Intro Physics Semester I **Name:**

Homework 17: Waves II: Sound One or two full answer questions will be marked. There will also be a mark for completeness. Homeworks are due usually the day after the chapter they are for is finished. Solutions will be posted soon thereafter. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

Answer Table

Name:

	a	b	c	d	e
1.	O	O	O	O	O
2.	O	O	O	O	O
3.	O	O	O	O	O
4.	O	O	O	O	O
5.	O	O	O	O	O
6.	O	O	O	O	O
7.	O	O	O	O	O
8.	O	O	O	O	O
9.	O	O	O	O	O
10.	O	O	O	O	O
11.	O	O	O	O	O
12.	O	O	O	O	O
13.	O	O	O	O	O
14.	O	O	O	O	O
15.	O	O	O	O	O
16.	O	O	O	O	O
17.	O	O	O	O	O
18.	O	O	O	O	O
19.	O	O	O	O	O
20.	O	O	O	O	O
21.	O	O	O	O	O
22.	O	O	O	O	O
23.	O	O	O	O	O
24.	O	O	O	O	O
25.	O	O	O	O	O
26.	O	O	O	O	O
27.	O	O	O	O	O
28.	O	O	O	O	O
29.	O	O	O	O	O
30.	O	O	O	O	O

	a	b	c	d	e
31.	O	O	O	O	O
32.	O	O	O	O	O
33.	O	O	O	O	O
34.	O	O	O	O	O
35.	O	O	O	O	O
36.	O	O	O	O	O
37.	O	O	O	O	O
38.	O	O	O	O	O
39.	O	O	O	O	O
40.	O	O	O	O	O
41.	O	O	O	O	O
42.	O	O	O	O	O
43.	O	O	O	O	O
44.	O	O	O	O	O
45.	O	O	O	O	O
46.	O	O	O	O	O
47.	O	O	O	O	O
48.	O	O	O	O	O
49.	O	O	O	O	O
50.	O	O	O	O	O
51.	O	O	O	O	O
52.	O	O	O	O	O
53.	O	O	O	O	O
54.	O	O	O	O	O
55.	O	O	O	O	O
56.	O	O	O	O	O
57.	O	O	O	O	O
58.	O	O	O	O	O
59.	O	O	O	O	O
60.	O	O	O	O	O

018 qfull 00700 1 3 0 easy math: derviving the Doppler effect

1. The Doppler effect is a shift in wave frequency depending on the motion of source and/or observer. It is common to all wave phenomena, but the actual formulae vary from case to case. A prominent difference is the one between the sound Doppler effect and the light Doppler effect. Here we derive the formulae for the sound Doppler effect. We are going to stick to 1-dimensional cases for simplicity.

   a) Draw a schematic diagram of transverse wave cycle traveling in the positve x direction with wavelength λ and phase velocity v'' (not v' since we need that symbol for something else below). The wave cycle takes time P' to pass a given x position in the reference frame in which v'' is observed. Derive, the formula

   $$f'\lambda = v'' \ ,$$

   where f' is the frequency of the wave.

   b) The formula $f'\lambda = v''$ derived in part (a) is obviously valid for a general frame of reference moving in the x direction. We will not consider frames moving in other directions. They are not so hard to deal with, but we want to keep this problem from getting too intricate. We are now in a position to to see how f' is related to f the frequency of the wave in the medium's frame: i.e., the frame in which the medium is at rest.

   First, note that the phase velocity v of the wave in medium's frame is the phase velocity that is derived from the medium properties and is just called the phase velocity without qualifications unless context says otherwise. The phase velocity in a general frame moving at velocity v' relative to the medium is

   $$v'' = v - v' \ .$$

   Second note that the wavelength λ of the wave is the same in all frames. Length is a frame invariant quantity in classical physics. This is not the case in relativistic physics which is needed for high relative velocities, but we are not going to consider those cases.

   Now derive the basic sound Doppler shift formulae

   $$f' = f\left(1 - \frac{v'}{v}\right) \ , \qquad f = \frac{f'}{1 - v'/v} \ .$$

   c) The frequency f' of formula

   $$f' = f\left(1 - \frac{v'}{v}\right)$$

   can be interpreted as the frequency observed in a frame moving at v' for a fixed f. Draw a plot of f' versus v' and explain what is happening in the cases $v < 0$, $v' = 0$, $0 < v' < v$, $v = v'$, and $v' > v$. Assume $f > 0$ and $v > 0$ as usual.

   d) The frequency f formula

   $$f = \frac{f'}{1 - v'/v}$$

   can be interpreted as the frequency observed in the medium frame for a source moving at v' with a fixed source frequency f'. Draw a plot of f versus v' and explain what is happening in the cases $v < 0$, $v' = 0$, $0 < v' < v$, $v = v'$, and $v' > v$. Assume $f' > 0$ and $v > 0$ as usual.

   e) Say a source is moving at velocity v'_1 and emitting at fixed frequency f'_1 and an observer is moving at velocity v'_2. Derive the formula for f'_2 in terms of v v'_1, v'_2, and f'_1.

   f) Taylor expand the result of part (e) to 1st order in small v_1/v and v_2/v, and derive the 1st order formula for

   $$\frac{f'_2 - f'_1}{f'_1} \ .$$

SUGGESTED ANSWER:

a) You will have to imagine the diagram. Well

$$P' = \frac{\lambda}{v''} \ ,$$

and thus

$$f' = \frac{1}{P'} = \frac{v''}{\lambda}$$

from which we obtain

$$f'\lambda = v'' \ .$$

b) We have

$$f'\lambda = v'' \ , \qquad f\lambda = v$$

and can divide the first by the second to obtain

$$\frac{f'}{f} = \frac{v''}{v} = 1 - \frac{v'}{v} \ .$$

Immediately, we now have

$$f' = f\left(1 - \frac{v'}{v}\right) \ , \qquad f = \frac{f'}{1 - v'/v} \ .$$

c) You will have to imagine the diagram for

$$f' = f\left(1 - \frac{v'}{v}\right) \ .$$

There will be a line of negative slope $-f/v$ with x intercept at $v' = v$ and y intercept of size f.

For $v' < 0$, the observer is moving head-on into the waves, and so observes frequency higher than f. In astrophysics jargon, the waves are blueshifted since visual light in this circumstance is shifted toward the blue end of the visual spectrum.

For $v' = 0$, the observer is at rest in the medium, and so observes frequency f.

For $0 < v' < v$, the observer is moving in the wave propagation direction. The waves are moving faster than the observer, and so he/she sees them passing him/her, but at a frequency lower than f. In astrophysics jargon, the waves are redshifted since visual light in this circumstance is shifted toward the blue end of the visual spectrum.

For $v' = v$, the observer is moving at the wave velocity, and observes the waves at rest. So the frequency $f' = 0$. The observer is moving at the sound speed.

For $v' > v$, the observer is supersonic and is passing the waves. The way we have set up the conventions this means he/she is observing a negative frequency. We don't usually think of frequency as being negative, but the conventions lead to that situation here—and the interpretation we've given.

d) You will have to imagine the diagram for

$$f = \frac{f'}{1 - v'/v} \ .$$

The derivative of the function is

$$\frac{df}{v'} = \frac{f/v}{(1 - v'/v)^2} \geq 0 \ .$$

So the function rises monotonically with v' with the only stationary points being a minimum at $v' = -\infty$ and a maximum at $v' = \infty$. The function has a singularity at $v' = v$. (A singularity is a point where a function does not formally exist, but in our sloppy physics way, we are often able to say it has infinite or negative infinite value at the singularity.) Below the singularity, the function rises asymptotically to infinity. Above singularity, the function rises asymptotically from negative infinity.

For $v' < 0$, the source is moving away the observer and the frequency is reduced below f'. It takes longer between wave cycles than if the source were at rest, because the source moves farther away between each crest emission. If the source is moving at $v' = -\infty$, the observed frequency $f = 0$ since no wave cycles ever get to the observer—there emission point is infinitely far away. In astrophysics jargon, the source is redshifted.

For $v' = 0$, the source is at rest in the medium, and the observer observes source frequency f'.

For $0 < v' < v$, the source is moving toward the observer. The frequency f is greater than f' since between emitting cycles, the source has moved closer to the observer. In astrophysics jargon, the source blueshifted.

For $v' = v$, the source is moving just at the wave velocity. All emitted wave cycles are in the same place, the current location of the source. The net wave pulse is a sonic boom.

For $v' > v$, the the source is moving faster than the wave cycles and leaving them trailing it. The observer fequency $f < 0$, but this is just because the wave cycles are being observed by the observer in the reverse order from the emission. He/she observes the later emitted ones before the earlier emitted ones. The $v' = \infty$ case can be interpreted as the cycles never being able to reach observer since they were all emitted when the source was infinitely far away. One moment the source is at negative infinity, the next at positive infinity. The cycles can't reach us from either place.

e) Well

$$f = \frac{f_1'}{1 - v_1'/v} \ , \qquad f = \frac{f_2'}{1 - v_2'/v} \ ,$$

and so

$$f_2' = f_1' \left(\frac{1 - v_2'/v}{1 - v_1'/v} \right) \ .$$

To go off on a tangent, we can define the relative velocity between observer and source as

$$\Delta v = v_2' - v_1' \ .$$

Now we ask does f_2' always redshift/blueshift with increasing/decreasing Δv. We take the derivative with respect to Δv (using partial differentiation and the chain rule) to get

$$\frac{df_2'}{d\Delta v} = \frac{df_2'}{dv_2'} \frac{dv_2'}{d\Delta v} = \frac{df_2'}{dv_1'} \frac{dv_1'}{d\Delta v} = -\frac{f_1'}{v} \frac{1}{(1 - v_1'/v)} - \frac{f_1'}{v} \frac{(1 - v_2'/v)}{(1 - v_1'/v)^2} \ .$$

So as long as as all motions are subsonic, the answer is yes. If there are supersonic motions, the answer is trickier.

f) If $v_1'/v << 1$ and $v_2'/v << 1$, then we can Taylor expand the part (e) formula to 1st order in small v_1'/v and v_2'/v. We obtain the 1st order formulae

$$f_2' = f_1' \left[1 - \frac{(v_2' - v_1')}{v} \right]$$

and

$$\frac{f_2' - f_1'}{f_1'} = -\frac{(v_2' - v_1')}{v} \ .$$

The last formula can be written in the simplified mnemonic form

$$\frac{\Delta f}{f} = -\frac{\Delta v}{v_{\rm sound}}$$

as long as we remember to interpret the symbols correctly. This formula clearly shows that there is always a redshift/blueshift with increasing/decreasing Δv.

Redaction: Jeffery, 2008jan01

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

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1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67384(80) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2012, CODATA)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\text{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

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2 Geometrical Formulae

$$C_{\text{cir}} = 2\pi r \qquad A_{\text{cir}} = \pi r^2 \qquad A_{\text{sph}} = 4\pi r^2 \qquad V_{\text{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\text{sphere}} = 4\pi \qquad d\Omega = \sin\theta\, d\theta\, d\phi$$

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3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1-x} \approx 1 + x \ : \ (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

5 Quadratic Formula

$$\text{If} \qquad 0 = ax^2 + bx + c\ , \qquad \text{then} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) \\ &= f(x_0) + (x - x_0) f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots \end{aligned}$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0 \qquad v^2 = v_0^2 + 2a(x - x_0)$$

$$x = \frac{1}{2}(v_0 + v)t + x_0 \qquad x = -\frac{1}{2}at^2 + vt + x_0 \qquad g = 9.8\,\text{m/s}^2$$

$$x_{\text{rel}} = x_2 - x_1 \qquad v_{\text{rel}} = v_2 - v_1 \qquad a_{\text{rel}} = a_2 - a_1$$

$$x' = x - v_{\text{frame}} t \qquad v' = v - v_{\text{frame}} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\text{avg}} = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0} t \qquad y = -\frac{1}{2} g t^2 + v_{y,0} t + y_0 \qquad v_{x,0} = v_0 \cos\theta \qquad v_{y,0} = v_0 \sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0 \cos\theta} \qquad y = y_0 + x \tan\theta - \frac{x^2 g}{2 v_0^2 \cos^2\theta}$$

$$x_{\text{for } y \text{ max}} = \frac{v_0^2 \sin\theta \cos\theta}{g} \qquad y_{\max} = y_0 + \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2 v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g} \qquad \theta_{\text{for max}} = \frac{\pi}{4} \qquad x_{\max}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0 \sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2 r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\tan} = r\alpha$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = v\omega$$

13 Very Basic Newtonian Physics

$$\vec{r}_{\rm cm} = \frac{\sum_i m_i \vec{r}_i}{m_{\rm total}} = \frac{\sum_{\rm sub} m_{\rm sub} \vec{r}_{\rm cm\ sub}}{m_{\rm total}} \qquad \vec{v}_{\rm cm} = \frac{\sum_i m_i \vec{v}_i}{m_{\rm total}} \qquad \vec{a}_{\rm cm} = \frac{\sum_i m_i \vec{a}_i}{m_{\rm total}}$$

$$\vec{r}_{\rm cm} = \frac{\int_V \rho(\vec{r})\vec{r}\, dV}{m_{\rm total}}$$

$$\vec{F}_{\rm net} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,{\rm m/s^2}$$

$$\vec{F}_{\rm normal} = -\vec{F}_{\rm applied} \qquad F_{\rm linear} = -kx$$

$$f_{\rm normal} = \frac{T}{r} \qquad T = T_0 - F_{\rm parallel}(s) \qquad T = T_0$$

$$F_{\rm f\ static} = \min(F_{\rm applied}, F_{\rm f\ static\ max}) \qquad F_{\rm f\ static\ max} = \mu_{\rm static} F_{\rm N} \qquad F_{\rm f\ kinetic} = \mu_{\rm kinetic} F_{\rm N}$$

$$v_{\rm tangential} = r\omega = r\frac{d\theta}{dt} \qquad a_{\rm tangential} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\rm centripetal} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\rm centripetal} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\rm drag,lin} = bv \qquad v_{\rm T} = \frac{mg}{b} \qquad \tau = \frac{v_{\rm T}}{g} = \frac{m}{b} \qquad v = v_{\rm T}(1 - e^{-t/\tau})$$

$$F_{\rm drag,quad} = bv^2 = \frac{1}{2}C\rho A v^2 \qquad v_{\rm T} = \sqrt{\frac{mg}{b}}$$

14 Energy and Work

$$dW = \vec{F}\cdot d\vec{s} \qquad W = \int \vec{F}\cdot d\vec{s} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\rm mechanical} = KE + PE$$

$$P_{\rm avg} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F}\cdot\vec{v}$$

$$\Delta KE = W_{\rm net} \quad \Delta PE_{\rm of\ a\ conservative\ force} = -W_{\rm by\ a\ conservative\ force} \quad \Delta E = W_{\rm nonconservative}$$

$$F = -\frac{dPE}{dx} \qquad \vec{F} = -\nabla PE \qquad PE = \frac{1}{2}kx^2 \qquad PE = mgy$$

15 Momentum

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta KE_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\text{cm}} = \vec{F}_{\text{net non-flux}} + (\vec{v}_{\text{flux}} - \vec{v}_{\text{cm}})\frac{dm}{dt} = \vec{F}_{\text{net non-flux}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_0 + v_{\text{ex}} \ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

16 Collisions

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\, dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\text{net}}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$KE_{\text{total } f} = KE_{\text{total } i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2 v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\text{rel}'} = -v_{\text{rel}} \qquad \text{1-d Elastic Collision Expressions}$$

17 Rotational Kinematics

$$2\pi = 6.2831853\ldots \qquad \frac{1}{2\pi} = 0.15915494\ldots$$

$$\frac{180^\circ}{\pi} = 57.295779\ldots \approx 60^\circ \qquad \frac{\pi}{180^\circ} = 0.017453292\ldots \approx \frac{1}{60^\circ}$$

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\omega = \alpha t + \omega_0 \qquad \Delta\theta = \frac{1}{2}\alpha t^2 + \omega_0 t \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t \qquad \Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$

18 Rotational Dynamics

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

$$L_z = RP_{xy} \sin \gamma_L \qquad \tau_z = RF_{xy} \sin \gamma_\tau \qquad L_z = I\omega \qquad \tau_{z,\text{net}} = I\alpha$$

$$I = \sum_i m_i R_i^2 \qquad I = \int R^2 \rho \, dV \qquad I_{\text{parallel axis}} = I_{\text{cm}} + mR_{\text{cm}}^2 \qquad I_z = I_x + I_y$$

$$I_{\text{cyl,shell,thin}} = MR^2 \qquad I_{\text{cyl}} = \frac{1}{2}MR^2 \qquad I_{\text{cyl,shell,thick}} = \frac{1}{2}M(R_1^2 + R_2^2)$$

$$I_{\text{rod,thin,cm}} = \frac{1}{12}ML^2 \qquad I_{\text{sph,solid}} = \frac{2}{5}MR^2 \qquad I_{\text{sph,shell,thin}} = \frac{2}{3}MR^2$$

$$a = \frac{g \sin \theta}{1 + I/(mr^2)}$$

$$KE_{\text{rot}} = \frac{1}{2}I\omega^2 \qquad dW = \tau_z \, d\theta \qquad P = \frac{dW}{dt} = \tau_z \omega$$

$$\Delta KE_{\text{rot}} = W_{\text{net}} = \int \tau_{z,\text{net}} \, d\theta \qquad \Delta PE_{\text{rot}} = -W = -\int \tau_{z,\text{con}} \, d\theta$$

$$\Delta E_{\text{rot}} = KE_{\text{rot}} + \Delta PE_{\text{rot}} = W_{\text{non,rot}} \qquad \Delta E = \Delta KE + KE_{\text{rot}} + \Delta PE = W_{\text{non}} + W_{\text{rot}}$$

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19 **Static Equilibrium**

$$\vec{F}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = \tau'_{\text{ext,net}} \quad \text{if } F_{\text{ext,net}} = 0$$

$$0 = F_{\text{net } x} = \sum F_x \qquad 0 = F_{\text{net } y} = \sum F_y \qquad 0 = \tau_{\text{net}} = \sum \tau$$

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20 **Gravity**

$$\vec{F}_{1 \text{ on } 2} = -\frac{Gm_1 m_2}{r_{12}^2}\hat{r}_{12} \qquad \vec{g} = -\frac{GM}{r^2}\hat{r} \qquad \oint \vec{g} \cdot d\vec{A} = -4\pi GM$$

$$PE = -\frac{Gm_1 m_2}{r_{12}} \qquad V = -\frac{GM}{r} \qquad v_{\text{escape}} = \sqrt{\frac{2GM}{r}} \qquad v_{\text{orbit}} = \sqrt{\frac{GM}{r}}$$

$$P^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \qquad P = \left(\frac{2\pi}{\sqrt{GM}}\right) r^{3/2} \qquad \frac{dA}{dt} = \frac{1}{2}r^2\omega = \frac{L}{2m} = \text{Constant}$$

$$R_{\text{Earth,mean}} = 6371.0\,\text{km} \qquad R_{\text{Earth,equatorial}} = 6378.1\,\text{km} \qquad M_{\text{Earth}} = 5.9736 \times 10^{24}\,\text{kg}$$

$$R_{\text{Earth mean orbital radius}} = 1.495978875 \times 10^{11}\,\text{m} = 1.0000001124\,\text{AU} \approx 1.5 \times 10^{11}\,\text{m} \approx 1\,\text{AU}$$

$$R_{\text{Sun,equatorial}} = 6.955 \times 10^{8} \approx 109 \times R_{\text{Earth,equatorial}} \qquad M_{\text{Sun}} = 1.9891 \times 10^{30}\,\text{kg}$$

21 Fluids

$$\rho = \frac{\Delta m}{\Delta V} \qquad p = \frac{F}{A} \qquad p = p_0 + \rho g d_{\text{depth}}$$

$$\begin{aligned}
\text{Pascal's principle} \quad & p = p_{\text{ext}} - \rho g(y - y_{\text{ext}}) \qquad \Delta p = \Delta p_{\text{ext}} \\
\text{Archimedes principle} \quad & F_{\text{buoy}} = m_{\text{fluid dis}} g = V_{\text{fluid dis}} \rho_{\text{fluid}} g \\
\text{equation of continuity for ideal fluid} \quad & R_V = Av = \text{Constant} \\
\text{Bernoulli's equation} \quad & p + \frac{1}{2}\rho v^2 + \rho g y = \text{Constant}
\end{aligned}$$

22 Oscillation

$$P = f^{-1} \qquad \omega = 2\pi f \qquad F = -kx \qquad PE = \frac{1}{2}kx^2 \qquad a(t) = -\frac{k}{m}x(t) = -\omega^2 x(t)$$

$$\omega = \sqrt{\frac{k}{m}} \qquad P = 2\pi\sqrt{\frac{m}{k}} \qquad x(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$E_{\text{mec total}} = \frac{1}{2}mv_{\text{max}}^2 = \frac{1}{2}kx_{\text{max}}^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$P = 2\pi\sqrt{\frac{I}{mgr}} \qquad P = 2\pi\sqrt{\frac{r}{g}}$$

23 Waves

$$\frac{d^2y}{dx^2} = \frac{1}{v^2}\frac{d^2y}{dt^2} \qquad v = \sqrt{\frac{F_{\text{T}}}{\mu}} \qquad y = f(x \mp vt)$$

$$y = y_{\text{max}}\sin[k(x \mp vt)] = y_{\text{max}}\sin(kx \mp \omega t)$$

$$\text{Period} = \frac{1}{f} \qquad k = \frac{2\pi}{\lambda} \qquad v = f\lambda = \frac{\omega}{k} \qquad P \propto y_{\text{max}}^2$$

$$y = 2y_{\max}\sin(kx)\cos(\omega t) \qquad n = \frac{L}{\lambda/2} \qquad L = n\frac{\lambda}{2} \qquad \lambda = \frac{2L}{n} \qquad f = n\frac{v}{2L}$$

$$v = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} \qquad n\lambda = d\sin(\theta) \qquad \left(n + \frac{1}{2}\right)\lambda = d\sin(\theta)$$

$$I = \frac{P}{4\pi r^2} \qquad \beta = (10\,\mathrm{dB}) \times \log\left(\frac{I}{I_0}\right)$$

$$f = n\frac{v}{4L} \ : \ n = 1, 3, 5, \ldots \qquad f_{\mathrm{medium}} = \frac{f_0}{1 - v_0/v_{\mathrm{medium}}}$$

$$f' = f\left(1 - \frac{v'}{v}\right) \qquad f = \frac{f'}{1 - v'/v}$$