, they tuck their legs under their bodies for landing (Wikipedia: High-rise syndrome; Halliday et al. e.g., 2001, p. 104–105).

Show stupid cat video: yes/no? It's at URL

http://news.nationalgeographic.com/news/2006/09/060928-cats-land-video.html .

ACKNOWLEDGMENTS

Support for this work has been provided by the Department of Physics of the University of Idaho and the Department of Physics of the University of Oklahoma.

REFERENCES

Arfken, G. 1970, Mathematical Methods for Physicists (New York: Academic Press)

Barger, V. D., & Olson, M. G. 1987, Classical Electricity and Magnetism (Boston: Allyn and Bacon, Inc.)

Barnhart, C. L. (editor) 1960, The American College Dictionary (New York: Random House) (Ba)

Enge, H. A. 1966, Introduction to Nuclear Physics (Reading, Massachusetts: Addison-Wesley Publishing Company)

Griffiths, D. J. 1999, Introduction to Electrodynamics (Upper Saddle River, New Jersey: Prentice Hall)

Halliday, D., Resnick, R., & Walker, J. 2001, Fundamentals of Physics, 6th Edition (New York: John Wiley & Sons, Inc.)

Ohanian, H. C. 1988, Classical Electrodynamics (Boston: Allyn and Bacon, Inc.)

Serway, R. A. & Jewett, J. W., Jr. 2008, Physics for Scientists and Engineers, 7th Edition (Belmont, California: Thomson)

Tipler, P. A., & Mosca, G. 2008, Physics for Scientists and Engineers, 6th Edition (New York: W.H. Freeman and Company)

Weber, H. J., & Arfken, G. B. 2004, Essential Mathematical Methods for Physicists (Amsterdam: Elsevier Academic Press)

Wolfson, R. & Pasachoff, J. M. 1990, Physics: Extended with Modern Physics (London: Scott, Foresman/Little, Brown Higher Education)

This preprint was prepared with the AAS LaTeX macros v5.2.