Intro Physics Semester I **Name:**

Homework 11: Rotational Dynamics: One or two full answer questions will be marked. There will also be a mark for completeness. Homeworks are due usually the day after the chapter they are for is finished. Solutions will be posted soon thereafter. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

Answer Table

Name:

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	31.	O	O	O	O	O
2.	O	O	O	O	O	32.	O	O	O	O	O
3.	O	O	O	O	O	33.	O	O	O	O	O
4.	O	O	O	O	O	34.	O	O	O	O	O
5.	O	O	O	O	O	35.	O	O	O	O	O
6.	O	O	O	O	O	36.	O	O	O	O	O
7.	O	O	O	O	O	37.	O	O	O	O	O
8.	O	O	O	O	O	38.	O	O	O	O	O
9.	O	O	O	O	O	39.	O	O	O	O	O
10.	O	O	O	O	O	40.	O	O	O	O	O
11.	O	O	O	O	O	41.	O	O	O	O	O
12.	O	O	O	O	O	42.	O	O	O	O	O
13.	O	O	O	O	O	43.	O	O	O	O	O
14.	O	O	O	O	O	44.	O	O	O	O	O
15.	O	O	O	O	O	45.	O	O	O	O	O
16.	O	O	O	O	O	46.	O	O	O	O	O
17.	O	O	O	O	O	47.	O	O	O	O	O
18.	O	O	O	O	O	48.	O	O	O	O	O
19.	O	O	O	O	O	49.	O	O	O	O	O
20.	O	O	O	O	O	50.	O	O	O	O	O
21.	O	O	O	O	O	51.	O	O	O	O	O
22.	O	O	O	O	O	52.	O	O	O	O	O
23.	O	O	O	O	O	53.	O	O	O	O	O
24.	O	O	O	O	O	54.	O	O	O	O	O
25.	O	O	O	O	O	55.	O	O	O	O	O
26.	O	O	O	O	O	56.	O	O	O	O	O
27.	O	O	O	O	O	57.	O	O	O	O	O
28.	O	O	O	O	O	58.	O	O	O	O	O
29.	O	O	O	O	O	59.	O	O	O	O	O
30.	O	O	O	O	O	60.	O	O	O	O	O

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012 qmult 00100 1 1 1 easy memory: rotational principles

1. Classical rotational dynamics principles are:

   a) secondary principles derived from the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, energy, etc.).
   b) independent postulates completely unrelated to the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, the energy concept, etc.).
   c) secondary principles derived from quantum mechanics.
   d) all gross approximations derived from the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, the energy concept, etc.).
   e) independent, but very approximate, postulates completely unrelated to the fundamental principles of Newtonian physics (i.e., Newton's three laws, force laws, the energy concept, etc.).

   SUGGESTED ANSWER: (a)

   This is, of course, the right answer for classical physics. But in quantum mechanics it seems to me that intrinsic angular momentum (electron spin, etc.) comes in as its own postulate.

   Wrong answers:
   b) Nope.

   Redaction: Jeffery, 2001jan01

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012 qmult 00200 1 1 3 easy memory: cross product definition

2. The mathematical operation

$$\vec{a} \times \vec{b} = ab \sin\theta \hat{n} ,$$

   where $\vec{a}$ and $\vec{b}$ are general vectors, θ is the angle between them (with their tails joined), and $\hat{n}$ is a unit vector normal to the plane defined by $\vec{a}$ and $\vec{b}$ and with sense determined by a right-hand rule (sweep fingers of the right hand from $\vec{a}$ to $\vec{b}$ and the thumb gives the sense), is the:

   a) outer product. b) dot product. c) cross product. d) inner product.
   e) angry product.

   SUGGESTED ANSWER: (c)

   It is also called the vector product since the product is a vector. But "vector" has two syllables and is less easy to say.

   Wrong answers:
   b) This is $\vec{a} \cdot \vec{b} = ab \cos\theta$.
   e) Sort of a more vehement cross product.

   Redaction: Jeffery, 2008jan01

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012 qmult 00220 1 1 2 easy memory: cross product standard values

3. Behold:

$$\vec{a} \times \vec{b} = \begin{cases} ab \sin\theta \hat{n} & \text{in general;} \\ 0 & \text{for } \theta = 0^\circ \text{ or } 180^\circ; \\ ab\hat{n} & \text{for } \theta = 90^\circ; \\ \underline{\qquad\qquad\qquad} & \text{in general.} \end{cases}$$

   a) $\vec{b} \times \vec{a}$. b) $-\vec{b} \times \vec{a}$. c) $-\vec{a} \times \vec{b}$. d) $-\vec{b} \cdot \vec{a}$. e) $\vec{a} \cdot \vec{b}$.

   SUGGESTED ANSWER: (b)

   It is also called the vector product since the product is a vector. But "vector" has two syllables and is less easy to say.

   Wrong answers:
   c) Not in general. In the special case of $\vec{a} \times \vec{b} = 0$, this is true.

   Redaction: Jeffery, 2008jan01

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012 qmult 00320 1 1 5 easy memory: rotational 2nd law specified

4. The rotational 2nd law for a particle or a system of particles is:

a) $\vec{A}\times\vec{B}$. b) $\dfrac{d\vec{\tau}_{\rm net}}{dt}=\dfrac{1}{2}I\vec{L}$. c) $\dfrac{d\vec{L}}{dt}=\dfrac{1}{2}I\tau_{\rm net}$. d) $\dfrac{d\vec{\tau}_{\rm net}}{dt}=\vec{L}$. e) $\dfrac{d\vec{L}}{dt}=\vec{\tau}_{\rm net}$.

SUGGESTED ANSWER: (e)

Wrong answers:
d) Exactly wrong.

Redaction: Jeffery, 2008jan01

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012 qmult 00330 1 1 2 easy memory: torque definition descriptive

5. Torque is:

a) that thing (for lack of a better word) needed for an angular **VELOCITY** to exist.
b) that thing (for lack of a better word) that can cause angular **ACCELERATION**.
c) the same as angular acceleration.
d) the same as angular velocity.
e) a minor rock star of the 1960's: the fairhaired Monkee.

SUGGESTED ANSWER: (b)

Torque is to angular acceleration what force is to (linear) acceleration.

Wrong answers:
e) That was Peter Tork. He and Mike Naismith were actual musicians. Davy Jones and Mickey Dolan were actual actors. Davy Jones caused David Bowie to be David Bowie since Britain wasn't big enough for two rock star David Joneses. How do I remember all this?

Redaction: Jeffery, 2001jan01

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012 qmult 00340 1 1 1 easy memory: vector addition
Extra keywords: new place

6. Angular momentum $\vec{L}$ and torque $\vec{\tau}$ are actually pseudovectors (AKA axial vectors), but our purposes they can be treated just like ordinary vectors—and so we will just call them vectors. In this class, we usually deal only with the z-components of angular momentum and torque which are scalars. The direction of these vectors can be determined from their definitions

$$\vec{L}=\vec{r}\times\vec{p} \qquad \text{and} \qquad \vec{\tau}=\vec{r}\times\vec{F} \ .$$

a) Vectors add like vectors. This can be done geometrically or using components in some coordinate system.
b) Vectors add just like scalars.
c) Vectors cannot be added at all. The concept of adding vectors is undefined.
d) Vectors add just like scalars **MULTIPLY**.
e) Vectors add just like scalars **DIVIDE**.

SUGGESTED ANSWER: (a)

The distinction of pseudovectors from ordinary vectors is beyond our scope.

Wrong answers:

Redaction: Jeffery, 2001jan01

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012 qmult 00570 2 1 2 moderate memory: conservation of ang. momentum
Extra keywords: new place

7. You are Katherina Witt (or Dorothy Hamill, Karen Magnussen, or even Barbara Ann Scott) at the Winter Olympics. After executing a flawless quad followed by a physically impossible horizontal leap, you torque yourself (using heal? toe?) into a spin. You now decide to speed up your spin (increase your ω). Remembering that without a net external torque, angular momentum ($L=I\omega$) is conserved and assuming that the ice friction torque on your blades is negligible, you do what? **HINT:** The more spread out from the axis a fixed amount of mass is, the greater the rotational inertia.

a) **PULL** your arms in to **INCREASE** your rotational inertia.
b) **PULL** your arms in to **DECREASE** your rotational inertia.
c) **FLING** your arms out to **DECREASE** your rotational inertia.
d) **FLING** your arms out to **INCREASE** your rotational inertia.
e) Belly flop.

SUGGESTED ANSWER: (b)

You might remember what skaters actually do. And Witt, Hamill, Magnuson, Scott? Skaters. Even I don't remember Scott: she was Canada's darling in the 1940s. But they went on happily one hopes, not having to lift "the still defended challenge cup." Do they use a heal or a toe or what? I'm a non-skating Canadian and don't know.

Wrong answers:
e) Well maybe, but it won't help your score.

Redaction: Jeffery, 2001jan01

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012 qmult 00660 1 3 4 easy math: rotational Newton's 2nd law simple example 1

8. There is a net torque of 3.0 N m on an object about a particular axis of the object. The object's rotational inertia for this axis is $10^2\,\text{kg}\,\text{m}^2$. What is the angular acceleration?

a) $\alpha = 3 \times 10^2\,\text{rad/s}^2$. b) $\alpha = \dfrac{1}{3} \times 10^2\,\text{rad/s}^2$. c) $\alpha = \dfrac{1}{3} \times 10^{-2}\,\text{rad/s}^2$.
d) $\alpha = 3 \times 10^{-2}\,\text{rad/s}^2$. e) $\alpha = 10^{-2}\,\text{rad/s}^2$.

SUGGESTED ANSWER: (d)

Note $\tau = I\alpha$ implies $\alpha = \tau/I$, and so

$$\alpha = \frac{\tau}{I} = \frac{3}{100} = 0.03\,\text{rad/s}^2 \ .$$

Wrong answers:

Redaction: Jeffery, 2001jan01

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012 qmult 00680 1 5 2 easy thinking: Archimedes

9. Archimedes (287?–212 BC) (the discoverer of the law of the lever) said something like "give me a lever long enough, a fulcrum (a support for a lever), and a place to stand, and I will move:

a) flea". b) the Earth". c) a horse to drink (after leading it to water)".
d) a man to drink". e) a minor rock star of the 1960's: the fairhaired Monkee."

SUGGESTED ANSWER: (b)

Wrong answers:
a) C'mon. Would he have said flea?
c) It's not a physics answer and I don't see how it could be done.
d) It's not a physics answer, but I do see how it could be done.
e) That was Peter Tork. He and Mike Naismith were actual musicians. Davy Jones and Mickey Dolan were actual actors. Davy Jones caused David Bowie to be David Bowie since Britain wasn't big enough for two rock star David Joneses. How do I remember all this?

Redaction: Jeffery, 2001jan01

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012 qmult 00720 1 4 5 easy deducto-memory: gravitational torque

10. "Let's play *Jeopardy*! For \$100, the answer is: Its torque about any origin can be calculated as if all the object's mass were located at the center of mass."

What is ____________, Alex?

a) a contact force b) friction c) a tension force d) a normal force
e) gravity near the Earth's surface

SUGGESTED ANSWER: (e)

For any system of particles of mass m_i and position r_i relative to any origin and with $\hat{y}$ specifying the upward direction, one has

$$\tau = \sum_i \vec{r}_i \times (-m_i g\hat{y}) = \left(\sum_i m_i \vec{r}_i\right) \times (-g\hat{y}) = \vec{r} \times (-mg\hat{y}) \ ,$$

where we have used the center of mass definition

$$\vec{r} = \frac{\sum_i m_i \vec{r}_i}{m}$$

with m being total mass.

Wrong answers:
 a) Contact force: no way.

Redaction: Jeffery, 2001jan01

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012 qmult 00850 1 1 4 easy memory: uniform solid ball rotational inertia

11. The rotational inertia of a uniform solid ball of mass M and radius r is:

 a) Mr^2. b) Mr. c) $\frac{2}{5}Mr$. d) $\frac{2}{5}Mr^2$. e) Mr^3.

 SUGGESTED ANSWER: (d)

 Wrong answers:
 a) This is for a thin ring or cylindrical shell.
 b) Not dimensionally correct.

 Redaction: Jeffery, 2008jan01

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012 qmult 01050 1 4 1 easy thinking: rotational analogs

12. For rigid body rotation about a fixed axis, the rotational analogs of inertial mass (m), momentum ($\vec{P} = m\vec{v}$), Newton's 2nd law ($\vec{F}_{\text{ext}} = m\vec{a}$), and translational (or center-of-mass) kinetic energy ($KE = (1/2)mv^2$) are, respectively:

 a) rotational inertia (I), angular momentum ($L = I\omega$), the rotational Newton's 2nd law ($\tau = I\alpha$), and rotational kinetic energy ($KE = (1/2)I\omega^2$).
 b) rotational inertia (I), the rotational Newton's 2nd law ($\tau_{\text{ext}} = I\alpha$), and angular momentum ($L = I\omega$).
 c) the rotational Newton's 2nd law ($\tau_{\text{ext}} = I\alpha$), angular momentum ($L = I\omega$), and rotational kinetic energy ($KE = (1/2)I\omega^2$).
 d) mass (m), angular velocity (ω), the static Newton's 2nd law ($0 = 0$), and rotational kinetic energy ($KE = (1/2)I\omega^2$).
 e) without clear definitions.

 SUGGESTED ANSWER: (a)

 Wrong answers:
 b) There are only three items in the answer list.
 c) There are only three items in the answer list.
 d) Moment of angular moment is a nonsense term and angular velocity does not fit.
 e) No they have clear definitions.

 Redaction: Jeffery, 2001jan01

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012 qmult 01200 1 1 3 easy memory: rolling motion

13. In rolling motion (e.g., a ball rolling along the ground), there is in general both rotational and:

 a) angular motion. b) centripetal motion. c) center-of-mass or translational motion.
 d) slow motion. e) blocked motion.

 SUGGESTED ANSWER: (c)

It could be argued that there can be rolling motion without center-of-mass motion. It depends on how you define terms. But in general there can be both and the statement is for the general case.

Note I actually prefer the more explanatory expression center-of-mass motion, but translational motion is the synonym.

Wrong answers:
a) Partially redundant with rolling motion.
b) Partially redundant with rolling motion.
e) A nonsense answer.

Redaction: Jeffery, 2008jan01

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012 qmult 01210 1 4 3 easy deducto-memory: no-slip condition

14. "Let's play *Jeopardy*! For $100, the answer is: The condition that is required for wheels in most ordinary circumstances: e.g., for car wheels."

What is the ____________ condition, Alex?

a) no-trip b) no-rip c) no-slip d) no-grip e) no-blip

SUGGESTED ANSWER: (c)

There are other useful conditions: no-tip (poor service), no-crip (no gangs), no-gyp (no swindling), no-kip (no salted herrings), no-flip (and no-flop), no-split (infinitives), no-spit (speaks for itself), ...

Wrong answers:
b) Well this one too actually.

Redaction: Jeffery, 2008jan01

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012 qmult 01230 1 1 1 easy memory: rolling motion relationships

15. For a roller of radius r and defining the center-of-mass or translation direction as the positive s direction and the counterclockwise as the positive rotational direction, the relationships between translational and rotational kinematic variables with the no-slip condition are

$$\Delta\theta = -\frac{\Delta s}{r} \ , \qquad \omega = -\frac{v}{r} \ ,$$

and:

a) $\alpha = -\dfrac{a}{r}$. b) $\alpha = -ar$. c) $\alpha = -\omega r$. d) $\alpha = \dfrac{a}{r}$. e) $\alpha = ar$.

SUGGESTED ANSWER: (a)

The minus sign in the formulae is annoying. One could make it go away by going against the usual convention and defining clockwise as the positive rotational direction.

Wrong answers:
d) This only relates the magnitudes.

Redaction: Jeffery, 2008jan01

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012 qmult 01240 1 5 1 easy thinking: rolling motion and static friction
Extra keywords: new place

16. A roller (i.e., a sphere or cylinder) needs static friction to enforce the no-slip condition in general. But it doesn't need static friction to keep rolling at constant velocity as it rolls along a level surface in the limit of no rolling friction. You know this (if for no other reason) because if the roller rolls off the table, it:

a) keeps spinning. b) stops spinning. c) goes into a parabolic trajectory.
d) goes into a straight line trajectory. e) drops straight down.

SUGGESTED ANSWER: (a)

It's a leading question if ever there was one. On a level surface with no forces acting and $v_{\rm tr} = v_{\rm rim}$ initially, then ideally $v_{\rm tr} = v_{\rm rim}$ is maintained by conservation of momentum and conservation of angular momentum: static friction doesn't turn on. Actually, rolling friction (due to deformation of ground and roller) will act to slow the translational motion and then static friction acts to maintain $v_{\rm tr} = v_{\rm rim}$. Ideally, on an incline static friction would turn on to maintain $v_{\rm tr} = v_{\rm rim}$ exactly as gravity tries to reduce (going uphill) or increase (going downhill) translation velocity. If one has $v_{\rm tr} \neq v_{\rm rim}$, then kinetic friction will try to bring about $v_{\rm tr} = v_{\rm rim}$.

Wrong answers:

b) You've seen a roller go off a table. It doesn't stop spinning.
c) True, but this is not the reason.
d) False, but it still not the reason.
e) C'mon, it's a parabolic path.

Redaction: Jeffery, 2001jan01

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012 qmult 01280 2 5 2 moderate thinking: mechanical energy conserved
Extra keywords: new place

17. In the idealized object-rolling-down-incline system, the rolling object is perfectly round (a cylinder or sphere), the object and incline are perfectly rigid, and there is a static frictional force that causes there to be no slipping. (The frictional force is a static frictional force since ideally at the point of contact there is no relative motion between object and incline surfaces.) Mechanical energy (the sum of potential energy, center-of-mass (translational) kinetic energy, rotational kinetic energy) is conserved. And yet there is a frictional force is doing work on the object. Why doesn't the frictional force cause loss of mechanical energy to heat energy (i.e., dissipation as it is called)?

a) There is never any frictional heating with round objects.
b) The frictional force does no net work. Going down the incline, the friction force (which points uphill) does negative work lowering the object's center-of-mass kinetic energy (which being increased by gravitational potential energy), but positive work to increase its rotational kinetic energy. The friction work contributions cancel as a calculation would show. Going up the incline, the frictional force is still uphill, but now converts rotational kinetic energy into center-of mass kinetic energy (which is disappearing into gravitational potential energy). Again the friction work contributions can cancel out. The static friction, in fact, acts as channel between center-of-mass and rotational kinetic energies. Kinetic friction dissipates energy by turning it into random microscopic kinetic and potential energy. The ideal static friction force does not do this, and so does not dissipate energy. Of course, in reality there is always some slipping and some non-elastic deformation of the bodies, and so some energy gets dissipated. This slipping and deformation is often not obvious to the eye.
c) Actually heat energy is being turned into macroscopic kinetic energy by friction in this case without rejection of any heat energy to a lower temperature thermal reservoir. This violates the 2nd law of thermodynamics, but that doesn't matter in a mechanics problem.
d) The frictional force **CAUSES** a torque and forces that cause torques **NEVER** cause dissipation.
e) The frictional force **DOES NOT CAUSE** a torque and forces that **DO NOT** cause torques **NEVER** cause dissipation.

SUGGESTED ANSWER: (b)

Note in the ideal case an object rolling at a constant velocity on a level with center-of-mass and rim speeds the same, there is no static friction at the contact point. Conservation of linear and angular momentum keep the center-of-mass and rim rotation speeds synchronized, and the object never applies a shearing frictional force on the ground and the ground never applies one back. The object would keep rotating if rolled over a ledge and went on parabolic trajectory Going uphill or downhill, the gravitational force tries to change the center-of-mass kinetic energy of the object and thus de-synchronize the center-of-mass speed and the rim speed. Now the object will apply a static frictional force at the point of contact to the ground and the ground reciprocates with an equal and opposite frictional force. This ground frictional force is uphill both for uphill and downhill motion. Going uphill, the object's center-of mass speed is being decreased and the friction force must slow it's rotation to maintain synchronization and no-slip. Going downhill, the object's translation speed is being increased and the friction force must speed up it's rotation to maintain synchronization and no-slip.

To show that net work by friction on the object is zero consider displacement of the object ds with s measured up the slope. The total work done by the friction frictional force is

$$dW = F\,ds + \tau\,d\theta \ ,$$

where $\tau = rF$ with r being the object radius. We take the direction of F (which is always uphill recall) as defining the positive sense for axis of rotation of the roller using a right-hand rule. Thus the torque is always positive. Going uphill or downhill, the change in θ with no-slip is given by $d\theta = -ds/r$: i.e., $d\theta < 0$ uphill and $d\theta > 0$ downhill according to our defined rotation positive direction. Thus

$$dW = F\,ds + \tau\,d\theta = F\,dx - rF\frac{ds}{r} = 0 \ .$$

Wrong answers:

a) O why would round objects be so blessed.
c) The argument (the incorrect argument) points to the wrong conclusion.
d) Forces that cause torques can cause dissipation too.
e) But it does cause a torque. And forces that don't cause torques do cause dissipation.

Redaction: Jeffery, 2001jan01

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012 qfull 00300 1 3 0 easy math: deriving the rotational 2nd law

18. We define the angular momentum and torque on a particle i by, respectively, the formulae

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i \qquad \text{and} \qquad \vec{\tau}_i = \vec{r}_i \times \vec{F}_i \ ,$$

where $\vec{r}_i$ is the particle position relative to a general origin, $\vec{p}_i$ is the particle momentum, and $\vec{F}_i$ is an external force on the paricle (but not necessarily the net external force on the particle). The term particle is used in the sense of a sample of matter sufficiently small that its extent in space can be neglected. Usually by particle in classical physics, one means a sample of matter that it macroscopically tiny—and so it doesn't matter exactly where you locate it's center of mass and other other properties as long as they are within the particle's actual extent—but still larger a microscopic particle (which requires quantum mechanics). Both angular momentum and torque are pseudovectors (AKA axial vectors) since they are formed by cross products of ordinary vectors. For our purposes, we can treat them just as ordinary vectors. As vectors, they have a direction in space space, but their extent is in their own abstract space. Where are they located in real space? Well nowhere really—it adds nothing to their utility to explicitly locate them. But for mental convenience, one often locates them at the origin.

From its definition, one can see that angular momentum is a sort of mass-weighted measure of rotation about an axis passing through the origin and perpendicular to $\vec{r}_i$ and $\vec{p}_i$. The particle with angular momentum does not have to be in rotation at all though: it could be going in a straight line and still have angular momentum about any origin. From its definition, one can see that torque is a sort of measure of the rotational acceleration effect of force $\vec{F}_i$ about the axis passing through the origin and perpendicular to $\vec{r}_i$ and $\vec{F}_i$. Since the definitions of angular momentum and torque are precise, they are precise measures and instead of saying "sort of something", we just say they are precisely what they are: angular momentum and torque.

Angular momentum and torque turn out to be quantitatively useful dynamical variables for analyzing rotational motion—and that's why we have defined them. In this, question we show the essentials of angular dynamics using angular momentum and torque by deriving, among other things, the rotational 2nd law. The rotational 2nd law is just called a law by convention since it is a derived result. It is called the rotational 2nd law since in rotationaly dynamics it plays a role analogous to Newton's 2nd law—which really is a law: it is a true axiom of classical mechanics.

a) What is the total angular momentum $\vec{L}$ of a system made up of particles i?

b) The dynamics of rotational systems is studied by studying the time variation of $\vec{L}$. To this end, we need formula for $d\vec{L}/dt$. Derive this formula and eliminate any zero terms. **HINT:** The product rule does apply to the cross product and the cross product of a vector with itself is zero.

c) Newton's 2nd all in general form is

$$\frac{d\vec{p}}{dt} = \vec{F}_{\text{ext}} \ ,$$

where $\vec{p}$ is the momentum of a system and $\vec{F}_{\text{ext}}$ is the net external force on the system. Use this law to replace the momentum derivative in the part (b) answer formula by the net external force on particle i

$$\vec{F}'_{\rm i} = \vec{F}_i + \sum_j \vec{F}_{ji} \ ,$$

where $\vec{F}_i$ net force on particle i from force sources outside the system of particles, F_{ji} is the force of particle j on particle i, and F_{ii} is zero the external force of particle i on itself logically zero.

d) Now show that

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i \ ,$$

making use of Newton's 3rd law and using the assumption that all inter-particle forces are central forces. A central force is one where the force of one particle/system on another is aligned with the line joining the particles/particle centers of mass. For solid objects and many others, the assumption of central forces is valid valid.

e) Now use the definition of torque to show that

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{ext}} \ .$$

where $\vec{\tau}_{\text{ext}}$ net torque on the system due to external forces on the system. This result is the general form of the rotational 2nd law. **HINT:** There is almost nothing to show now.

f) The general rotational 2nd law is generally useful. But we can specialize it to a form which is useful for rigid body rotation about a fixed axis which conventionally is the z-axis of 3-dimensional Cartesian coordinate. As first step to this form, prove that the z component of

$$\vec{C} = \vec{A} \times \vec{B}$$

is given by

$$C_z = (\vec{A}_{xy} \times \vec{B}_{xy}) \cdot \hat{z} \ ,$$

where $\vec{A}_{xy}$ is the component of $\vec{A}$ in the xy plane $\vec{B}_{xy}$ is the component of $\vec{B}$ in the xy plane, and $\hat{z}$ is the unit vector in the z direction. The last formula show that the z component of z is independent of the z components of $\vec{A}$ and $\vec{B}$ and depends only on there xy plane componets.

g) Using the part (f) result, show that

$$L_z = \sum_i (\vec{r}_{xy,i} \times \vec{p}_{xy,i}) \cdot \hat{z} = \sum_i r_{xy,i} p_{xy,i} \sin\theta_i \ , \qquad \tau_z = \sum_i (\vec{r}_{xy,i} \times \vec{F}_{xy,i}) \cdot \hat{z} = \sum_i r_{xy,i} p_{xy,i} \sin\theta_i \ .$$

where θ_i is the angular coordinate of the second vector relative to the first measured positive in the counterclockwise direction. **HINT:** The proof is by inspection.

h) For rigid body rotation about the z axis, at least in simple analysis, you do not need consider the angular momentum or torque components for the x and y axes. In general, they will not be zero, but you just do not need them to describe the motion—in a simple analysis. For rigid body rotation about the z axis, show that

$$L_z = I\omega \ ,$$

where

$$I = \sum_i m_i r_{xy,i}^2$$

is the rotational inertia (or moment of inertai) of the body about the z axis and ω is the common angular acceleration of the body. For a rigid body, I is a constant, of course.

i) Now proof the special case rotational 2nd law

$$I\alpha = \tau_z \ .$$

where α is the angular acceleration. **HINT:** The proof is by inspection.

j) Moment arm is the name for the length of a perpendicular from the z axis of rotation to a line aligned with $\vec{F}_{xy,i}$. A signed moment arm is one where multiple the moment arm by ± 1, where the upper case is for a counterclockwise torque and the lower case for a clockwise torque. Prove that

$$r_{xy,i} F_{xy,i} \sin\theta_i = d_i F_{xy,i} \ ,$$

where d_i is the signed moment arm for $\vec{F}_{xy,i}$. What if $\theta_i = 0$? What is τ_z evaluated using signed moment arms? In many cases, using moment arms or signed moment arrms simplifies the calculation of τ_z. **HINT:** A diagram might help.

SUGGESTED ANSWER:

a) Behold:

$$\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \vec{p}_i \ .$$

b) Behold:

$$\frac{d\vec{L}}{dt} = \sum_i \vec{v}_i \times \vec{p}_i + \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt} = \sum_i m_i(\vec{v}_i \times \vec{v}_i) = 0 + \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt} \ ,$$

and thus we obtain

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt} \ .$$

c) Behold:

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt} = \sum_i \vec{r}_i \times \vec{F}_i' = \sum_i \vec{r}_i \times \vec{F}_i + \sum_{ij} \vec{r}_i \times \vec{F}_{ji} \ ,$$

where we have used Newton's 2nd law and the prescribed form for the nex external force on particle i.

d) In

$$\sum_{ij} \vec{r}_i \times \vec{F}_{ji} \ ,$$

all the terms have to fall into pairs of the form

$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij}$$

since there are no $\vec{F}_{ii}$ terms. By the 3rd law, we have

$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} \ .$$

Now $\vec{r}_i - \vec{r}_j$ is the displacement vector from particle j to particle i. By the assumption of central forces, $\vec{F}_{ji}$ is aligned with $\vec{r}_i - \vec{r}_j$, and thus

$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} = 0 \ .$$

So all the terms in

$$\sum_{ij} \vec{r}_i \times \vec{F}_{ji}$$

cancel pairwise. Consequently,

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i \ ,$$

e) Behold:

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i = \sum_i \vec{\tau}_i = \vec{\tau}_{\text{ext}} \ .$$

f) Well

$$\vec{A} = \vec{A}_{xy} + \vec{A}_z \ , \qquad \vec{B} = \vec{B}_{xy} + \vec{B}_z \ .$$

Thus,

$$\vec{C} = \vec{A}\times\vec{B} = (\vec{A}_{xy}\times\vec{B}_{xy})+(\vec{A}_{xy}\times\vec{B}_z)+(\vec{A}_z\times\vec{B}_{xy})+(\vec{A}_z\times\vec{B}_z) = (\vec{A}_{xy}\times\vec{B}_{xy})+(\vec{A}_{xy}\times\vec{B}_z)+(\vec{A}_z\times\vec{B}_{xy}) \ ,$$

where the last equality follows since the cross product of aligned vectors is zero. Now the vectors $(\vec{A}_{xy} \times \vec{B}_z)$ and $(\vec{A}_z \times \vec{B}_{xy})$ have no component in the z direction by the nature of the cross product. Thus,

$$C_z = \vec{C} \cdot \hat{=} (\vec{A}_{xy} \times \vec{B}_{xy}) \cdot \hat{z} \ ,$$

QED.

g) By inspection,

$$L_z = \sum_i (\vec{r}_{xy,i} \times \vec{p}_{xy,i}) \cdot \hat{z} = \sum_i r_{xy,i} p_{xy,i} \sin\theta_i \ , \qquad \tau_z = \sum_i (\vec{r}_{xy,i} \times \vec{F}_{xy,i}) \cdot \hat{z} = \sum_i r_{xy,i} p_{xy,i} \sin\theta_i \ .$$

h) Behold:

$$L_z = \sum_i r_{xy,i} p_{xy,i} \sin\theta_i = \sum_i m_i r_{xy,i} v_{xy,i} \sin(\pm 90^\circ) = \sum_i m_i r_{xy,i} v_{\mathrm{tan},i} = \sum_i m_i r_{xy,i}^2 \omega = I\omega \ .$$

i) From the general rotational 2nd law, we know that

$$\frac{dL_z}{dt} = \tau_z \ .$$

From the part (h) result, we find for rigid body rotation about the z axis that

$$\frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha \ .$$

Thus, we have

$$I\alpha = \tau_z \ ,$$

the rotational 2nd law for rigid body rotation about the z axis.

j) From a diagram that you will have to imagine, it is clear that the moment arm is the opposite to $|\theta_i|$ of a triangle formed by the moment arm, $r_{xy,i}$, and a line segment through the line aligned with force $\vec{F}_{xy,i}$. Thus,

$$d_i = r_{xy,i} |\sin\theta_i| \ ,$$

and thus

$$r_{xy,i} F_{xy,i} \sin\theta_i = d_i F_{xy,i} \ ,$$

If $\theta_i = 0$, the moment are and signed moment arm is zero and the torque for that force is zero. Now we see that

$$\tau_z = \sum_i r_{xy,i} F_{xy,i} \sin\theta_i = \sum_i d_i F_{xy,i} \ .$$

Redaction: Jeffery, 2008jan01

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012 qfull 00310 1 3 0 easy math: rigid-body rotation and total angular momentum

19. Let us consider some intereting features of rigid body rotation.

a) Starting from the general angular momentum definition for a system of particles

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i \ .$$

prove for a rigid body rotating around the z axis that

$$\vec{L} = I\omega\hat{z} - \left(\sum_i m_i \vec{r}_{xy,i} r_{z,i}\right)\omega \ ,$$

where I is the rotational inertia, $\vec{r}_{xy,i}$ is the xy component vector of $\vec{r}_i$ (which is a time-dependent quantity is the vector is rotating about the z axis with the particle), $r_{z,i}$ is the z component of $\vec{r}_i$, and ω is the angular velocity (which is not necessarily constant). The second term in the formula for $\vec{L}$ is not necessarily zero. This shows that rigid body rotators do not necessarily have the angular momentum vector aligned with the rotation z axis. However, if you are only studying the rotation, usually the studying the z component of angular momemtum suffices.

b) Give two simple cases where the second term in the part (a) result vanishes.

SUGGESTED ANSWER:

a) Behold:

$$\begin{aligned}\vec{L} &= \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (\vec{r}_{xy,i} \times \vec{r}_{z,i}) \times (\vec{p}_{xy,i} + \vec{p}_{z,i}) \\ &= \sum_i [(\vec{r}_{xy,i} \times \vec{p}_{xy,i}) + (\vec{r}_{xy,i} \times \vec{p}_{z,i}) + (\vec{r}_{z,i} \times \vec{p}_{xy,i}) + (\vec{r}_{z,i} \times \vec{p}_{z,i})] \\ &= \sum_i [(\vec{r}_{xy,i} \times \vec{p}_{xy,i}) + (\vec{r}_{z,i} \times \vec{p}_{xy,i}) = \sum_i [(\vec{r}_{xy,i} \times \vec{m}_i v_{xy,i}) + (\vec{r}_{z,i} \times \vec{m}_i v_{xy,i}) \\ &= \left(\sum_i m_i r_{xy,i}^2\right)\omega\hat{z} - \left(\sum_i m_i \vec{r}_{xy,i} r_{z,i}\right)\omega \qquad = I\omega\hat{z} - \left(\textstyle\sum_i m_i \vec{r}_{xy,i} r_{z,i}\right)\omega \ .\end{aligned}$$

where we have used the fact that $\hat{z} \times \hat{\theta}_i = -\hat{r}_{xy,i}$ and I is the rotational inertial which is constant for a rigid body. The first term can only depend on time through ω, but the second can depend on time through the $\hat{r}_{xy,i}$ vectors.

The second term will not in general be zero and it will not be constant in general even if ω is constant. However, the 2nd term has no component in the z direction. Thus, if one wants just a description of the z rotation of the rigid rotator, one generally does not need to consider it. But if one actually wanted to know what x and y constraint torques are needed for the system, you would have to consider it. If you building a rigid rotator, you might well want to know what they are in order for your system to be strong enough to supply them.

b) A first case is when the center of mass for every z level is on the z axis. This makes the contribution for every z level zero by the definition of the center of mass. So the whole second term vanishes. In this case, the center of mass of the whole rigid body would be on the z axis necessarily. However, just having the center of mass on the z axis does not imply that the second term will vanish.
indent A second case is when the body is symmetrical about the $z = 0$ plane. Then the contributions to the second term from levels z and $-z$ would cancel and the second term will vanish.
indent Recall that the body is rigid. So if the conditions cited above hold at one instant in time, they hold for all time.
indent An interesting question is whether a torqueless rigid rotor can always be described as rotating about a fixed axis. The answer is no as one learns from Wikipedia: Poinsot's ellipsoid. It will in many special cases, of course. I don't know of simple proof of the result—but I havn't looked very hard. Since the rotation axis is not fixed in general and the angular momentum vector has a fixed direction, clearly the axis of rotation is not aligned with the angular momentum vector.

Redaction: Jeffery, 2008jan01

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012 qfull 00350 1 5 0 easy thinking: moment arm

20. The general rule for torque is

$$\vec{\tau} = \vec{r} \times \vec{F} \ .$$

However, there is a simple rule that provides an alternative way of calculating the magnitude of the torque. This rule is particularly useful when calculating the torque for forces illustrated on 2-dimensional diagrams. The magnitude of the torque recall is

$$\tau = rF \sin\theta \ ,$$

where θ is the angle between the radius and force vectors. Draw a general radius $\vec{r}$ from an origin to a point where a force $\vec{F}$ is applied. Then find an obviously relevant displacement on the figure that has magnitude $r\sin\theta$. This displacement is called the moment arm (HRW-230). Finding the magnitude of the moment arm on a diagram often leads to simple calculation for the torque.

SUGGESTED ANSWER:

You will have to imagine the diagram. Draw a line through the force extend in both directions to infinity. The displacement from the origin to the nearest point on this line hits it at a right angle and has length $r\sin\theta$. The length has the same formula whether θ is less than, equal to, or greater than 90°. On a diagram for a torque calculation, one just locates the moment arm and determines its length by some obvious geometrical means and then finds the magnitude of the torque by multiplying the magnitudes of its moment arm and force. The direction of the torque is usually geometrically obvious.

Redaction: Jeffery, 2001jan01

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012 qfull 00610 2 3 0 moderate math: LLNL door and torque
Extra keywords: (HR-243:53p)

21. The shield door of the neutron test facility at Lawrence Livermore National Laboratory (where they design the bombs) is (or was), according to HRW-243, the world's most massive hinged door: mass $44,000\,\mathrm{kg}$, rotational inertia about its hinge axis $8.7\times10^4\,\mathrm{kg\,m^2}$, door width 2.4 meters. Assuming zero friction, what constant force perpendicular to the door at its outer edge is needed to move the door 90° in $10\,\mathrm{s}$ starting from **REST**? Show your calculations. Could you open this door in the time given? Would you want to open this door?

SUGGESTED ANSWER:

The required angular acceleration is

$$\alpha = \frac{2\Delta\theta}{t^2} \approx 0.0314\,\mathrm{radians/s^2} \ .$$

The required force is

$$F = \frac{\tau_{\rm net}}{r} = \frac{I\alpha}{r} \approx 1\times10^3\,\mathrm{N} \approx 200\,\mathrm{lb}$$

to about 1-digit accuracy. A very strong person with something solid to push on could do this. But I wouldn't bet on me, not for $10\,\mathrm{s}$ straight. Of course, if you were willing to take more time it could be done easily. If you pushed for $100\,\mathrm{s}$, it would only take $\sim 10\,\mathrm{N} \approx 2\,\mathrm{lb}$ of force. But I wouldn't want to open this door when the neutron flux was inside.

Redaction: Jeffery, 2001jan01

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012 qfull 00640 3 3 0 tough math: Pippa on merry-go-round

22. Wee Pippa Passing runs up to a playground merry-go-round, initially at rest, and jumps radially onto the rim.

    a) What is the torque she exerts about the rotational axis of the merry-go-round? Does the merry-go-round start to rotate? Why or why not? Does the merry-go-round move at all? Why or why not?

    b) Pippa and the merry-go-round both can have angular momentum about the merry-go-round axis. She and the merry-go-round are coupled together by the static frictional force between her feet and the surface. But Pippa can directly control the relative velocity between herself and the merry-go-round by walking or running: thus she can change the coupling condition. When she is at rest on the merry-go-round, she and the merry-go-round constitute one rigid rotator. But when she moves they constitute two rigid rotators about the merry-go-round axis.

Say Pippa starts running just on the rim of the merry-go-round just after jumping on. The merry-go-round axis is frictionless: thus the total angular momentum of the system about the axis cannot change. Using conservation of angular momentum for an isolated system find an expression for the merry-go-round angular frequency $\omega_{\rm m}$ in terms of Pippa's relative angular frequency $\omega_{\rm p\ rel}$ and the rotational inertias about the axis of Pippa $I_{\rm p}$ and the merry-go-round $I_{\rm m}$. Note that $\omega_{\rm p\ rel} = \omega_{\rm p} - \omega_{\rm m}$. Show your derivation.

c) Give the expression for Pippa's final angular velocity relative to the ground using the part (b) results. What would Pippa's final angular velocity be in the limits that $I_{\rm m} \to \infty$ and $I_{\rm m} \to 0$. Show your derivation.

d) Pippa runs on the rim at 3.0 m/s relative to the rim. The radius of the merry-go-round is 3 m. The tangential rim velocity of the merry-go-round is −2.0 m/s when Pippa is running. Pippa has a mass of 40 kg. Assuming the merry-go-round is a uniform disk, what is its mass. Show your calculation.

e) Is it at all possible with Pippa and merry-go-round starting from rest relative to the ground that both Pippa and the merry-go-round could be made to spin in the same direction relative to the ground without external torques about the merry-go-round axis? Why or why not?

SUGGESTED ANSWER:

a) Since she jumps on radially, she exerts zero torque about the axis. Therefore, merry-go-round doesn't move. Her linear momentum becomes the linear momentum of the Pippa-merry-go-round-Earth system since the merry-go-round is rigidly attached to the Earth Thus, effectively the linear momentum disappears into the great Earth momentum sink and the merry-go-round doesn't noticeably move. Actually, there might be some flexing that is not apparent to the eye.

b) For the system the total angular momentum L is given by

$$\begin{aligned} L &= I_{\rm m}\omega_{\rm m} + I_{\rm p}\omega_{\rm p} \\ &= I_{\rm m}\omega_{\rm m} + I_{\rm p}(\omega_{\rm p\ rel} + \omega_{\rm m}) \\ &= (I_{\rm m} + I_{\rm p})\omega_{\rm m} + I_{\rm p}\omega_{\rm p\ rel} \ . \end{aligned}$$

Sans external torques, L is conserved. Thus we can write the equation for both before and after Pippa starts running and solve for the final $\omega_{\rm m}$. Doing so gives

$$\begin{aligned} \omega_{\rm m} &= \frac{L - I_{\rm p}\omega_{\rm p\ rel}}{I_{\rm m} + I_{\rm p}} \\ &= \frac{(I_{\rm m} + I_{\rm p})\omega_{\rm m,0} + I_{\rm p}\omega_{\rm p\ rel,0} - I_{\rm p}\omega_{\rm p\ rel}}{I_{\rm m} + I_{\rm p}} \\ &= \omega_{\rm m,0} + \left(\frac{I_{\rm p}}{I_{\rm m} + I_{\rm p}}\right)(\omega_{\rm p\ rel,0} - \omega_{\rm p\ rel}) \ , \end{aligned}$$

where we denote the inital values with subscript 0. Note the initial values in equation are for after Pippa has jumped on merry-go-round, but before she starts running. In this case, of course, $\omega_{\rm m,0} = \omega_{\rm p\ rel,0} = 0$: thus

$$\omega_{\rm m} = -\frac{I_{\rm p}}{I_{\rm m} + I_{\rm p}}\omega_{\rm p\ rel} \ .$$

c) Pippa's final angular velocity relative to the ground is

$$\omega_{\rm p} = \omega_{\rm p\ rel} + \omega_{\rm m} = \omega_{\rm p\ rel}\left(1 - \frac{I_{\rm p}}{I_{\rm m} + I_{\rm p}}\right) \ .$$

If $I_{\rm m} \to \infty$, $\omega_{\rm p} \to \omega_{\rm p\ rel}$. Thus, if the merry-go-round has infinite rotational inertia, it will not move relative to the ground. If $I_{\rm m} \to 0$, then $\omega_{\rm p} \to 0$. Thus, if the merry-go-round rotational inertia is vanishingly small, Pippa cannot move relative to the ground by running around the rim. Weird isn't it.

d) Multiplying the part (b) answer by r gives

$$v_{\rm m} = -\frac{I_{\rm p}}{I_{\rm m} + I_{\rm p}} v_{\rm p\ rel}$$

for the merry-go-round rim velocity. Rearranging gives

$$I_{\rm m} = -I_{\rm p}\left(1 + \frac{v_{\rm p\ rel}}{v_{\rm m}}\right) .$$

Now the rotational inertias about the merry-go-round axis are $I_{\rm m} = (1/2)m_{\rm m}r^2$ and $I_{\rm p} = m_{\rm p}r^2$ where the radii are the same since Pippa is running on the rim. Thus one finds

$$m_{\rm m} = -2m_{\rm p}\left(1 + \frac{v_{\rm p\ rel}}{v_{\rm m}}\right) = 40\,{\rm kg} .$$

It is a very light merry-go-round.

e) No. The system has zero total angular momentum at the start and sans external torques must continue to have zero total angular momentum. If both Pippa and the merry-go-round spun in the same sense, the total z angular momentum would not be zero.

Redaction: Jeffery, 2001jan01

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012 qfull 00840 2 5 0 moderate thinking: hollow sphere rotational inertia

23. The rotational inertia of a solid uniform sphere about an axis through its center of mass is

$$I = \frac{2}{5}mr^2 ,$$

where m is its mass and r is its radius.

a) What is the mass of sphere of uniform density ρ and radius r?

b) What is the rotational inertia of a uniform-density spherical shell of mass m and inner radius r_1 and outer radius r_2? Write the answer in terms of m, r_1, and r_2

c) The binomial theorem is

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^n .$$

Expand $(x + \Delta x)^n$ using the binomial theorem for $n = 3$ and $n = 5$ and truncate the series to 1st order in Δx. The truncated expansions are accurate to 1st order in small Δx.

d) Making use of the parts (b) and (c) results what is the rotational inertia of a differentially thin hollow shell of mass m and radius r? **HINT:** Let $r_1 = r$ and $r_2 = r + \Delta r$ in the part (b) result.

SUGGESTED ANSWER:

a) Behold

$$m = \frac{4\pi}{3} r^3 \rho .$$

b) If the hollow in the shell had been filled it would have contributed

$$\frac{4\pi}{3} r^3 \rho \left(\frac{2}{5}\right) r_1^2$$

to a total rotational inertia of

$$\frac{4\pi}{3} r^3 \rho \left(\frac{2}{5}\right) r_2^2 .$$

Thus the rotational inertial of the shell is

$$I = \frac{4\pi}{3} \rho \left(\frac{2}{5}\right) (r_1^5 - r^5) .$$

The mass of the shell from the part (a) result is

$$m = \frac{4\pi}{3}\rho(r_2^3 - r_1^3) \ .$$

Eliminating ρ using the part (a) result, we get

$$I = \frac{2}{5}m\left(\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right)$$

which can't be simplified further I think.

c) The binomial theorem is

$$(a+b)^n = \begin{cases} \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^n & \text{in general;} \\ 1 & \text{for } n = 0; \\ a+b & \text{for } n = 1; \\ a^2 + 2ab + b^2 & \text{for } n = 2; \\ a^3 + 3a^2b + 3ab^2 + b^3 & \text{for } n = 3; \\ a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 & \text{for } n = 4; \\ a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 & \text{for } n = 5. \end{cases}$$

Now we find

$$(x + \Delta x)^3 = x^3 + 3x^2\Delta x \ , \qquad (x + \Delta x)^5 = x^5 + 5x^5\Delta x \ .$$

d) Letting $r_1 = r$ and $r_2 = r + \Delta r$, we find to 1st order in small Δr that

$$I = \frac{2}{5}m\left(\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right) = \frac{2}{5}m\left(\frac{5r^4\Delta r}{3r^3\Delta r}\right) = \frac{2}{3}mr^2 \ .$$

If the shell becomes differentially thin, the higher order terms vanish and one obtains

$$I = \frac{2}{3}mr^2$$

as the exact rotational inertia of a differentially thin shell.

Redaction: Jeffery, 2001jan01

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012 qfull 00940 2 3 0 moderate math: perpendicular axis theorem, etc.

Extra keywords:

24. Do the following problems.

a) The perpendicular axis theorem applies exactly to infinitely thin planar objects and approximately to just thin planar objects. You have a planar object with area density σ. You choose an origin and define a set of coordinates with the x and y axes in the object plane and the z axis perpendicular to the object plane. Show that

$$I_z = I_x + I_y \ ,$$

where I_x, I_y, and I_z are the rotational inertias about, respectively, the x, y, and z axes.

b) The rotational inertia of an infinitely thin, uniform rod about axis through its center and perpendicular to the rod is

$$\frac{1}{12}Ma^2 \ ,$$

where M is the rod mass and a is the rod length. Now consider an infinitely thin, uniform rectangular plate. The plate has mass M. Let the z axis pass through the center of the plate and be perpendicular to the plate. The x and y axes are parallel to the sides of the plate. The length of the plate in the y direction is a. Derive the rotational inertia about the x axis? **HINT:** A diagram might help.

c) The length of the length of the plate from part (b) in the x direction is length b. Derive the rotational inertia about the y axis?

d) What is the rotational inertia of the plate about the z axis?

e) Now let's extend the plate in the z direction to create a rectangular parallelepiped. A rectangular box has 6 sides all of which are rectangles and the sides all meet at right angles. The z length of our box is c and the mass is M: the x and y lengths are the same as for the plate. Derive the rotational inertia about the y axis?

SUGGESTED ANSWER:

a) Behold:

$$I_z = \int r^2 \sigma\, dA = \int (x^2 + y^2)\sigma\, dA = \int x^2 \sigma\, dA + \int y^2 \sigma\, dA = I_x + I_y \ ,$$

where $r = \sqrt{x^2 + y^2}$ is the cylindrical coordinate for the z axis and x and y are the cylindrical coordinates for the x and y axes for the special case of our planar object.

b) The described plate is just a stack of infinitely thin rods of the kind described in the preamble. Thus the rotational inertia of the x axis of the plate is just

$$I_x = \frac{1}{12} M a^2 \ .$$

c) The y axis rotational inertia is just like the x axis rotational inertia with b replacing a. So the y axis rotational inertia is

$$I_y = \frac{1}{12} M b^2 \ .$$

d) From the perpendicular axis theorem, it is

$$I_z = I_x + I_y = \frac{1}{12} M (a^2 + b^2) \ .$$

e) The box is just a stack of plates, and thus

$$I_z = \frac{1}{12} M (a^2 + b^2) \ .$$

Redaction: Jeffery, 2008jan01

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012 qfull 01050 1 3 0 easy math: rotational energetics, car crankshaft

25. A car crankshaft rotating at 1500 rpm transfers energy from the car engine to the wheel axle at 74.6 kW (or 100 hp). What's the average torque of the crankshaft? What's the average **TANGENTIAL FORCE** of the crankshaft on the drum (or whatever one calls that thing on the axle that the crankshaft turns) assuming it has radius 0.10 m? Show your calculations.

SUGGESTED ANSWER:

First, note that

$$\omega = 1500\,\text{rpm} = 1500\,\text{rpm} \times \left(\frac{2\pi}{1\,\text{rev}}\right) \times \left(\frac{1\,\text{min}}{60\,\text{s}}\right) \approx 150\,\text{rad/s} \ .$$

The power transferred is given by

$$P = \tau_{\text{avg}} \omega \ ,$$

and so

$$\tau_{\text{avg}} = \frac{P}{\omega} \approx 500\,\text{N}\,\text{m}$$

to 1-digit accuracy. The average tangential force is given by

$$F_{\text{avg tan}} = \frac{\tau}{r} \approx 5000\,\text{N} \approx 1000\,\text{lb}$$

which seems like an awful lot, but how would I know.

Redaction: Jeffery, 2001jan01

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012 qfull 01220 2 3 0 moderate math: rolling ball on incline into gooey stuff

26. A uniform solid ball of mass 0.5 kg starting from rest rolls down an incline of angle $\theta = 30°$. There is no slipping between the ball and incline: i.e., the no-slip condition is imposed.

a) Calculate the ball's center-of-mass acceleration given the downhill roller acceleration formula

$$a = \frac{g\sin\theta}{1 + I/(mr^2)} \ .$$

Show your calculation.

b) What is its center-of-mass velocity after 30 s and how far has it gone down the slope? Show your calculation.

c) At the 30 s mark the ball completely inelastically collides with a blob of gooey stuff of mass 5.0 kg resting and stuck on a piece of wax paper of negligible mass. The ball-blob-paper then slides 20 m further down the incline and comes to a stop. What is the kinetic friction coefficient μ_{k} of the paper with the incline? Show your calculation.

Why does the ball-blob-paper stay at rest after it stops? To answer that question, ask yourself this question have the forces on it changed?

d) How far would the ball-blob-paper have gone if $\mu_{\text{k}} = 0.1$? Why does it go this far?

SUGGESTED ANSWER:

a) We are given the downhill roller acceleration formula

$$a = \frac{g\sin\theta}{1 + I/(mr^2)}$$

which is derived somewhere else. Applying this result, we find:

$$a = \frac{g\sin\theta}{1 + I/(mr^2)} \approx \frac{10 \times 1/2}{1 + 2/5} \approx 3.5\,\text{m/s}^2 \ ,$$

to about 2-digit accuracy.

b) From the appropriate constant-acceleration kinematic equations

$$v = at \approx 105\,\text{m/s}$$

and

$$x = \frac{1}{2}at^2 \approx 1600\,\text{m}$$

to about 2-digit accuracy.

c) Since the collision is completely inelastic, the initial velocity of the post-collision ball-blob-paper from conservation of linear momentum is

$$v_i = \frac{m_{\text{ball}}v}{m_{\text{ball}} + m_{\text{blob}}} \approx 10\,\text{m/s} \ .$$

We can use either $\vec{F}_{\text{net}} = m\vec{a}$ or energetics to solve for kinetic friction coefficient: they are virtually the same thing. Let's use energetics. The net force in the downward direction is

$$F = m_{\text{ball-blob}}g\sin\theta - \mu_{\text{k}}m_{\text{ball-blob}}g\cos\theta$$

and, using the work-kinetic-energy theorem

$$\Delta E = \Delta KE + \Delta PE = W_{\text{non}} \ ,$$

we find

$$= -\frac{1}{2} m_{\text{ball-blob}} v_i^2 = F\Delta x \ ,$$

Rearranging we find

$$\mu_k = \tan\theta + \frac{v_i^2}{2g\Delta x \cos\theta} \approx 0.58 + 0.3 \approx 0.9$$

to about 1-digit accuracy. When the ball-blob-paper comes to a stop, it is held in place by static friction.

As the ball-blob-paper comes to a stop, kinetic friction changes into static friction. The actual change is gradual, but in our crude approximation it happens at the instant when velocity goes to zero. The static frictional force is weaker than the kinetic frictional force. It just balances gravity. Thus, it won't accelerate the ball-blob-paper up the slope which the kinetic friction force would have done if it had magically stayed on.

Of course, static friction can become greater than kinetic friction since the coefficient of static friction is larger than the coefficient of kinetic friction. Static friction increases with the applied force on the object up to the point where the applied force exceeds the upper limit of static firction $\mu_{\text{st}} F_{\text{N}}$, where μ_{st} is the coefficient of static friction and F_{N} on the object.

But you ask, what if the upper limit on static friction is exceeded? Well this is unlikely since commonly (though probably not universally) $\mu_{\text{st}} > \mu_{\text{k}}$. If this were true in our case and kinetic friction could stop the object being pulled by gravity, then the upper limit will not be exceeded. But what if we have the strange situation that $\mu_{\text{st}} < \mu_{\text{k}}$. Well the static friction upper limit may still be big enough to keep the object at rest. But if that is not true, the object is going to arrive in final state where the friction force is somewhere between the ideal kinetic and static friction limits. In this state, the object may slide down at a constant velocity. The friction force just matching the gravitational force. If the object is perturbed to slow down, friction decreases and the object speeds up. If the object is perturbed to speed up, friction increases and slows it down. But I don't really know.

d) The downward force is

$$F = m_{\text{ball-blob}} g \sin\theta - \mu_{\text{k}} m_{\text{ball-blob}} g \cos\theta \approx 22\,\text{N} > 0\,\text{N}$$

to about 2-digit accuracy. With a net force downhill, the acceleration downhill is positive and the object moves downhill with increasing velocity forever. So it never comes to a stop and its stopping distance is therefore infinity.

Redaction: Jeffery, 2001jan01

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012 qfull 01310 3 5 0 tough thinking: double incline, rotational inertia

27. You have a triangular block which gives you two inclines (i.e., a double incline). Incline 1 is at θ_1 to the horizontal and incline 2 at θ_2. The kinetic friction coefficients are $\mu_{\text{k}\,1}$ and $\mu_{\text{k}\,2}$ for, respectively, inclines 1 and 2. You have a pulley at the apex and a ideal, massless rope connecting two blocks: block 1 (with mass m_1) on incline 1 and block 2 (with m_2 on incline 2. The rope is taut and is parallel to each incline. The rope over the pulley wheel does not slip and the pulley wheel has rotational inertia I about its axis and radius r. The axle of the pulley is frictionless. Block 1 slides up the slope and Block 2 slides down the slope. Call both these directions positive. You know all the variables except the acceleration of the blocks along the slope and the tension forces in the two regions of the rope.

a) Draw the free body diagrams for the blocks and write down their equations of motion (i.e., Newton's 2nd law applied to their special cases). Why can't you solve for acceleration now?

b) What is the rotational equation of motion for the pulley?

c) Now solve for the general acceleration formula in terms of the knowns: i.e., sans tension forces.

d) Specialize the general formula for the case where $\mu_{\text{k}\,1} = \mu_{\text{k}\,2} = 0$.

e) Specialize the general formula for the case where $\theta_1 = 0°$.

f) Specialize the general formula for the case where $\theta_1 = 0^\circ$ and $\theta_2 = 90^\circ$.

g) Specialize the general formula for the case where $\theta_1 = \theta_2 = 90^\circ$.

h) Specialize the general formula for the case where $I = 0$.

i) Why is the general formula we derived wrong for the case when the blocks slide in the opposite directions? What simple demarche corrects the expression?

SUGGESTED ANSWER:

a) You will have to imagine the diagrams. Taking up slope 1 and down slope 2 as both positive, the equations of motion are:

$$m_1 a = F_{\rm T1} - m_1 g \sin\theta_1 - \mu_{\rm k\,1} m_1 g \cos\theta_1$$

and

$$m_2 a = m_2 g \sin\theta_2 - F_{\rm T2} - \mu_{\rm k\,2} m_2 g \cos\theta_2 \ .$$

The acceleration is the same for both rope regions because the rope is taut. Acceleration can't be solved for since we have only two equations and three unknowns: a, $F_{\rm T1}$, and $F_{\rm T2}$.

b) Because there is no slip, the rope exerts a torque on the pulley. The no-slip frictional force (a static friction force) provides the torque and it is entirely tangential to the pulley. The no-slip frictional force is in fact exactly the difference in tension forces between the two rope regions. The best way to see this is consider just the two points where the rope first touches the pulley on either side. Consider the rest of the rope that wraps around the pulley as part of the internal system of the pulley: a massless part, and so it doesn't add to the rotational inertia. Now the ropes clearly provide two opposing external torques about the pulley axis with r being radius vector for both torques and the angle for both torques is 90°.

From the rotational version of the second law specialized to rigid body rotation about a single axis, we obtain

$$I\alpha = I\frac{a}{r} = rF_{\rm T2} - rF_{\rm T1} \ ,$$

where the no-slip condition forces $\alpha = a/r$. Next we obtain

$$I\frac{a}{r^2} = F_{\rm T2} - F_{\rm T1} \ .$$

There are a couple of points to ponder. First, the no-slip condition is the opposite limiting case from the case of a pulley with a frictionless surface. Both limiting cases are comparatively easy to treat. It's the intermediate cases that are tough to treat. Fortunately, most pulley systems are designed to be nearly exactly no-slip.

Second, friction is not a conservative force, but the rope-pulley friction in this case does not dissipate energy even though it does "work." At the macroscopic level the trick is to see that it does no net work. The rope-on-pulley friction is a positively directed force on the pulley and the pulley-on-rope friction is opposite in sign but equal in magnitude. Because of no-slip, the pulley and rope are moved through the same distances by the friction forces. The net work done must be zero. At the microscopic level, no-slip ideally means that no microscopic motion between surfaces occurs, and so no energy is dissipated to heat. Actually, of course, there is always some dissipation due to microscopic slipping and probably other effects.

c) We simply add the equations from the part (a) answer and substitute from the part (b) answer to eliminate the tension forces:

$$\left(m_1 + m_2 + \frac{I}{r^2}\right) a = g\left[m_2(\sin\theta_2 - \mu_{\rm k\,2}\cos\theta_2) - m_1(\sin\theta_1 + \mu_{\rm k\,1}\cos\theta_1)\right] \ .$$

Then the general expression and all the requested special cases from parts (d)–(h) are given:

$$a = g \times \begin{cases} \dfrac{m_2(\sin\theta_2 - \mu_{\mathrm{k}\,2}\cos\theta_2) - m_1(\sin\theta_1 + \mu_{\mathrm{k}\,1}\cos\theta_1)}{m_1 + m_2 + I/r^2} & \text{general;} \\ \dfrac{m_2\sin\theta_2 - m_1\sin\theta_1}{m_1 + m_2 + I/r^2} & \mu_{\mathrm{k}\,1} = \mu_{\mathrm{k}\,2} = 0; \\ \dfrac{m_2(\sin\theta_2 - \mu_{\mathrm{k}\,2}\cos\theta_2) - m_1\mu_{\mathrm{k}\,1}}{m_1 + m_2 + I/r^2} & \theta_1 = 0^\circ; \\ \dfrac{m_2 - m_1\mu_{\mathrm{k}\,1}}{m_1 + m_2 + I/r^2} & \theta_1 = 0^\circ \text{ and } \theta_2 = 90^\circ; \\ \dfrac{m_2 - m_1}{m_1 + m_2 + I/r^2} & \theta_1 = \theta_2 = 90^\circ; \\ \dfrac{m_2(\sin\theta_2 - \mu_{\mathrm{k}\,2}\cos\theta_2) - m_1(\sin\theta_1 + \mu_{\mathrm{k}\,1}\cos\theta_1)}{m_1 + m_2} & I = 0. \end{cases}$$

i) All the forces have the same direction whether the velocity is positive or negative, except for the friction forces. Thus, all the analysis we've done is unchanged by a switch to a negative velocity case, except for the friction forces. The difficulty is simply fixed by changing the sign in front of the kinetic coefficients in the general formula. With this fix, the general formula becomes

$$a = g\left[\frac{m_2(\sin\theta_2 + \mu_{\mathrm{k}\,2}\cos\theta_2) - m_1(\sin\theta_1 - \mu_{\mathrm{k}\,1}\cos\theta_1)}{m_1 + m_2 + I/r^2}\right]$$

when $a < 0$.

Of course, besides the positive and negative velocity cases, there is the static case. The static case is different because there we a different formula for friction: i.e., the static friction formula

$$F_{\text{static friction}} = \min(F_{\text{applied}}, \mu_{\text{st}} F_{\text{normal}}) \ .$$

As long as the static force is not at its upper limit, we really don't have an intrinsic formula for friction: we must rely on knowing the applied force.

Let investigate our system in the static case. The equations of motion for the two objects become

$$0 = F_{\mathrm{T}1} - m_1 g\sin\theta_1 + F_{\text{static friction 1}}$$

and

$$0 = m_2 g\sin\theta_2 - F_{\mathrm{T}2} + F_{\text{static friction 2}} \ ,$$

where we have taken the fiducial direction of the frictional forces to be positive for both slopes. The equation of motion for the pulley in this case gives $0 = F_{\mathrm{T}2} - F_{\mathrm{T}1}$, since the pulley is static. Thus we can define $F_{\mathrm{T}} = F_{\mathrm{T}1} = F_{\mathrm{T}2}$. We now have

$$0 = F_{\mathrm{T}} - m_1 g\sin\theta_1 + F_{\text{static friction 1}}$$

and

$$0 = m_2 g\sin\theta_2 - F_{\mathrm{T}} + F_{\text{static friction 2}} \ .$$

Thus we have two equations and three unknowns: F_{T}, $F_{\text{static friction 1}}$, and $F_{\text{static friction 2}}$. What can we do? Well nothing really. The system is, I believe, indeterminate by the equilibrium rules and force laws we have available (HRW-282). If we studied the elasticity properties of materials and knew how much the inclines and rope deformed, we could calculate the unknowns. But, dear reader, I leave that job for you.

Redaction: Jeffery, 2001jan01

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012 qfull 01320 2 3 0 moderate math: Atwood's machine, rotational inertia

Extra keywords: On tests this would be hard question

28. You have an Atwood's machine: a pulley with two dangling blocks. Block 1 has $m_1 = 0.500\,\mathrm{kg}$ and block 2 has $m_2 = 0.520\,\mathrm{kg}$. The pulley has mass $m_{\mathrm{pu}} = 2.0\,\mathrm{kg}$ and effective radius $r = 0.070\,\mathrm{m}$. The rope is massless.

a) Assume the pulley wheel magically has zero rotational inertia and a frictionless axle. What is the acceleration of the block system? Starting from rest how long does it take for the blocks to move 1 m?

b) Now assume the pulley does have rotational inertia. Estimate the rotational inertia of the pulley assuming it is a uniform disk. Answer the questions from part (a) in this case.

SUGGESTED ANSWER:

a) We can use a special case of the general acceleration expression for these pulley systems that we found in an earlier problem—if indeed we did find it. Using the special case where the pulley wheel has zero rotational inertia,

$$a = g\left(\frac{m_2 - m_1}{m_1 + m_2}\right) \approx 0.20\,\mathrm{m/s^2}$$

to about 2-digit accuracy. From the appropriate kinematic equation the time to fall $\Delta y = 1\,\mathrm{m}$ is

$$t = \sqrt{\frac{2\Delta y}{a}} \approx 3\,\mathrm{s}$$

to about 1-digit accuracy.

b) Assuming the pulley is a uniform disk

$$I = \frac{1}{2}m_{\mathrm{pu}}r^2 \approx 4.9\times10^{-3}\,\mathrm{kg\,m^2}\ .$$

Again can use a special case of the general acceleration expression for these pulley systems that we found in an earlier problem:

$$a = g\left(\frac{m_2 - m_1}{m_1 + m_2 + I/r^2}\right) = g\left[\frac{m_2 - m_1}{m_1 + m_2 + (1/2)m_{\mathrm{pu}}}\right] \approx 0.10\,\mathrm{m/s^2}$$

to about 2-digit accuracy. The fall time is now

$$t = \sqrt{\frac{2\Delta y}{a}} \approx 4.5\,\mathrm{s}$$

to about 2-digit accuracy.

Redaction: Jeffery, 2001jan01

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

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1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67428(67) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2006, CODATA)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\mathrm{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23),\mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

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2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\mathrm{sphere}} = 4\pi \qquad d\Omega = \sin\theta\, d\theta\, d\phi$$

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3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a - b) - \cos(a + b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a - b) + \sin(a + b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

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4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1 - x} \approx 1 + x \ : \ (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

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5 Quadratic Formula

If $\quad 0 = ax^2 + bx + c\ ,\quad$ then $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\dfrac{b}{2a} \pm \sqrt{\left(\dfrac{b}{2a}\right)^2 - \dfrac{c}{a}}$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) \\ &= f(x_0) + (x - x_0) f^{(1)}(x_0) + \frac{(x-x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x-x_0)^3}{3!} f^{(3)}(x_0) + \ldots \end{aligned}$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0 \qquad v^2 = v_0^2 + 2a(x - x_0)$$

$$x = \frac{1}{2}(v_0 + v)t + x_0 \qquad x = -\frac{1}{2}at^2 + vt + x_0 \qquad g = 9.8\,\mathrm{m/s^2}$$

$$x_{\text{rel}} = x_2 - x_1 \qquad v_{\text{rel}} = v_2 - v_1 \qquad a_{\text{rel}} = a_2 - a_1$$

$$x' = x - v_{\text{frame}} t \qquad v' = v - v_{\text{frame}} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\text{avg}} = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0} t \qquad y = -\frac{1}{2} g t^2 + v_{y,0} t + y_0 \qquad v_{x,0} = v_0 \cos\theta \qquad v_{y,0} = v_0 \sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0 \cos\theta} \qquad y = y_0 + x \tan\theta - \frac{x^2 g}{2 v_0^2 \cos^2\theta}$$

$$x_{\text{for } y \text{ max}} = \frac{v_0^2 \sin\theta \cos\theta}{g} \qquad y_{\max} = y_0 + \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2 v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g} \qquad \theta_{\text{for max}} = \frac{\pi}{4} \qquad x_{\max}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0 \sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2 r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\tan} = r\alpha$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = v\omega$$

13 Very Basic Newtonian Physics

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i \vec{r}_i}{m_{\text{total}}} = \frac{\sum_{\text{sub}} m_{\text{sub}} \vec{r}_{\text{cm sub}}}{m_{\text{total}}} \qquad \vec{v}_{\text{cm}} = \frac{\sum_i m_i \vec{v}_i}{m_{\text{total}}} \qquad \vec{a}_{\text{cm}} = \frac{\sum_i m_i \vec{a}_i}{m_{\text{total}}}$$

$$\vec{r}_{\text{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\, dV}{m_{\text{total}}}$$

$$\vec{F}_{\text{net}} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,\text{m/s}^2$$

$$\vec{F}_{\text{normal}} = -\vec{F}_{\text{applied}} \qquad F_{\text{linear}} = -kx$$

$$f_{\text{normal}} = \frac{T}{r} \qquad T = T_0 - F_{\text{parallel}}(s) \qquad T = T_0$$

$$F_{\text{f static}} = \min(F_{\text{applied}}, F_{\text{f static max}}) \qquad F_{\text{f static max}} = \mu_{\text{static}} F_{\text{N}} \qquad F_{\text{f kinetic}} = \mu_{\text{kinetic}} F_{\text{N}}$$

$$v_{\text{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\text{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\text{centripetal}} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\text{drag,lin}} = bv \qquad v_{\text{T}} = \frac{mg}{b} \qquad \tau = \frac{v_{\text{T}}}{g} = \frac{m}{b} \qquad v = v_{\text{T}}(1 - e^{-t/\tau})$$

$$F_{\text{drag,quad}} = bv^2 = \frac{1}{2}C\rho A v^2 \qquad v_{\text{T}} = \sqrt{\frac{mg}{b}}$$

14 Energy and Work

$$dW = \vec{F} \cdot d\vec{s} \qquad W = \int \vec{F} \cdot d\vec{s} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\text{mechanical}} = KE + PE$$

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F} \cdot \vec{v}$$

$$\Delta KE = W_{\text{net}} \quad \Delta PE_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \quad \Delta E = W_{\text{nonconservative}}$$

$$F = -\frac{dPE}{dx} \qquad \vec{F} = -\nabla PE \qquad PE = \frac{1}{2}kx^2 \qquad PE = mgy$$

15 Momentum

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta KE_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\text{cm}} = \vec{F}_{\text{net non-flux}} + (\vec{v}_{\text{flux}} - \vec{v}_{\text{cm}})\frac{dm}{dt} = \vec{F}_{\text{net non-flux}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_0 + v_{\text{ex}} \ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

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16 **Collisions**

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\,dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\text{net}}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$KE_{\text{total } f} = KE_{\text{total } i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\text{rel}'} = -v_{\text{rel}} \qquad \text{1-d Elastic Collision Expressions}$$

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17 **Rotational Kinematics**

$$2\pi = 6.2831853\ldots \qquad \frac{1}{2\pi} = 0.15915494\ldots$$

$$\frac{180^\circ}{\pi} = 57.295779\ldots \approx 60^\circ \qquad \frac{\pi}{180^\circ} = 0.017453292\ldots \approx \frac{1}{60^\circ}$$

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\omega = \alpha t + \omega_0 \qquad \Delta\theta = \frac{1}{2}\alpha t^2 + \omega_0 t \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t \qquad \Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$

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18 **Rotational Dynamics**

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{\tau} = \vec{r} \times \vec{F} \qquad \vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

$$L_z = RP_{xy} \sin\gamma_L \qquad \tau_z = RF_{xy} \sin\gamma_\tau \qquad L_z = I\omega \qquad \tau_{z,\text{net}} = I\alpha$$

$$I = \sum_i m_i R_i^2 \qquad I = \int R^2 \rho \, dV \qquad I_{\text{parallel axis}} = I_{\text{cm}} + mR_{\text{cm}}^2 \qquad I_z = I_x + I_y$$

$$I_{\text{cyl,shell,thin}} = MR^2 \qquad I_{\text{cyl}} = \frac{1}{2}MR^2 \qquad I_{\text{cyl,shell,thick}} = \frac{1}{2}M(R_1^2 + R_2^2)$$

$$I_{\text{rod,thin,cm}} = \frac{1}{12}ML^2 \qquad I_{\text{sph,solid}} = \frac{2}{5}MR^2 \qquad I_{\text{sph,shell,thin}} = \frac{2}{3}MR^2$$

$$a = \frac{g \sin\theta}{1 + I/(mr^2)}$$

$$KE_{\text{rot}} = \frac{1}{2}I\omega^2 \qquad dW = \tau_z \, d\theta \qquad P = \frac{dW}{dt} = \tau_z \omega$$

$$\Delta KE_{\text{rot}} = W_{\text{net}} = \int \tau_{z,\text{net}} \, d\theta \qquad \Delta PE_{\text{rot}} = -W = -\int \tau_{z,\text{con}} \, d\theta$$

$$\Delta E_{\text{rot}} = KE_{\text{rot}} + \Delta PE_{\text{rot}} = W_{\text{non,rot}} \qquad \Delta E = \Delta KE + KE_{\text{rot}} + \Delta PE = W_{\text{non}} + W_{\text{rot}}$$

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19 Static Equilibrium

$$\vec{F}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = 0 \qquad \vec{\tau}_{\text{ext,net}} = \tau'_{\text{ext,net}} \qquad \text{if } F_{\text{ext,net}} = 0$$

$$0 = F_{\text{net }x} = \sum F_x \qquad 0 = F_{\text{net }y} = \sum F_y \qquad 0 = \tau_{\text{net}} = \sum \tau$$