Intro Physics Semester I **Name:**

Homework 9: Momentum: One or two full answer questions will be marked. There will also be a mark for completeness. Homeworks are due usually the day after the chapter they are for is finished. Solutions will be posted soon thereafter. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

Answer Table

Name:

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	31.	O	O	O	O	O
2.	O	O	O	O	O	32.	O	O	O	O	O
3.	O	O	O	O	O	33.	O	O	O	O	O
4.	O	O	O	O	O	34.	O	O	O	O	O
5.	O	O	O	O	O	35.	O	O	O	O	O
6.	O	O	O	O	O	36.	O	O	O	O	O
7.	O	O	O	O	O	37.	O	O	O	O	O
8.	O	O	O	O	O	38.	O	O	O	O	O
9.	O	O	O	O	O	39.	O	O	O	O	O
10.	O	O	O	O	O	40.	O	O	O	O	O
11.	O	O	O	O	O	41.	O	O	O	O	O
12.	O	O	O	O	O	42.	O	O	O	O	O
13.	O	O	O	O	O	43.	O	O	O	O	O
14.	O	O	O	O	O	44.	O	O	O	O	O
15.	O	O	O	O	O	45.	O	O	O	O	O
16.	O	O	O	O	O	46.	O	O	O	O	O
17.	O	O	O	O	O	47.	O	O	O	O	O
18.	O	O	O	O	O	48.	O	O	O	O	O
19.	O	O	O	O	O	49.	O	O	O	O	O
20.	O	O	O	O	O	50.	O	O	O	O	O
21.	O	O	O	O	O	51.	O	O	O	O	O
22.	O	O	O	O	O	52.	O	O	O	O	O
23.	O	O	O	O	O	53.	O	O	O	O	O
24.	O	O	O	O	O	54.	O	O	O	O	O
25.	O	O	O	O	O	55.	O	O	O	O	O
26.	O	O	O	O	O	56.	O	O	O	O	O
27.	O	O	O	O	O	57.	O	O	O	O	O
28.	O	O	O	O	O	58.	O	O	O	O	O
29.	O	O	O	O	O	59.	O	O	O	O	O
30.	O	O	O	O	O	60.	O	O	O	O	O

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005 qmult 00600 1 1 4 easy memory: center of mass definition

1. The center of mass is the:

   a) position-weighted mean mass of an object.
   b) object-weighted mean mass of the position.
   c) mean of mass an weighted object position of.
   d) mass-weighted mean position of an object.
   e) simple center of the object.

   SUGGESTED ANSWER: (d)

   For most objects, one really needs to do a integration of position weighted by density to find the center of mass.

   Wrong answers:
   c) Word jumble.
   e) Simple center has no precise physics definition.

   Redaction: Jeffery, 2001jan01

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005 qmult 00610 1 4 2 easy deducto-memory: center of mass, reference frame

2. The center of mass (i.e., the actual physical position of the center of mass in space relative to the physical system it is the center of mass of) is:

   a) a function of the coordinate system.
   b) independent of the coordinate system.
   c) dependent on the coordinate system.
   d) both independent of and a function of the coordinate system.
   e) neither independent of nor a function of the coordinate system.

   SUGGESTED ANSWER: (b)

   Wrong answers:
   a) Absolutely wrong.
   c) Absolutely wrong and meaning the same thing as answer (a).
   d) Not logically possible.
   e) Not logically possible again.

   Redaction: Jeffery, 2001jan01

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005 qmult 00620 1 4 5 easy deducto-memory: center of mass, 3-d symmetry 1

3. If an object is symmetric in 3 dimensions about some point (i.e., its geometric center), its center of mass must be:

   a) outside of the object.
   b) neither inside nor outside the object.
   c) at the point about which the object is symmetric in 2 of the dimensions, but not in the 3rd.
   d) at the point about which the object is symmetric in 1 of the dimensions, but not in the other 2.
   e) at the geometric center.

   SUGGESTED ANSWER: (e)

   Wrong answers:
   a) Centers of mass can be outside of bodies, but they don't have to be in general nor for objects symmetric in 3 dimensions.

   Redaction: Jeffery, 2001jan01

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005 qmult 00640 1 5 3 easy thinking: center of mass, symmetric sphere

4. The center of mass of sphere of radius R with a density given by $\rho = Cr^2$ (where r is the radial coordinate) is at:

   a) $r = R/2$. b) $r = R/3$. c) $r = 0$. d) $r = 3R$. e) $r = 2R$.

   SUGGESTED ANSWER: (c)

The sphere has perfect symmetrical symmetry, and so the mass-weighted mean position must be at the center.

Wrong answers:

Redaction: Jeffery, 2001jan01

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005 qmult 00654 1 5 5 easy thinking: hoop center of mass

5. Where is the center of mass of a hoop?

   a) At the end of the hoop.
   b) At the top of the hoop.
   c) At the left side of hoop.
   d) Nowhere since a center of mass must be physically inside an object to be a center of mass.
   e) On the axis of the hoop at the geometrical center of the hoop.

   SUGGESTED ANSWER: (e)

   Wrong answers:
   a) The end of a hoop?

   Redaction: Jeffery, 2001jan01

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005 qmult 00660 1 4 5 easy deducto-memory: hanging center of mass

6. "Let's play *Jeopardy*! For \$100, the answer is: If one hangs a rigid object from a freely turning pivot point and lets it come to stable static equilibrium, the center of mass is directly below the pivot point. Thus, center of mass can be found from the intersection of two lines through the object that start at two points used as pivot points and that go in the direction through the object that was downward when each of the points was the pivot point. The method fails if the two pivot points and the center of mass happen to be collinear."

   a) What is an **EMPIRICAL** method for finding gravitational torque, Alex?
   b) What is a **THEORETICAL** method for finding gravitational torque, Alex?
   c) What is gravitational torque, Alex?
   d) What is a center of mass, Alex?
   e) What is an **EMPIRICAL** method for finding the center of mass of a rigid object, Alex?

   SUGGESTED ANSWER: (e)

   Wrong answers:
   d) The answer is not a definition of center of mass, but only a way of determining it.

   Redaction: Jeffery, 2001jan01

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009 qmult 00100 1 1 5 easy memory: definition of vector momentum

7. Momentum (or linear momentum) is defined by the formula ____________, where m is system mass and $\vec{v}$ is system center-of-mass velocity.

   a) $\vec{p} = \dfrac{m}{\vec{v}}$ b) $\vec{p} = \dfrac{\vec{v}}{m}$ c) $\vec{p} = \dfrac{1}{2}mv^2$ d) $\vec{p} = \dfrac{1}{2}m\vec{v}$ e) $\vec{p} = m\vec{v}$

   SUGGESTED ANSWER: (e)

   Wrong answers:
   a) Division by a vector is not defined.
   c) The right-hand side is kinetic energy.

   Redaction: Jeffery, 2001jan01

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009 qmult 00110 1 1 5 easy memory: definition of 1-d momentum
Extra keywords: physci

8. The momentum (or linear momentum) for a 1-dimensional system is given by the formula ____________, where m is system mass and v is system center-of-mass velocity for the single dimension.

   a) $p = \dfrac{m}{v}$ b) $p = \dfrac{v}{m}$ c) $p = \dfrac{1}{2}mv^2$ d) $p = \dfrac{1}{2}mv$ e) $p = mv$

SUGGESTED ANSWER: (e)

No vector signs are needed if one is restricting the definition to one dimension. The signs of the 1-dimensional components give the direction.

Wrong answers:
c) The right-hand side is kinetic energy.

Redaction: Jeffery, 2001jan01

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009 qmult 00150 1 1 3 easy memory: momentum is not energy
Extra keywords: physci KB-93-15
9. Linear momentum is **NOT**:

a) a physical quantity. b) dependent on velocity. c) a kind of energy.
d) dependent on mass. e) given by $p = mv$ for 1-dimensional cases.

SUGGESTED ANSWER: (c)

Momentum is closely related to kinetic energy. The two both calculated from mass and velocity, but momentum is not energy. For one thing, momentum is a vector and energy is a scalar.

Wrong answers:
e) But it is so given.

Redaction: Jeffery, 2001jan01

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009 qmult 00160 2 5 1 moderate thinking: KE change and momentum change
Extra keywords: physci KB-94-13
10. If the kinetic energy of an object is doubled, the momentum magnitude changes by a factor of:

a) $\sqrt{2}$. b) 2. c) $1/2$. d) $1/\sqrt{2}$. e) 1.

SUGGESTED ANSWER: (a)

Recall $\vec{p} = m\vec{v}$, and thus $\vec{v} = \vec{p}/m$. Thus, $KE = mv^2/2 = p^2/(2m)$, and thus $p = \sqrt{2mKE}$. Thus, momentum magnitude increases as the square root of KE. Thus, if KE increases by 2, momentum magnitude increases by $\sqrt{2}$.

Wrong answers:
b) Not a good guess, but better than some others anyway.

Redaction: Jeffery, 2001jan01

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009 qmult 00170 1 1 5 easy memory: general form of Newton's 2nd law
11. The general form of ____________ is

$$\frac{d\vec{p}}{dt} = \vec{F}_{\text{ext}} ,$$

where $\vec{p}$ is the total momentum of a system and $\vec{F}_{\text{ext}}$ is the net external force on the system. The net external force includes ordinary forces (field forces and contact forces), and what can be called momentum flux forces.

A momentum flux force is just the momentum added to a system by adding mass with its own momentum to the system. If all the added mass had the same velocity $\vec{v}_{\text{flux}}$, then the momentum flux force would be

$$\vec{v}_{\text{flux}}\frac{dm}{dt} .$$

Note dm/dt can be positive or negative: if negative, mass is actually be lost from the system. The added mass can interact with the rest of mass of the system after becoming part of it by ordinary forces or not. Using our special case momentum flux force and the chain rule, we can specialize ____________ to the formula

$$m\vec{a} = \vec{F}_{\text{ext,ordinary}} + (\vec{v}_{\text{flux}} - \vec{v})\frac{dm}{dt} ,$$

were $\vec{a}$ is the center-of-mass acceleration and $\vec{v}$ is the center-of-mass velocity.

If $\vec{F}_{\rm ext} = 0$, then $\vec{p}$ is a constant: i.e., momentum is conserved. This is really the general form of Newton's 1st law: in the absence of a net external force, momentum is conserved. If $dm/dt = 0$ too (and it is hard to arrange $\vec{F}_{\rm ext} = 0$ without having $dm/dt = 0$), then $\vec{v}$ is a constant: this is Newton's 1st law as usually stated. The 1st law is not really a law (i.e., an axiom) of classical mechanics: it is a result that is called a law because Newton called it that—maybe even for him it was a traditional law.

Since our general form of ____________ is an axiom of classical mechanics, it must apply to the parts of a system treated as systems in their own right. Imagine a system broken into two parts 1 and 2. We must have

$$\frac{d\vec{p}_1}{dt} = \vec{F}'_{\rm ext,1} + \vec{F}_{21} \qquad \text{and} \qquad \frac{d\vec{p}_2}{dt} = \vec{F}'_{\rm ext,2} + \vec{F}_{12}$$

where 1 and 2 label the parts, $\vec{F}'_{\rm ext,1}$ the net external force on part 1 excluding the force part 2 exerts on part 1, $\vec{F}_{21}$ is the force part 2 exerts on part 1, $\vec{F}'_{\rm ext,2}$ the net external force on part 2 excluding the force part 1 exerts on part 2, and $\vec{F}_{12}$ is the force part 1 exerts on part 2. If we add the two part expressions and make use of the general form for the whole system itself, we find

$$0 = \vec{F}_{21} + \vec{F}_{12} \ , \qquad \text{and thus} \qquad \vec{F}_{21} = -\vec{F}_{12} \ .$$

The last result as it stands is not quite Newton's 3rd law since $\vec{F}_{12}$ and $\vec{F}_{21}$ are net forces, not particular force (e.g., gravity). However, since in special cases only particular forces exist and the last result must be true for them, the 3rd law is proven. So in reality even the 3rd law is not an axiom either: it is a consequence of ____________.

Having established the 3rd law from ____________, one can now see that there is no inconsistency in applications of ____________. Say you divide a body into any number of parts and apply ____________ to each one. You then sum up these applications and all the internal forces cancel pairwise and the sum expression is ____________ applied to the body as a whole—you do not get a different formula—there is no inconsistency.

The 3rd law, in fact, is not always valid even in classical mechanics. It is violated by the magnetic force in certain cases. For these cases, one needs an even more general form of ____________ in which the fields that cause forces are assigned a field momentum. But this more general form is well beyond the scope of intro physics.

a) the principle of inertia b) Newton's 4th law c) Nernst's theorem
d) Newton's zeroth law e) Newton's 2nd law

SUGGESTED ANSWER: (e)

Wrong answers:
c) This the usually known as the 3rd law of thermodynamics.

Redaction: Jeffery, 2008jan01

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009 qmult 00200 1 1 2 easy memory: conservation of momentum
Extra keywords: physci

12. For a system on which no net external force acts, momentum is:

a) not conserved. b) conserved. c) zero. d) never zero. e) always negative.

SUGGESTED ANSWER: (b)

Wrong answers:
c) Sometimes, but not always.
e) A negative momentum really only makes sense in 1-dimensional problems where one dispenses with vector notation and makes one sense positive and one sense negative.

Redaction: Jeffery, 2001jan01

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009 qmult 00260 2 4 1 mod. deducto-memory: collision, explosion, momentum
Extra keywords: physci

13. A collision or explosion is an event in which relatively strong forces act between objects for a relatively short time. If one considers all the objects involved in the collision or explosion as constituting one system, then frequently in calculations it is useful to use the ____________ principle provided the

external forces acting on the system can be considered negligible compared to the internal forces of the system during the collision or explosion.

a) conservation of momentum b) conservation of mechanical energy c) cosmological
d) anthropic e) Peter

SUGGESTED ANSWER: (a)

Wrong answers:
b) Mechanical energy is not necessarily conserved. It can be dissipated to heat or generated from explosion energy.
c) Not this time.
d) I fail to see how the fact of the existence humankind can lead to any useful information.
e) Back in the 1970s there was this guy named Peter who discovered/invented his own principle: "people tend to rise in life to their level of incompetence." It's probably as true as it is false.

Redaction: Jeffery, 2001jan01

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009 qmult 00272 1 3 4 easy math: diver from boat, momentum conserved
Extra keywords: physci KB-96-25

14. A 50 kg girl (who is initially at rest) dives horizontally at 2.0 m/s in the **POSITIVE** direction from a 200 kg boat that initially is at rest. Given that the event is an ideal collision event, what is the initial recoil **VELOCITY** of the boat?

a) 0.25 m/s. b) 0.70 m/s. c) −0.70 m/s. d) −0.50 m/s. e) infinite.

SUGGESTED ANSWER: (d)

The total momentum was zero before the collision event and must be so after by conservation of momentum since it is an ideal collision event. Thus

$$0 = m_{\rm girl}v_{\rm girl} + m_{\rm boat}v_{\rm boat}$$

from which it immediately follows that

$$v_{\rm boat} = -\frac{m_{\rm girl}}{m_{\rm boat}}v_{\rm girl} = -0.5\,{\rm m/s}\ .$$

Hm, 2.0 m/s more of a stroll off than a dive.

Wrong answers:
a) It looks like you forgot to multiply by velocity.
b) Wrong sign. Wrong absolute value.
e) As Lurch would say: "Aaaarh."

Redaction: Jeffery, 2001jan01

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010 qmult 00100 1 4 5 easy deducto-memory: ideal collision defined

15. "Let's play *Jeopardy*! For $100, the answer is: An interaction between two or more bodies where the interaction forces are so strong and duration so short that only the interaction forces between the bodies are significant during the interaction: other forces that arise from sources external to the interaction can be neglected during the interaction. Often the interaction can be regarded as instantaneous on the time scale of the evolution of the system of bodies before and after the interaction."

What is ____________, Alex?

a) linear momentum b) uniform circular motion c) proper motion
d) a weekend in Las Vegas e) an ideal collision in physics

SUGGESTED ANSWER: (e)

Wrong answers:
c) Proper motion is the angular motion of stars on the dome of the sky (Frak-335).
d) In a context other than a physics course. I wanted to use Brighton, but North American students wouldn't get it. Ah, fateful Brighton.

Redaction: Jeffery, 2001jan01

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010 qmult 00200 2 4 2 moderate deducto-memory: impulse definition calculus-based

16. An impulse $\vec{I}$ is:

a) simply the sum of the momentum components of a momentum vector.
b) a force on an object times the time Δt it acts on the object:

$$\vec{I} = \vec{F}\Delta t \ .$$

If the force is not constant (which actually the usual case in collisions for example), then

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\, dt \ ,$$

where the integral is over Δt. Net impulse yields a new version of Newton's 2nd law:

$$\vec{I}_{\rm net} = \int_{\Delta t} \vec{F}_{\rm net}(t)\, dt = \int_{\Delta t} \frac{d\vec{p}}{dt}\, dt = \Delta\vec{p} \ :$$

i.e., $\vec{I}_{\rm net}$ equals the change in momentum of the object. The concept of impulse is used mainly for strong, short-duration forces (i.e., impulsive forces) such as those that occur in collisions and explosions. In such cases, the new version of the 2nd law can be approximated by just using the impulse of the impulsive forces on the left-hand side if they give the only significant contributions to the net impulse.
c) a sudden desire.
d) the opposite of an expulse.
e) an effect used to drive an impulse engine. If your warp drive fails and you are on impulse and 30,000 light-years from Rigel 8, then it's time to get out the card games.

SUGGESTED ANSWER: (b) There are plenty of clues.

Wrong answers:
a) Nah. This sum is physically meaningless.
c) Not the best answer in this context. Also this kind of an impulse is never to my knowledge considered a vector.
d) A nonsense answer.
e) Not the best answer in this context.

Redaction: Jeffery, 2001jan01

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010 qmult 00300 1 1 3 easy memory: collision conservation of momentum

17. In a collision, the impulsive forces are internal to the system of colliding objects: i.e., they do not sum to a net force on the system. Assuming external net force on the system for the short duration of the collision is insignificant compared to the internal forces, then by Newton's 2nd law there is no change in the total momentum of the system. This means that ____________ conserved through the collision.

a) total kinetic energy **IS** b) total mass **IS** c) total momentum **IS**
d) total momentum is **NOT** e) total mass is **NOT**

SUGGESTED ANSWER: (c)

Wrong answers:
a) Kinetic energy is only conserved in elastic collisions. In any case this is not the completion to the remark.
b) Total mass is conserved in non-relativistic cases, but the line is not the completion of the remark. It's not what "This" means.
d) False.
e) False, except in relativistic systems.

Redaction: Jeffery, 2001jan01

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010 qmult 00400 1 1 1 easy memory: collision types

18. In elementary discussions, collisions are usually divided into three types:

    a) inelastic, completely inelastic, elastic. b) inelastic, hardly inelastic, elastic.
    c) inelasticated, completely inelasticated, elasticated.
    d) inelasticated, hardly inelasticated, elasticated. e) good, bad, ugly.

    SUGGESTED ANSWER: (a)

    Wrong answers:

    e) This is an old Clint Eastwood film.

    Redaction: Jeffery, 2008jan01

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010 qmult 00420 1 1 4 easy memory: completely inelastic collision

19. In a completely (or perfectly) inelastic collision:

    a) total kinetic energy is conserved. b) the colliding objects bounce off each other.
    c) total momentum is **NOT** conserved. d) the colliding objects stick together.
    e) the colliding objects have no hair.

    SUGGESTED ANSWER: (d)

    Wrong answers:

    a) Kinetic energy is only conserved in elastic collisions.
    b) Exactly wrong.
    c) False.
    e) It's black holes which have no hair.

    Redaction: Jeffery, 2001jan01

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010 qmult 00710 1 5 1 easy thinking: identical masses collide

20. Two identical point masses elastically collide in a 1-dimensional setup. Mass 1 had velocity v before the collision and velocity zero after. Mass 2's velocity was initially zero and after the collision it was:

    a) v. b) $2v$. c) $-v$. d) $-2v$. e) still zero.

    SUGGESTED ANSWER: (a)

    Conservation of momentum alone allows only answer (a). Actually, you did not need the information that mass 1's velocity was zero after the collision. In 1-dimensional elastic collisions, conservation of momentum and kinetic energy completely determine the outcome of a collision. The outcome of mass 1 at rest and mass 2 having velocity v is consistent with conservation of momentum and kinetic energy, and so it is the only possible outcome. (We are neglecting the ghost solution where mass 1 goes through mass 2 without an interaction at all.)

    The above result can be obtained from the full formulae for post-collision velocities of a 1-dimensional elastic collision of point masses 1 and 2:

$$v_1' = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} \ ,$$
$$v_2' = \frac{(m_2 - m_1)v_2 + 2m_1v_1}{m_2 + m_1} \ ,$$

    where the prime indicates post-collision and the unprime pre-collision. If $m_1 = m_2$, then $v_1' = v_2$ and $v_2' = v_1$ always.

    Wrong answers:

    e) Where did all the momentum go.

    Redaction: Jeffery, 2001jan01

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010 qmult 00720 2 1 5 moderate memory: relative velocity rule

21. In a 1-dimensional elastic collision the initial velocity $= v_{\rm rel,i}$ and final relative $v_{\rm rel,f}$ are related by:

a) $v_{\text{rel,f}} = v_{\text{rel,i}}^2$. b) $v_{\text{rel,f}} = 2v_{\text{rel,i}}$. c) $v_{\text{rel,f}} = -2v_{\text{rel,i}}$. d) $v_{\text{rel,f}} = \frac{1}{2}v_{\text{rel,i}}$.
e) $v_{\text{rel,f}} = -v_{\text{rel,i}}$.

SUGGESTED ANSWER: (e)

One "false" answer I once had was $v_{\text{rel}f} = v_{\text{rel}i}$, but that is arguably a true answer. It corresponds to the ghost solution where the particles exert no forces on each other in a collision. For example, a hoop and billiard ball can collide in the sense of their center of masses passing through each other, but no collision forces act. If one says that an ideal physical collision must exert strong forces, then that rules out the ghost solution. But, of course, real collisions are not ideal and I think we need to keep track of the ghost solution. After all, the ghost solution is exactly a 1-d elastic collision.

Wrong answers:
a) Hard to believe it could be unchanged by the collision.

Redaction: Jeffery, 2001jan01

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010 qmult 00850 1 1 5 easy memory: ball bounces off wall
22. A ball hits a wall with an incident angle of 30° measured from a normal to the wall. Consider the normal as a pole: angles measured from the pole are polar angles; those around the pole are azimuthal angles. The ball rebounds at polar angle:

a) 30° in a plane **PERPENDICULAR** to the one defined by the incident path and the normal.
b) 30°, but in a random azimuthal direction.
c) 60°, but in a random azimuthal direction.
d) 60° in a plane **PERPENDICULAR** to the one defined by the incident path and the normal.
e) 30° in the plane **DEFINED** by the incident path and the normal.

SUGGESTED ANSWER: (e)

Wrong answers:
a) Perpendicular!

Redaction: Jeffery, 2001jan01

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009 qmult 00910 1 1 1 easy memory: operation of rocket
Extra keywords: physci KB-92-9
23. The operation of a rocket in space is based on:

a) conservation of momentum. b) conservation of angular momentum. c) jet fuel pushing on the vacuum. d) starlight pressure. e) running an internal treadmill.

SUGGESTED ANSWER: (a)

Wrong answers:
d) It has been suggested by some, like Arthur C. Clarke, that light pressure could be used for sailing in space. But this is probably only possible within solar systems.
e) This is the hamster theory of space travel.

Redaction: Jeffery, 2001jan01

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009 qmult 00940 1 4 2 easy deducto-memory: rocket thrust force
24. The force that accelerates a rocket in the vacuum of space is the:

a) force of the vacuum. b) force exerted by the ejected fuel on the rocket: the **THRUST** force. c) force exerted by the ejected fuel on the rocket: the **THRUSH** force. d) force of light pressure. e) Force.

SUGGESTED ANSWER: (b)

Thrust force is right. But one has to remember that what is usually specified as the thrust force $v_{\text{exhaust}}|dm/dt|$ (where v_{exhaust} is the exhaust speed relative to the rocket frame and dm/dt is the rocket mass change rate) is only actually the thrust force in an inertial frame that is instantaneously at rest with respect to the rocket. Note v_{exhaust} is usually taken a positive quantity. The thrust force is thus just specified by its magnitude. Its direction is obvious: opposite to the rocket's motion.

Wrong answers:

c) Shades of the *Man from U.N.C.L.E*: ah, Napoleon Solo and Ilya Kuragin where are you now?

d) There's been a lot of speculation about using the light pressure from the Sun for solar sailing. But that doesn't describe how a rocket goes.

e) May it be with you.

Redaction: Jeffery, 2001jan01

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005 qfull 00610 1 3 0 easy math: center-of-mass integration of a rod

25. You have a 1-dimensional object of length a. It is a non-uniform thin rod. It's linear density (i.e., mass per unit length) is given by

$$\lambda = Cx^p \ ,$$

where C is a constant, $p > -1$ (i.e., p is a real number greater than -1) and the origin is at one end of the rod. Derive the formula for the center of mass of the rod. Sketch a plot of x_{CM} versus p.

SUGGESTED ANSWER:

Behold:

$$x_{\text{CM}} = \frac{\int_0^a \lambda x\, dx}{\int_0^a \lambda\, dx} = \frac{\int_0^a x^{p+1}\, dx}{\int_0^a x^p\, dx} = \frac{[x^{p+2}/(p+2)]|_0^a}{[x^{p+1}/(p+1)]|_0^a} = a\left(\frac{p+1}{p+2}\right) \ .$$

You will have to imagine the plot. We can describe it though. First let's take the derivative:

$$\frac{dx_{\text{CM}}}{dp} = a\left[\frac{1}{p+2} - \frac{p+1}{(p+2)^2}\right] = \frac{a}{(p+2)^2} \ .$$

So for $p > -1$, the derivative is always greater than or equal to zero and it only equals zero for $p = \infty$. So the x_{CM} curve rises from zero at from $p = -1$ (which is a valid limiting case) and approaches a asymmtotically as p goes to infinity. For $p < 0$, $x_{\text{CM}} < a/2$ as you would expect for a rod with the higher density on the left-hand side. Note the density actually goes to infinity at $x = 0$ for $p < 0$, but x_{CM} is only pushed to the left end of the rod when $p \leq -1$. For $p = 0$, $x_{\text{CM}} = a/2$ as you would expect for a uniform rod. For $p > 0$, $x_{\text{CM}} > a/2$ as you would expect for a rod with the higher density on the left-hand side. Only for $p = \infty$ is x_{CM} pushed to the right end of the rod.

What of the cases for $p < -1$? Our formula is invalid there since we have integrated up to a divergence in linear density. It still gives the right result for $p = -1$ as limiting case as is evident. The formula gives, a couple of hyperbolas, in fact. Going to the right of $p = -2$, the formula curve rises from $-\infty$ to approach a asymptotically from below. Going to the left of $p = -2$, the curve falls from ∞ to approach a asymptotically from above.

Just from the $p \to -1$ case for $p > -1$, we know the correct limiting behavior for $p \leq -1$ is $x_{\text{CM}} = 0$. How do we study this mathematically? Well we don't integrate to $x = 0$, but to $x = \epsilon$ which is a small non-zero number. Unfortunately, there are four distinct casess.

First case, $p = -1$. Here, we get

$$x_{\text{CM}} = \frac{a - \epsilon}{\ln(a) - \ln(\epsilon)}$$

which in the limit of small ϵ becomes

$$x_{\text{CM}} \approx \frac{a}{-\ln(\epsilon)}$$

which goes to 0 as $\epsilon \to 0$.

Second case, $p \in (-2, -1)$ range. Here we get

$$x_{\text{CM}} = \left(\frac{p+1}{p+2}\right)\left(\frac{a^{p+2} - \epsilon^{p+2}}{a^{p+1} - \epsilon^{p+1}}\right)$$

which in the limit of small ϵ becomes

$$x_{\rm CM} \approx \left(\frac{p+1}{p+2}\right)\left(\frac{a^{p+2}}{-\epsilon^{p+1}}\right)$$

(which is positive since $p+1<0$) which goes to 0 as $\epsilon \to 0$.

Third case, $p=-2$. Here, we get

$$x_{\rm CM} = -\left(\frac{\ln(a)-\ln(\epsilon)}{a^{-1}-\epsilon^{-1}}\right)$$

(which is positive since $\epsilon^{-1} > a^{-1}$) which in the limit of small ϵ becomes

$$x_{\rm CM} \approx -\frac{\ln(\epsilon)}{\epsilon^{-1}}$$

(which is positive since $\ln(\epsilon)<0$). Now

$$\lim_{\epsilon\to 0}\frac{\ln(\epsilon)}{1/\epsilon} = \lim_{\epsilon\to 0} -\frac{1/\epsilon}{1/\epsilon^2} = 0 \ .$$

So $x_{\rm CM} \to 0$ as ϵ goes to 0 for $p=-2$.

Fourth case, $p<-2$. Here we get

$$x_{\rm CM} = \left(\frac{p+1}{p+2}\right)\left(\frac{a^{p+2}-\epsilon^{p+2}}{a^{p+1}-\epsilon^{p+1}}\right)$$

which in the limit of small ϵ becomes

$$x_{\rm CM} \approx \left(\frac{p+1}{p+2}\right)\epsilon$$

which goes to 0 as $\epsilon \to 0$.

Now wasn't that tedious.

Redaction: Jeffery, 2001jan01

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009 qfull 00270 1 5 0 easy thinking: conservation of momentum, exploding shell

Extra keywords: fragmenting shell

26. A shell of mass m is at a certain instant is exactly in horizontal flight with velocity v in the positive x-direction. At that same instant the shell explodes instantaneously (virtually instantaneously to be more realistic) into 3 fragments: two fragments have zero x velocity, one going straight up, the other straight down; the 3rd fragment, which has mass m_3, just at the instant after explosion continues in the x direction with only its x velocity nonzero.

a) What is the x-direction velocity v_3 of the 3rd fragment in terms of given variables? Derive the formula for v_3 from a basic principle.

b) Just after the explosion, what must the total y-direction momentum be of the two fragments that went straight up and down? Explain why it is this value.

c) How far does the 3rd fragment go in the x direction if it starts a height y_0 above a flat plain assuming no air drag? Derive the formula for the distance including a derivation of the the horizontal-launch projectile range formula. Give the answer in terms of given variables and g.

SUGGESTED ANSWER:

a) By conservation of momentum in the x-direction, we have

$$p = mv = m_3v_3$$

for the x direction. Thus,

$$v_3 = \left(\frac{m}{m_3}\right)v \ .$$

b) Just before the explosion the shell has zero y-direction momentum. By conservation of momentum in the absence of external forces or, as in this case, when external forces have no time to act, the y-direction momentum just after the explosion must still be zero. The 3rd fragment is given as having no y-direction momentum just after the explosion, and so the two other fragments must zero total y-direction mometum too. Of course, individually those two fragments clearly do have y-direction momentum. And also of course the y-direction momenta of all the fragments evolves under the action of gravity at finite times after the explosion.

c) From the horizontal-launch projectile range formula (which is not the horizontal launch formula),

$$x = v\left(\frac{m}{m_3}\right)\sqrt{\frac{2y_0}{g}} \ .$$

Since $m/m_3 > 1$, the 3rd fragment goes farther than the shell would have if it had not exploded.

Now we will derive the horizontal-launch projectile range formula. Note that

$$x = v_3 t$$

by one of the constant-acceleration kinematics formulae. The time for the fall satisfies

$$y_0 = \frac{1}{2}gt^2$$

by another of the constant-acceleration kinematics formulae. Thus, the fall time is

$$t = \sqrt{\frac{2y_0}{g}} \ ,$$

and we get

$$x = v_3\sqrt{\frac{2y_0}{g}} = v\left(\frac{m}{m_3}\right)\sqrt{\frac{2y_0}{g}}$$

all over again.

Redaction: Jeffery, 2001jan01

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009 qfull 00280 3 3 0 tough thinking: conservation of momentum in a boat with King

Extra keywords: boat-King-you system

27. Your dog King—a she, but somehow that wasn't obvious before the puppies ("Ooh, arn't they sweethearts.")—is on a long flat-bottomed boat (10 m in length to be precise) on a placid lake. King now muscles in at 90 kg—you must not have been listening when her grandsire Gross Hans the St. Bernard was being aired—"he could haul an SUV". You, also on the boat, however, are a petite 50 kg—which makes opening the door when you get home from work, King keening just inside, its own demonstration in macroscopic conservation of momentum. The boat is 60 kg: it's made of that light-weight stuff that the astronauts ate on the Moon. It is fore-aft symmetric: i.e., its center of mass is exactly **MIDSHIPS**. Assume there is no net external force acting on the boat-King-you system and that the system is **1-DIMENSIONAL** along the boat axis.

a) Act 1, Scene I: The boat is at rest (or even becalmed): King's motionless (thankfully) at the bow (bowwow?); you are at the stern likewise enjoying a breather. Give the formula in terms of **VARIABLES** (i.e., **SYMBOLS**) for the center-of-mass position of the total boat-King-you system using the stern as the **ORIGIN** and recall the system is 1-dimensional. Now what is the center-of-mass position (i.e., what is the numerical value)? **HINT:** Draw a diagram.

b) Act 1, Scene II: King in a sudden attack of nautical anxiety needs some loving and, after a brief and inconsequential acceleration phase, bounds toward you at 6.0 m/s **RELATIVE** to the boat: this, of course, should be -6.0 m/s if the positive direction is aft-to-fore. (We are really treating King as free-flying object that is not constantly pounding her paws on the deck. She's an ideal dog.) Find the formula in terms of **VARIABLES** for the boat's velocity relative to the water now. Show how you got the formula.

What is the boat's velocity relative to the water? What is your velocity relative to the water? What is King's velocity relative to the water? What is the boat-King-you system's center of mass velocity relative to the water? Take the aft-to-fore direction as positive. Numerical values are expected.

c) Act 1, Scene III: King makes a **COMPLETELY INELASTIC COLLISION** with you at the stern: i.e., she sticks to you with a deep, warm feeling—"Belay King, Belay!" Assuming you don't all tip backward into the drink, what are the velocities of you, King, the boat, and the boat-King-you system center of mass now? Briefly explain your answers.

SUGGESTED ANSWER:

a) To simplify the notation you are 1, the boat is 2, and King is 3. Behold:

$$x_{\rm cm} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{0 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = 6\,\mathrm{m}\ .$$

b) Hm, tricky. What equations do we have relating the variables?

Well the total momentum p of the the boat-King-you system was zero in Scene I and no external forces have acted. Ergo total momentum is still zero:

$$p = 0 = m_1 v_1 + m_2 v_2 + m_3 v_3\ ,$$

where all velocities are relative to the water. That is one equation. A second eqution is

$$v_1 = v_2$$

and a third is

$$v_{3,\rm rel} = v_3 - v_2$$

or

$$v_3 = v_{3\,\rm rel} + v_2\ .$$

We now have three equations and three unkowns (i.e., v_1, v_2, and v_3), and so a soltution is possible.

We now substitute into the momentum equation and get an expression with v_2 as the only unknown

$$0 = p = m_1 v_2 + m_2 v_2 + m_3(v_{3,\rm rel} + v_2)\ .$$

Solving for the unknowns gives

$$v_1 = v_2 = v_1 = \frac{p - m_3 v_{3,\rm rel}}{m_1 + m_2 + m_3} = 2.7\,\mathrm{m/s}$$

and

$$v_3 = -3.3\,\mathrm{m/s}\ .$$

Now the total momentum $p = mv$ where v is the center-of-mass velocity. Since $p = 0$ at all times in problem, so does v. So the center-of-mass velocity is

$$v = 0\ .$$

There is a subltety here. You may ask why you can't used the King's momentum relative to the boat in a direct way to solve for the motions. Before King moves, there is no momemtum relative to the boat and after she starts bounding their is King's momentum relative to the boat, but no other momentum relative to the boat. This relative momentum is **NOT** conserved. The resolution of the paradox is that the boat is not continuously an inertial frame. It's an inertial frame initially and after King is in uniform motion, but when King accelerates, the boat accelerates. Since the boat frame is not continuously an inertial frame, we can't use it as a reference frame when making use of Newtonian physics without tricky corrections.

c) Well none of the objects is moving relative to each other, and therefore none is moving relative to the center of mass. Since the center of mass is still at rest relative to the water, all the objects are at rest relative to the water.

In a more mathematical form,

$$0 = p = mv = m_1v_1 + m_2v_2 + m_3v_3 \ ,$$

with all velocities relative to the water. If you, the boat, and King all have zero velocities relative to each other, then they all have the same velocity relative to the water. To satisfy the last equation then, all the velocities are zero relative to the water.

Redaction: Jeffery, 2001jan01

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010 qfull 00200 2 5 0 moderate thinking: goats, completely inelastic coll.

Extra keywords: goats Richthofen and Mannock

28. Richthofen and Mannock are two goats with masses 40 kg and 50 kg, respectively. They charge each other head-on on a icy pond—amazingly they stay upright, but goats are sure-hooféd. Just before impact Richthofen is moving at 10.0 m/s—he's way out of control—and Mannock's moving at −5.0 m/s. On impact they lock horns literally (i.e., stick together). Treat the goats as point masses and their collision as an ideal collision (i.e., one in which only the collision forces are non-negligible). The problem is entirely 1-dimensional.

a) What type of collision has occurred? It is one of the basic kinds of ideal collisions.

b) What is the center-of-mass velocity of the Richthofen-Mannock system just **BEFORE** and just **AFTER** the collision? Make the collision approximation. Be explicit: which velocity is before and which is after?

c) Assuming that the coefficient of kinetic friction μ_{kin} between goat and ice is 0.13, what is the displacement of the unhappy Richthofen-Mannock system from the impact site to where it stops moving?

SUGGESTED ANSWER:

a) Because they stick together their collision is a **COMPLETELY INELASTIC COLLISION**. Also because they stick together, the problem remains entirely 1-dimensional. It is impossible for the post-collision Richthofen-Mannock system to leave the initial axis of collision and still conserve momentum. If the goats hadn't stuck together this could have happened. They would simply go in opposite directions along any axis and momentum conservation would be possible.

b) Behold:

$$v_{\text{cm}} = \left(\frac{m_{\text{Ri}}v_{\text{Ri}} + m_{\text{Ma}}v_{\text{Ma}}}{m_{\text{Ri}} + m_{\text{Ma}}} \right)_{\text{pre-col}} \approx 1.7\,\text{m/s}$$

is the center of mass velocity both before and after the collision. This follows from conservation of total momentum through the collision (in the collision approximation) and conservation of mass:

$$\vec{p}_{\text{total}} = (m_{\text{Ri}} + m_{\text{Ma}})v_{\text{cm}} \ ,$$

and so

$$v_{\text{cm}} = \frac{\vec{p}_{\text{total}}}{m_{\text{Ri}} + m_{\text{Ma}}}$$

is conserved too.

c) The work friction does to stop the system must cancel the initial kinetic energy. The work-kinetic-energy theorem

$$\Delta KE = W_{\text{net}}$$

in our case leads to

$$\Delta KE = 0 - \frac{1}{2}(m_{\text{Ri}} + m_{\text{Ma}})v_{\text{cm}}^2 = W_{\text{fr}} = F_{\text{fr,kin}} \cdot \Delta s = -\mu_{\text{kin}}(m_{\text{Ri}} + m_{\text{Ma}})g\Delta s \ ,$$

where the work of friction must be negative since kinetic friction always opposes the direction of motion and Δs is the unknown displacement. Note we have used the rule for kinetic friction that $F_{\text{fr,kin}} = \mu_{\text{kin}}F_{\text{normal}}$ where the rule relates force magnitudes since, of course, kinetic friction and normal force are perpendicular. In this case, the normal force magnitude is mg since it must cancel gravity. Solving for Δs gives

$$\Delta s = \frac{v_{\rm cm}^2}{2\mu_{\rm kin} g} \approx 1.1\,{\rm m}$$

to about 2-digit accuracy.

The problem is entirely 1-dimensional, and so the direction of the displacement is given by its sign. It's a positive displacement.

Fortran-95 Code

```
        xm1=40.d0
        v1=10.d0
        xm2=50.d0
        v2=-5.d0
        xmu=0.13
        gg=9.8d0
        xm=xm1+xm2
        vcen=(xm1*v1+xm2*v2)/xm
        xkecen=.5d0*xm*vcen**2
        dels=vcen**2/(2.d0*xmu*g)
        print*,'vcen,xkecen,dels'
        print*,vcen,xkecen,dels
!  1.6666666666666667 125.00000000000001 1.0901796803779013
```

Redaction: Jeffery, 2001jan01

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010 qfull 00210 1 3 0 easy math: two objects collide elastically

29. Two objects collide elastically in a 1-dimensional setup. The first has mass $m_1 = 1.00\,{\rm kg}$ and initial velocity $v_1 = 6.00\,{\rm m/s}$ and the second, mass $m_2 = 8.00\,{\rm kg}$ and initial velocity $v_2 = 3.00\,{\rm m/s}$. Note initial velocities are given, not speeds. What are the **TWO** final velocities?

SUGGESTED ANSWER:

This is a cinch:

$$\begin{aligned} v_{1'} &= \frac{(m_1-m_2)v_1 + 2m_2v_2}{m_1+m_2} = \frac{(1-8)\times 6 + 2\times 8\times 3}{1+8} \\ &= \frac{-42+48}{9} = \frac{2}{3}\,{\rm m/s} \end{aligned}$$

and

$$\begin{aligned} v_{2'} &= \frac{(m_2-m_1)v_2 + 2m_1v_1}{m_1+m_2} \\ &= v_{{\rm rel}'\,f} + v_{1'} = -v_{{\rm rel}\,i} + v_{1'} = 3 + \frac{2}{3} = \frac{11}{3}\,{\rm m/s}\ . \end{aligned}$$

Redaction: Jeffery, 2001jan01

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010 qfull 00300 2 5 0 moderate thinking: Jupiter gravity assist

30. Now you can't (to good approximation) push or pull on space. But one can take a swing about a celestial body. This maneuver in NASAese is called a gravity assist or a swingby or a slingshot encounter. Say we have a space probe on Jupiter's orbital path opposite to Jupiter's motion. The path can be regarded as straight line over the distance scale of our problem. The probe mass has $m = 1000\,{\rm kg}$ and Jupiter has $m_{\rm Ju} = 1.8988\times 10^{27}\,{\rm kg} \approx 318$ Earth masses. The probe velocity $v_{{\rm probe}\,i} = -10\,{\rm km/s}$ and Jupiter's velocity $v_{{\rm Ju}\,i} = 13.1\,{\rm km/s}$: both velocities are relative to the Sun's rest frame. The encounter is planned such that, if all goes well, the probe does 180° swing around Jupiter (i.e., a U-turn). Regarding the encounter as a 1-dimensional elastic collision what is the outgoing probe speed? Now why does NASA uses slingshot shot encounters? Give two reasons.

SUGGESTED ANSWER:

This is essentially a 1-d elastic collision with one collider having effectively an infinite mass. Thus the final velocity for the probe is given by

$$v_{\text{probe }f} = -v_{\text{probe }i} + 2v_{\text{Ju }i} = 36\,\text{km/s} \ .$$

where we have set Jupiter's mass to infinity. Jupiter's velocity is super-negligibly changed by the encounter.

Slingshot encounters can be used to change probe directions and to change their speeds: often one boosts their speeds. Extra kinetic energy in a boost comes at the expense of the celestial body's kinetic energy, but the expense is very, very nugatory. There's no concern that NASA is degrading our solar system: not in this way anyhow.

Redaction: Jeffery, 2001jan01

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010 qfull 00400 2 5 0 moderate thinking: teenagers, symmetrical coll.

Extra keywords: This is a 2-d problem

31. Teenagers on ice—thin ice—each of mass 50 kg, slide together and cling. They are both moving at speed 3 m/s prior to the collision. An old codger of 75 kg moving at 2 m/s then collides with them and the whole conglomerate then comes to a dead stop. What is the angle between the initial velocity vectors of the two teenagers. **HINT:** A diagram might help.

SUGGESTED ANSWER:

Take the line that the clinging teenagers move on to be the positive x-axis and their point of collision to be the origin. Since their masses are equal and their initial speeds were equal, their initial y-velocity components had to be equal in magnitude and opposite in sign in order to cancel on collision. Their initial x-velocity components were equal and add up on collision. The angles of the initial velocity vectors from the x-axis had to be θ and $-\theta$.

Since the codger stops the lot, he had to have had zero y momentum and his x momentum had to have the same magnitude and opposite sign from the clinging teenagers: i.e., $p_{\text{condger }x} = -150\,\text{kg}\,\text{m}^2 = -p_{\text{teenagers }x}$. Thus each teenager had x momentum $75\,\text{kg}\,\text{m}^2$ before the first collision. Since the magnitudes of their initial momenta were both $150\,\text{kg}\,\text{m}^2$, it follows that $\cos\theta = 1/2$ and $\theta = 60°$. Thus the angle between the initial velocity vectors was $120°$.

Redaction: Jeffery, 2001jan01

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009 qfull 00910 2 3 0 moderate math: rocket speed in space

32. Lost deep in the void of eternal space there is a rocket in an inertial frame. At time zero, it's initial velocity is v_0. Suddenly its engine fires ejecting fuel at a constant exhaust speed v_{ex} (i.e., a constant relative velocity with respect to the rocket).

   a) What is the general formula for the ratio of mass to initial mass (i.e., m/m_0) in terms of velocity v, v_0, and v_{ex}? We are not asking for a derivation. Just write down the formula. Of course, if you don't know it or can't locate it somewhere, you could derive it.

   b) What is the mass ratio m/m_0 when v is equal to the exhaust speed? After the time of this event what is the direction of motion of ejected fuel relative to the inertial frame?

   c) What is the mass ratio when v is equal to the exhaust speed if $v_0 = 0$?

SUGGESTED ANSWER:

a) Since

$$v = v_0 + v_{\text{ex}} \ln\left(\frac{m_0}{m}\right) \ ,$$

the general formula for the mass ratio is

$$\frac{m}{m_0} = e^{-(v-v_0)/v_{\text{ex}}} \ .$$

b) If $v = v_{\text{ex}}$, then

$$\frac{m}{m_0} = e^{-(v_{\text{ex}}-v_0)/v_{\text{ex}}} = e^{-1+v_0/v_{\text{ex}}} \ .$$

The velocity of the ejected fuel relative to the inertial frame is always $v - v_{\rm ex}$. So before the equality time, the ejected fuel is moving opposite the rocket direction in the inertial frame in which the velocities are being measured. After the equality time, the ejected fuel is moving in the rocket direction. So it is actually tracking the rocket in the inertial frame.

c) If v_0 and $v = v_{\rm ex}$, then

$$\frac{m}{m_0} = e^{-1} = \frac{1}{2.7182818\ldots} = 0.3678794\ldots\ .$$

Redaction: Jeffery, 2001jan01

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

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1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67428(67) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2006, CODATA)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\mathrm{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

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2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\mathrm{sphere}} = 4\pi \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

---

3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a - b) - \cos(a + b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a - b) + \sin(a + b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1 - x} \approx 1 + x \;:\; (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

5 Quadratic Formula

$$\text{If} \qquad 0 = ax^2 + bx + c\;, \qquad \text{then} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

$$\text{Taylor's series} \qquad f(x) = \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$$

$$= f(x_0) + (x - x_0)f^{(1)}(x_0) + \frac{(x-x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x-x_0)^3}{3!} f^{(3)}(x_0) + \ldots$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0 \qquad v^2 = v_0^2 + 2a(x - x_0)$$

$$x = \frac{1}{2}(v_0 + v)t + x_0 \qquad x = -\frac{1}{2}at^2 + vt + x_0 \qquad g = 9.8\,\mathrm{m/s^2}$$

$$x_{\text{rel}} = x_2 - x_1 \qquad v_{\text{rel}} = v_2 - v_1 \qquad a_{\text{rel}} = a_2 - a_1$$

$$x' = x - v_{\text{frame}} t \qquad v' = v - v_{\text{frame}} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\text{avg}} = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0} t \qquad y = -\frac{1}{2} g t^2 + v_{y,0} t + y_0 \qquad v_{x,0} = v_0 \cos\theta \qquad v_{y,0} = v_0 \sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0 \cos\theta} \qquad y = y_0 + x \tan\theta - \frac{x^2 g}{2 v_0^2 \cos^2\theta}$$

$$x_{\text{for } y \text{ max}} = \frac{v_0^2 \sin\theta \cos\theta}{g} \qquad y_{\text{max}} = y_0 + \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2 v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g} \qquad \theta_{\text{for max}} = \frac{\pi}{4} \qquad x_{\text{max}}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0 \sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2 r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\tan} = r\alpha$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = v\omega$$

13 Very Basic Newtonian Physics

$$\vec{r}_{\rm cm} = \frac{\sum_i m_i \vec{r}_i}{m_{\rm total}} = \frac{\sum_{\rm sub} m_{\rm sub} \vec{r}_{\rm cm\ sub}}{m_{\rm total}} \qquad \vec{v}_{\rm cm} = \frac{\sum_i m_i \vec{v}_i}{m_{\rm total}} \qquad \vec{a}_{\rm cm} = \frac{\sum_i m_i \vec{a}_i}{m_{\rm total}}$$

$$\vec{r}_{\rm cm} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\rm total}}$$

$$\vec{F}_{\rm net} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,{\rm m/s^2}$$

$$\vec{F}_{\rm normal} = -\vec{F}_{\rm applied} \qquad F_{\rm linear} = -kx$$

$$f_{\rm normal} = \frac{T}{r} \qquad T = T_0 - F_{\rm parallel}(s) \qquad T = T_0$$

$$F_{\rm f\ static} = \min(F_{\rm applied}, F_{\rm f\ static\ max}) \qquad F_{\rm f\ static\ max} = \mu_{\rm static} F_{\rm N} \qquad F_{\rm f\ kinetic} = \mu_{\rm kinetic} F_{\rm N}$$

$$v_{\rm tangential} = r\omega = r\frac{d\theta}{dt} \qquad a_{\rm tangential} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\rm centripetal} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\rm centripetal} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\rm drag,lin} = bv \qquad v_{\rm T} = \frac{mg}{b} \qquad \tau = \frac{v_{\rm T}}{g} = \frac{m}{b} \qquad v = v_{\rm T}(1 - e^{-t/\tau})$$

$$F_{\rm drag,quad} = bv^2 = \frac{1}{2}C\rho A v^2 \qquad v_{\rm T} = \sqrt{\frac{mg}{b}}$$

14 Energy and Work

$$dW = \vec{F}\cdot d\vec{s} \qquad W = \int \vec{F}\cdot d\vec{s} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\rm mechanical} = KE + PE$$

$$P_{\rm avg} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F}\cdot\vec{v}$$

$$\Delta KE = W_{\rm net} \quad \Delta PE_{\rm of\ a\ conservative\ force} = -W_{\rm by\ a\ conservative\ force} \quad \Delta E = W_{\rm nonconservative}$$

$$F = -\frac{dPE}{dx} \qquad \vec{F} = -\nabla PE \qquad PE = \frac{1}{2}kx^2 \qquad PE = mgy$$

15 Momentum

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta KE_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\text{cm}} = \vec{F}_{\text{net non-flux}} + (\vec{v}_{\text{flux}} - \vec{v}_{\text{cm}})\frac{dm}{dt} = \vec{F}_{\text{net non-flux}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_0 + v_{\text{ex}} \ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

16 Collisions

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\,dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\text{net}}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$KE_{\text{total}\ f} = KE_{\text{total}\ i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\text{rel}'} = -v_{\text{rel}} \qquad \text{1-d Elastic Collision Expressions}$$

17 Rotational Kinematics

$$2\pi = 6.2831853\ldots \qquad \frac{1}{2\pi} = 0.15915494\ldots$$

$$\frac{180^\circ}{\pi} = 57.295779\ldots \approx 60^\circ \qquad \frac{\pi}{180^\circ} = 0.017453292\ldots \approx \frac{1}{60^\circ}$$

$$\theta = \frac{s}{r} \qquad \omega = \frac{d\theta}{dt} = \frac{v}{r} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{a}{r} \qquad f = \frac{\omega}{2\pi} \qquad P = \frac{1}{f} = \frac{2\pi}{\omega}$$

$$\omega = \alpha t + \omega_0 \qquad \Delta\theta = \frac{1}{2}\alpha t^2 + \omega_0 t \qquad \omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{1}{2}(\omega_0 + \omega)t \qquad \Delta\theta = -\frac{1}{2}\alpha t^2 + \omega t$$