Intro Physics Semester I **Name:**

Homework 12: Equilibrium: One or two full answer questions will be marked. There will also be a mark for completeness. Homeworks are due usually the day after the chapter they are for is finished. Solutions will be posted soon thereafter. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

Answer Table

Name:

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	31.	O	O	O	O	O
2.	O	O	O	O	O	32.	O	O	O	O	O
3.	O	O	O	O	O	33.	O	O	O	O	O
4.	O	O	O	O	O	34.	O	O	O	O	O
5.	O	O	O	O	O	35.	O	O	O	O	O
6.	O	O	O	O	O	36.	O	O	O	O	O
7.	O	O	O	O	O	37.	O	O	O	O	O
8.	O	O	O	O	O	38.	O	O	O	O	O
9.	O	O	O	O	O	39.	O	O	O	O	O
10.	O	O	O	O	O	40.	O	O	O	O	O
11.	O	O	O	O	O	41.	O	O	O	O	O
12.	O	O	O	O	O	42.	O	O	O	O	O
13.	O	O	O	O	O	43.	O	O	O	O	O
14.	O	O	O	O	O	44.	O	O	O	O	O
15.	O	O	O	O	O	45.	O	O	O	O	O
16.	O	O	O	O	O	46.	O	O	O	O	O
17.	O	O	O	O	O	47.	O	O	O	O	O
18.	O	O	O	O	O	48.	O	O	O	O	O
19.	O	O	O	O	O	49.	O	O	O	O	O
20.	O	O	O	O	O	50.	O	O	O	O	O
21.	O	O	O	O	O	51.	O	O	O	O	O
22.	O	O	O	O	O	52.	O	O	O	O	O
23.	O	O	O	O	O	53.	O	O	O	O	O
24.	O	O	O	O	O	54.	O	O	O	O	O
25.	O	O	O	O	O	55.	O	O	O	O	O
26.	O	O	O	O	O	56.	O	O	O	O	O
27.	O	O	O	O	O	57.	O	O	O	O	O
28.	O	O	O	O	O	58.	O	O	O	O	O
29.	O	O	O	O	O	59.	O	O	O	O	O
30.	O	O	O	O	O	60.	O	O	O	O	O

1. To be in rotational equilibrium relative to some origin in an inertial frame, an object must have (relative to that origin):

   a) zero angular momentum. b) non-zero angular momentum.
   c) constant angular momentum. d) non-constant angular momentum. e) no hair.

2. In **STATIC** equilibrium:

   a) there is no center-of-mass or rotational acceleration, but there can be **NONZERO** center-of-mass velocity and angular velocity.
   b) there is no translational or rotational acceleration, and **NO** center-of-mass or rotational velocity. If static equilibrium exists in a specific reference frame, it exists in **ALL** reference frames no matter how those reference frames may be moving.
   c) there is no center-of-mass or rotational acceleration, and **NO** center-of-mass or rotational velocity. If static equilibrium exists in a specific reference frame, it exists **ONLY** in reference frames **NOT** moving with respect to the specific reference frame.
   d) there are no forces at all.
   e) there are no torques at all.

3. "Let's play *Jeopardy*! For \$100, the answer is: The net gravitational torque on an object about any origin at all is the same as if all the mass of the object were concentrated at the center of mass: i.e.,

$$\vec{\tau}_{\text{grav}} = \vec{r}_{\text{cm}} \times mg\hat{g} \ ,$$

   where $\vec{r}_{\text{cm}}$ is measured from the origin and $\hat{g}$ is a unit vector in the direction of the gravitational force. Only when $\vec{r}_{\text{cm}}$ and $\hat{g}$ are aligned or when $\vec{r}_{\text{cm}} = 0$ does the gravitational torque vanish. Thus for an object hanging from a frictionless pivot, equilibrium only exists for the center of mass directly above, on, or directly below the pivot. Above is unstable because any perturbation causes a gravitational torque away from the equilibrium. On is a neutral equilibrium since there is no gravitational torque whatever the orientation of the object. Below is stable since the gravitational torque then tries to pull the center of mass back to the equilibrium point. With any damping to kill rotational kinetic energy (but insufficient static friction in the pivot point to oppose any gravitational torque), the object will come to a static stable equilibrium with the center of mass directly below the pivot."

   a) Why does the center of mass of an object tend to come to rest directly **BELOW** a freely turning pivot from which the object is hanging, Alex?
   b) Why does the center of mass of an object tend to come to rest directly **ABOVE** a freely turning pivot from which the object is hanging, Alex?
   c) Why is there a center of mass, Alex?
   d) Why is there a center, Alex?
   e) Why is there a universe, Alex?

4. An object of mass 1 kg sits on a horizontal beam at 1 m from a pivot point. What is the torque about the pivot point that the weight of the mass causes?

   a) 1 N m. b) 2 N m. c) 3 N m. d) 4 N m. e) 9.8 N m.

5. Two objects are sitting on a uniform horizontal beam. The beam rests on a pivot at its center of mass. The beam is free to rotate about the pivot. Object 1 sits on the left-hand side of the pivot at a distance ℓ_1 from the pivot. Object 2 sits on the right-hand side at a distance ℓ_2. Given $m_1 = Nm_2$, what is ℓ_2 in terms of ℓ_1? **HINT:** Draw a diagram.

   a) $\ell_2 = N\ell_1$. b) $\ell_2 = \ell_1/N$. c) $\ell_2 = \ell_1$. d) $\ell_2 = 2\ell_1$. e) $\ell_2 = 0$.

6. In a planar or 2-dimensional case of static equilibrium with no special rules relating forces, can you solve for four unknown forces assuming perfectly rigid objects?

   a) No. The system is **INDETERMINATE**: you only have **FOUR** equilibrium equations.
   b) Yes. The system is **DETERMINATE** since you have **FOUR** equilibrium equations.
   c) No. The system is **INDETERMINATE**: you only have **THREE** equilibrium equations.
   d) Yes. The system is **DETERMINATE**: you have **THREE** equilibrium equations.
   e) No. The system is **INDETERMINATE**: you only have **TWO** equilibrium equations.

7. There are some results that are useful to know in studying equilibrium for a rigid body. We will not specialize to a rigid system initially—we will say when we do so specialize. We only consider inertial frames in our derivations and discussions. Non-inertial frames can be treated, but they are trickier.

   a) The net external torque on a system about a first general origin O in an inertial frame is

   $$\vec{\tau}_{\text{ext}} = \sum_i \vec{r}_i \times \vec{F}_i \ .$$

   The net external force on the system is

   $$\vec{F}_{\text{ext}} = \sum_i \vec{F}_i \ .$$

   A second general origin O′ is located at $\Delta\vec{r}$ relative to the first origin. The vector $\Delta\vec{r}$ can in general depend linearly on time, and thus the second origin can define a second inertial frame. The displacements relative to the second origin are related to displacements relative to the first origin by

   $$\vec{r}\,'_i = \vec{r}_i - \Delta\vec{r}$$

   —and yes, the minus sign is right: you should draw a diagram to see this. Derive the expression for the net external torque $\vec{\tau}\,'_{\text{ext}}$ about the second origin in terms of the first origin quantities. Note that forces are frame-invariant quantities in classical mechanics.

   b) If $\vec{F}_{\text{ext}} = 0$, what is the relationship of $\vec{\tau}\,'_{\text{ext}}$ and $\vec{\tau}_{\text{ext}}$. What is the relationship between the two angular momenta relative to the two origins in this case? **HINT:** You'll have to do a trivial integral to answer the second question.

   c) Now we specialize to a rigid body of constant mass m and consider conditions imposed one after another. Explain what each condition implies about the system and why it does so? First, we impose $\vec{F}_{\text{ext}} = 0$. Second, we impose that the system is examined in the rest frame of its center of mass. Third, we impose that $\tau_{\text{ext}} = 0$ using the center of mass as the origin. Fourth, we impose that system is not rotating in anyway about the center of mass.

   d) Are $\vec{F}_{\text{ext}} = 0$ and $\vec{\tau}_{\text{ext}} = 0$ sufficient or necessary conditions for overall equilibrium for a rigid body? Are they sufficient or necessary conditions for overall static equilibrium? Explain your answers.

8. The general formula for the net torque about a z axis is

   $$\tau_z = \sum_i r_{xy,i} F_{xy,i} \sin\theta_i \ ,$$

   where i indexes the applied forces, $r_{xy,i}$ is the xy component of radial vector from the origin to where a force is applied, $F_{xy,i}$ is the xy component of an applied force, and θ_i is the angular coordinate of $\vec{F}_{xy,i}$ relative to $r_{xy,i}$ measured positive in the counterclockwise direction usually.

   a) Consider a horizontal beam with a supporting fulcrum and weights of mass m_i put along it with their center of masses at x_i. The beam itself is one of the weights. The beam system is aligned with the x direction and is symmetric about the xy plane through the origin. This means there are no x and y torques. This is a usual setup for a balance scale with the rotation axis assigned to the the z axis. Draw a diagram of the horizontal-beam system.

   b) Specialize the general formula for the net torque about a z axis to the horizontal-beam system with the origin taken at the fulcrum point. The fulcrum is ideal and exerts no torque.

   c) There are no forces in the x or z directions for the horizontal beam system. What is the general formula for the force in the y direction. **HINT:** Do not forget the normal force of the fulcrum.

   d) What are the conditions for static equilibrium for the horizontal beam system? We assume the beam and weights act as a rigid body. The beam can pivot about the fulrum.

   e) Usually, we can only solve for F_{N} from the static equilibrium equations themselves. Thus, we can only really solve the two equations of equilibrium for one other unknown, either an x_j or an m_j. Using $\sum_{i,i\neq j}$ to mean sum over i excluding j, solve for unknown x_j and F_{N} and then for unknown m_j and F_{N}.

9. A horizontal beam is balanced on a point fulcrum at $x_{\text{fulcrum}} = 0.20\,\text{m}$. The beam is has mass $0.10\,\text{kg}$ and its center of mass is at $x_{\text{beam cm}} = 0.50\,\text{m}$. There is also a single object of mass $2.0\,\text{kg}$ on the beam. Where is its center of mass?

10. A ladder leans against a wall in static equilibrium. Ladder, wall, and ground are perfectly rigid. The ladder has mass m_{ladder}, length ℓ_{ladder}, and center of located at ℓ_{cm} along its length measuring from its base. The angle between the ladder and the **VERTICAL** is θ. **HINT:** Draw a good diagram marking on all possible forces: gravity, the normal forces of the ground and wall, and the friction forces of the ground and wall. Draw the ladder leaning to the right so that we are all consistent.

    a) Write out the equations of equilibrium including all possible forces. Just so we are all on the same wavelength take the origin for the torque equation to be the contact point between ladder and ground. Why is this a good choice? Can you obtain explicit formulae in terms of the knowns (i.e., m_{lad}, ℓ_{lad}, $\ell_{\text{lad cm}}$ and θ) for the normal forces of the ground and wall, and the static friction forces of the ground and wall? **HINT:** In setting up the equations you must adopt some conventions about which directions are positive for which forces and what is the positive torque direction. As long as you are consistent everything works out the same physically no matter what conventions you adopt.

    b) Assuming the wall is frictionless, what are the expressions for the normal forces of the ground and wall, and the static friction force of the ground? Are these intrinsic expressions for these forces? Given the static friction between ordinary ground and ladders what must happen as θ becomes large?

    c) Again assume the wall is frictionless. Say the ladder is just on the verge of slipping at θ_{slip}. What is the formula for the static friction coefficient of the ground?

    d) Assuming the floor is frictionless, what are the expressions for the normal forces of the ground and wall, and the static friction force of the wall? What ordinary friction rule is violated in this case?

11. A roller (either a spherically symmetric ball or a cylindrically symmetric cylinder) is rolled up to a step and rests against it. The roller has radius R and mass m. The step has height y with $y \leq R$. A force F_{app} is applied horizontally to the roller at height R to try to push the roller over the step. A line drawn through the applied force through passes the roller's center of mass. Find the expression for F_{app} that just marginally lifts the roller. This is a static equilibrium situation where the normal force of the ground has just gone to zero, but the roller is still marginally touching the ground. What are the F_{app} values for $y = 0$ and $y = R$? **HINTS:** Draw a good diagram, use geometry, and identify the best pivot point for a torque calculation.

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

1 Constants

$$
\begin{aligned}
c &= 2.99792458\times10^{8}\,\mathrm{m/s} \approx 2.998\times10^{8}\,\mathrm{m/s} \approx 3\times10^{8}\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition} \\
e &= 1.602176487(40)\times10^{-19}\,\mathrm{C} \\
G &= 6.67428(67)\times10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2006, CODATA)} \\
g &= 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value} \\
k &= \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots\times10^{9} \approx 8.99\times10^{9} \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition} \\
k_{\mathrm{Boltzmann}} &= 1.3806504(24)\times10^{-23}\,\mathrm{J/K} = 0.8617343(15)\times10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K} \\
m_e &= 9.10938215(45)\times10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV} \\
m_p &= 1.672621637(83)\times10^{-27}\,\mathrm{kg} = 938.272013(23),\mathrm{MeV} \\
\varepsilon_0 &= \frac{1}{\mu_0 c^2} = 8.8541878176\ldots\times10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)} \\
\mu_0 &= 4\pi\times10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}
\end{aligned}
$$

2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\mathrm{sphere}} = 4\pi \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

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4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1-x} \approx 1 + x \;:\; (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

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5 Quadratic Formula

If $\quad 0 = ax^2 + bx + c\,, \quad$ then $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\dfrac{b}{2a} \pm \sqrt{\left(\dfrac{b}{2a}\right)^2 - \dfrac{c}{a}}$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series
$$f(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0)$$
$$= f(x_0) + (x - x_0) f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0 \qquad v^2 = v_0^2 + 2a(x - x_0)$$

$$x = \frac{1}{2}(v_0 + v)t + x_0 \qquad x = -\frac{1}{2}at^2 + vt + x_0 \qquad g = 9.8\,\mathrm{m/s^2}$$

$$x_{\rm rel} = x_2 - x_1 \qquad v_{\rm rel} = v_2 - v_1 \qquad a_{\rm rel} = a_2 - a_1$$

$$x' = x - v_{\rm frame}t \qquad v' = v - v_{\rm frame} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\rm avg} = \frac{\Delta\vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\rm avg} = \frac{\Delta\vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0}t \qquad y = -\frac{1}{2}gt^2 + v_{y,0}t + y_0 \qquad v_{x,0} = v_0\cos\theta \qquad v_{y,0} = v_0\sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0\cos\theta} \qquad y = y_0 + x\tan\theta - \frac{x^2 g}{2v_0^2\cos^2\theta}$$

$$x_{\text{for } y \text{ max}} = \frac{v_0^2\sin\theta\cos\theta}{g} \qquad y_{\rm max} = y_0 + \frac{v_0^2\sin^2\theta}{2g}$$

$$x(y=y_0) = \frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin(2\theta)}{g} \qquad \theta_{\text{for max}} = \frac{\pi}{4} \qquad x_{\rm max}(y=y_0) = \frac{v_0^2}{g}$$

$$x(\theta=0) = \pm v_0\sqrt{\frac{2(y_0-y)}{g}} \qquad t(\theta=0) = \sqrt{\frac{2(y_0-y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\rm tan} = r\alpha$$

$$\vec{a}_{\rm centripetal} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\rm centripetal} = \frac{v^2}{r} = r\omega^2 = v\omega$$

13 Very Basic Newtonian Physics

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i \vec{r}_i}{m_{\text{total}}} = \frac{\sum_{\text{sub}} m_{\text{sub}} \vec{r}_{\text{cm sub}}}{m_{\text{total}}} \qquad \vec{v}_{\text{cm}} = \frac{\sum_i m_i \vec{v}_i}{m_{\text{total}}} \qquad \vec{a}_{\text{cm}} = \frac{\sum_i m_i \vec{a}_i}{m_{\text{total}}}$$

$$\vec{r}_{\text{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\text{total}}}$$

$$\vec{F}_{\text{net}} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,\text{m/s}^2$$

$$\vec{F}_{\text{normal}} = -\vec{F}_{\text{applied}} \qquad F_{\text{linear}} = -kx$$

$$f_{\text{normal}} = \frac{T}{r} \qquad T = T_0 - F_{\text{parallel}}(s) \qquad T = T_0$$

$$F_{\text{f static}} = \min(F_{\text{applied}}, F_{\text{f static max}}) \qquad F_{\text{f static max}} = \mu_{\text{static}} F_{\text{N}} \qquad F_{\text{f kinetic}} = \mu_{\text{kinetic}} F_{\text{N}}$$

$$v_{\text{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\text{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\text{centripetal}} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\text{drag,lin}} = bv \qquad v_{\text{T}} = \frac{mg}{b} \qquad \tau = \frac{v_{\text{T}}}{g} = \frac{m}{b} \qquad v = v_{\text{T}}(1 - e^{-t/\tau})$$

$$F_{\text{drag,quad}} = bv^2 = \frac{1}{2}C\rho A v^2 \qquad v_{\text{T}} = \sqrt{\frac{mg}{b}}$$

14 Energy and Work

$$dW = \vec{F}\cdot d\vec{s} \qquad W = \int \vec{F}\cdot d\vec{s} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\text{mechanical}} = KE + PE$$

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F}\cdot\vec{v}$$

$$\Delta KE = W_{\text{net}} \quad \Delta PE_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \quad \Delta E = W_{\text{nonconservative}}$$

$$F = -\frac{dPE}{dx} \qquad \vec{F} = -\nabla PE \qquad PE = \frac{1}{2}kx^2 \qquad PE = mgy$$

15 Momentum

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta KE_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\text{cm}} = \vec{F}_{\text{net non-flux}} + (\vec{v}_{\text{flux}} - \vec{v}_{\text{cm}})\frac{dm}{dt} = \vec{F}_{\text{net non-flux}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_0 + v_{\text{ex}} \ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

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16 **Collisions**

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\,dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\text{net}}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$KE_{\text{total }f} = KE_{\text{total }i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\text{rel}'} = -v_{\text{rel}} \qquad \text{1-d Elastic Collision Expressions}$$

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17 **Rotational Kinematics**

$$2\pi = 6.2831853\ldots \qquad \frac{1}{2\pi} = 0.15915494\ldots$$

$$\frac{180^\circ}{\pi} = 57.295779\ldots \approx 60^\circ \qquad \frac{\pi}{180^\circ} = 0.017453292\ldots \approx \frac{1}{60^\circ}$$

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\omega = \alpha t + \omega_0 \qquad \Delta\theta = \frac{1}{2}\alpha t^2 + \omega_0 t \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t \qquad \Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$

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18 **Rotational Dynamics**

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

$$L_z = RP_{xy} \sin \gamma_L \qquad \tau_z = RF_{xy} \sin \gamma_\tau \qquad L_z = I\omega \qquad \tau_{z,\text{net}} = I\alpha$$

$$I = \sum_i m_i R_i^2 \qquad I = \int R^2 \rho \, dV \qquad I_{\text{parallel axis}} = I_{\text{cm}} + mR_{\text{cm}}^2 \qquad I_z = I_x + I_y$$

$$I_{\text{cyl,shell,thin}} = MR^2 \qquad I_{\text{cyl}} = \frac{1}{2}MR^2 \qquad I_{\text{cyl,shell,thick}} = \frac{1}{2}M(R_1^2 + R_2^2)$$

$$I_{\text{rod,thin,cm}} = \frac{1}{12}ML^2 \qquad I_{\text{sph,solid}} = \frac{2}{5}MR^2 \qquad I_{\text{sph,shell,thin}} = \frac{2}{3}MR^2$$

$$a = \frac{g \sin \theta}{1 + I/(mr^2)}$$

$$KE_{\text{rot}} = \frac{1}{2}I\omega^2 \qquad dW = \tau_z \, d\theta \qquad P = \frac{dW}{dt} = \tau_z \omega$$

$$\Delta KE_{\text{rot}} = W_{\text{net}} = \int \tau_{z,\text{net}} \, d\theta \qquad \Delta PE_{\text{rot}} = -W = -\int \tau_{z,\text{con}} \, d\theta$$

$$\Delta E_{\text{rot}} = KE_{\text{rot}} + \Delta PE_{\text{rot}} = W_{\text{non,rot}} \qquad \Delta E = \Delta KE + KE_{\text{rot}} + \Delta PE = W_{\text{non}} + W_{\text{rot}}$$

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19 Static Equilibrium

$$\vec{F}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = \tau'_{\text{ext,net}} \qquad \text{if } F_{\text{ext,net}} = 0$$

$$0 = F_{\text{net }x} = \sum F_x \qquad 0 = F_{\text{net }y} = \sum F_y \qquad 0 = \tau_{\text{net}} = \sum \tau$$