Intro Physics Semester I **Name:**

Homework 9: Momentum: One or two full answer questions will be marked. There will also be a mark for completeness. Homeworks are due usually the day after the chapter they are for is finished. Solutions will be posted soon thereafter. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

Answer Table

Name:

	a	b	c	d	e		a	b	c	d	e
1.	O	O	O	O	O	31.	O	O	O	O	O
2.	O	O	O	O	O	32.	O	O	O	O	O
3.	O	O	O	O	O	33.	O	O	O	O	O
4.	O	O	O	O	O	34.	O	O	O	O	O
5.	O	O	O	O	O	35.	O	O	O	O	O
6.	O	O	O	O	O	36.	O	O	O	O	O
7.	O	O	O	O	O	37.	O	O	O	O	O
8.	O	O	O	O	O	38.	O	O	O	O	O
9.	O	O	O	O	O	39.	O	O	O	O	O
10.	O	O	O	O	O	40.	O	O	O	O	O
11.	O	O	O	O	O	41.	O	O	O	O	O
12.	O	O	O	O	O	42.	O	O	O	O	O
13.	O	O	O	O	O	43.	O	O	O	O	O
14.	O	O	O	O	O	44.	O	O	O	O	O
15.	O	O	O	O	O	45.	O	O	O	O	O
16.	O	O	O	O	O	46.	O	O	O	O	O
17.	O	O	O	O	O	47.	O	O	O	O	O
18.	O	O	O	O	O	48.	O	O	O	O	O
19.	O	O	O	O	O	49.	O	O	O	O	O
20.	O	O	O	O	O	50.	O	O	O	O	O
21.	O	O	O	O	O	51.	O	O	O	O	O
22.	O	O	O	O	O	52.	O	O	O	O	O
23.	O	O	O	O	O	53.	O	O	O	O	O
24.	O	O	O	O	O	54.	O	O	O	O	O
25.	O	O	O	O	O	55.	O	O	O	O	O
26.	O	O	O	O	O	56.	O	O	O	O	O
27.	O	O	O	O	O	57.	O	O	O	O	O
28.	O	O	O	O	O	58.	O	O	O	O	O
29.	O	O	O	O	O	59.	O	O	O	O	O
30.	O	O	O	O	O	60.	O	O	O	O	O

1. The center of mass is the:

   a) position-weighted mean mass of the an object.
   b) object-weighted mean mass of the position.
   c) mean of mass an weighted object position of.
   d) mass-weighted mean position of an object.
   e) simple center of the object.

2. The center of mass (i.e., the actual physical position of the center of mass in space relative to the physical system it is the center of mass of) is:

   a) a function of the coordinate system.
   b) independent of the coordinate system.
   c) dependent on the coordinate system.
   d) both independent of and a function of the coordinate system.
   e) neither independent of nor a function of the coordinate system.

3. If an object is symmetric in 3 dimensions about some point (i.e., its geometric center), its center of mass must be:

   a) outside of the object.
   b) neither inside nor outside the object.
   c) at the point about which the object is symmetric in 2 of the dimensions, but not in the 3rd.
   d) at the point about which the object is symmetric in 1 of the dimensions, but not in the other 2.
   e) at the geometric center.

4. The center of mass of sphere of radius R with a density given by $\rho = Cr^2$ (where r is the radial coordinate) is at:

   a) $r = R/2$. b) $r = R/3$. c) $r = 0$. d) $r = 3R$. e) $r = 2R$.

5. Where is the center of mass of a hoop?

   a) At the end of the hoop.
   b) At the top of the hoop.
   c) At the left side of hoop.
   d) Nowhere since a center of mass must be physically inside an object to be a center of mass.
   e) On the axis of the hoop at the geometrical center of the hoop.

6. "Let's play *Jeopardy*! For \$100, the answer is: If one hangs a rigid object from a freely turning pivot point and lets it come to stable static equilibrium, the center of mass is directly below the pivot point. Thus, center of mass can be found from the intersection of two lines through the object that start at two points used as pivot points and that go in the direction through the object that was downward when each of the points was the pivot point. The method fails if the two pivot points and the center of mass happen to be collinear."

   a) What is an **EMPIRICAL** method for finding gravitational torque, Alex?
   b) What is a **THEORETICAL** method for finding gravitational torque, Alex?
   c) What is gravitational torque, Alex?
   d) What is a center of mass, Alex?
   e) What is an **EMPIRICAL** method for finding the center of mass of a rigid object, Alex?

7. Momentum (or linear momentum) is defined by the formula ____________, where m is system mass and $\vec{v}$ is system center-of-mass velocity.

   a) $\vec{p} = \dfrac{m}{\vec{v}}$ b) $\vec{p} = \dfrac{\vec{v}}{m}$ c) $\vec{p} = \dfrac{1}{2}mv^2$ d) $\vec{p} = \dfrac{1}{2}m\vec{v}$ e) $\vec{p} = m\vec{v}$

8. The momentum (or linear momentum) for a 1-dimensional system is given by the formula ____________, where m is system mass and v is system center-of-mass velocity for the single dimension.

   a) $p = \dfrac{m}{v}$ b) $p = \dfrac{v}{m}$ c) $p = \dfrac{1}{2}mv^2$ d) $p = \dfrac{1}{2}mv$ e) $p = mv$

9. Linear momentum is **NOT**:

a) a physical quantity. b) dependent on velocity. c) a kind of energy.
d) dependent on mass. e) given by $p = mv$ for one-dimensional cases.

10. If the kinetic energy of an object is doubled, the momentum magnitude changes by a factor of:

a) $\sqrt{2}$. b) 2. c) 1/2. d) $1/\sqrt{2}$. e) 1.

11. The general form of ____________ is

$$\frac{d\vec{p}}{dt} = \vec{F}_{\text{ext}} ,$$

where $\vec{p}$ is the total momentum of a system and $\vec{F}_{\text{ext}}$ is the net external force on the system. The net external force includes ordinary forces (field forces and contact forces), and what can be called momentum flux forces.

A momentum flux force is just the momentum added to a system by adding mass with its own momentum to the system. If all the added mass had the same velocity $\vec{v}_{\text{flux}}$, then the momentum flux force would be

$$\vec{v}_{\text{flux}}\frac{dm}{dt} .$$

Note dm/dt can be positive or negative: if negative, mass is actually be lost from the system. The added mass can interact with the rest of mass of the system after becoming part of it by ordinary forces or not. Using our special case momentum flux force and the chain rule, we can specialize ____________ to the formula

$$m\vec{a} = \vec{F}_{\text{ext,ordinary}} + (\vec{v}_{\text{flux}} - \vec{v})\frac{dm}{dt} ,$$

were $\vec{a}$ is the center-of-mass acceleration and $\vec{v}$ is the center-of-mass velocity.

If $\vec{F}_{\text{ext}} = 0$, then $\vec{p}$ is a constant: i.e., momentum is conserved. This is really the general form of Newton's 1st law: in the absence of a net external force, momentum is conserved. If $dm/dt = 0$ too (and it is hard to arrange $\vec{F}_{\text{ext}} = 0$ without having $dm/dt = 0$), then $\vec{v}$ is a constant: this is Newton's 1st law as usually stated. The 1st law is not really a law (i.e., an axiom) of classical mechanics: it is a result that is called a law because Newton called it that—maybe even for him it was a traditional law.

Since our general form of ____________ is an axiom of classical mechanics, it must apply to the parts of a system treated as systems in their own right. Imagine a system broken into two parts 1 and 2. We must have

$$\frac{d\vec{p}_1}{dt} = \vec{F}'_{\text{ext},1} + \vec{F}_{21} \qquad \text{and} \qquad \frac{d\vec{p}_2}{dt} = \vec{F}'_{\text{ext},2} + \vec{F}_{12}$$

where 1 and 2 label the parts, $\vec{F}'_{\text{ext},1}$ the net external force on part 1 excluding the force part 2 exerts on part 1, $\vec{F}_{21}$ is the force part 2 exerts on part 1, $\vec{F}'_{\text{ext},2}$ the net external force on part 2 excluding the force part 1 exerts on part 2, and $\vec{F}_{12}$ is the force part 1 exerts on part 2. If we add the two part expressions and make use of the general form for the whole system itself, we find

$$0 = \vec{F}_{21} + \vec{F}_{12} , \qquad \text{and thus} \qquad \vec{F}_{21} = -\vec{F}_{12} .$$

The last result as it stands is not quite Newton's 3rd law since $\vec{F}_{12}$ and $\vec{F}_{21}$ are net forces, not particular force (e.g., gravity). However, since in special cases only particular forces exist and the last result must be true for them, the 3rd law is proven. So in reality even the 3rd law is not an axiom either: it is a consequence of ____________.

Having established the 3rd law from ____________, one can now see that there is no inconsistency in applications of ____________. Say you divide a body into any number of parts and apply ____________ to each one. You then sum up these applications and all the internal forces cancel pairwise and the sum expression is ____________ applied to the body as a whole—you do not get a different formula—there is no inconsistency.

The 3rd law, in fact, is not always valid even in classical mechanics. It is violated by the magnetic force in certain cases. For these cases, one needs an even more general form of ____________ in which the fields that cause forces are assigned a field momentum. But this more general form is well beyond the scope of intro physics.

a) the principle of inertia b) Newton's 4th law c) Nernst's theorem
d) Newton's zeroth law e) Newton's 2nd law

12. For a system on which no net external force acts, momentum is:

a) not conserved. b) conserved. c) zero. d) never zero. e) always negative.

13. A collision or explosion is an event in which relatively strong forces act between objects for a relatively short time. If one considers all the objects involved in the collision or explosion as constituting one system, then frequently in calculations it is useful to use the ____________ principle provided the external forces acting on the system can be considered negligible compared to the internal forces of the system during the collision or explosion.

    a) conservation of momentum b) conservation of mechanical energy c) cosmological
    d) anthropic e) Peter

14. A 50 kg girl (who is initially at rest) dives horizontally at 2.0 m/s in the **POSITIVE** direction from a 200 kg boat that initially is at rest. Given that the event is an ideal collision event, what is the initial recoil **VELOCITY** of the boat?

    a) 0.25 m/s. b) 0.70 m/s. c) −0.70 m/s. d) −0.50 m/s. e) infinite.

15. "Let's play *Jeopardy*! For $100, the answer is: An interaction between two or more bodies where the interaction forces are so strong and duration so short that only the interaction forces between the bodies are significant during the interaction: other forces that arise from sources external to the interaction can be neglected during the interaction. Often the interaction can be regarded as instantaneous on the time scale of the evolution of the system of bodies before and after the interaction."

    What is ____________, Alex?

    a) linear momentum b) uniform circular motion c) proper motion
    d) a weekend in Las Vegas e) an ideal collision in physics

16. An impulse $\vec{I}$ is:

    a) simply the sum of the momentum components of a momentum vector.
    b) a force on an object times the time Δt it acts on the object:

    $$\vec{I} = \vec{F}\Delta t \ .$$

    If the force is not constant (which actually the usual case in collisions for example), then

    $$\vec{I} = \int_{\Delta t} \vec{F}(t)\,dt \ ,$$

    where the integral is over Δt. Net impulse yields a new version of Newton's 2nd law:

    $$\vec{I}_{\rm net} = \int_{\Delta t} \vec{F}_{\rm net}(t)\,dt = \int_{\Delta t} \frac{d\vec{p}}{dt}\,dt = \Delta\vec{p} \ :$$

    i.e., $\vec{I}_{\rm net}$ equals the change in momentum of the object. The concept of impulse is used mainly for strong, short-duration forces (i.e., impulsive forces) such as those that occur in collisions and explosions. In such cases, the new version of the 2nd law can be approximated by just using the impulse of the impulsive forces on the left-hand side if they give the only significant contributions to the net impulse.
    c) a sudden desire.
    d) the opposite of an expulse.
    e) an effect used to drive an impulse engine. If your warp drive fails and you are on impulse and 30,000 light-years from Rigel 8, then it's time to get out the card games.

17. In a collision, the impulsive forces are internal to the system of colliding objects: i.e., they do not sum to a net force on the system. Assuming external net force on the system for the short duration of the collision is insignificant compared to the internal forces, then by Newton's 2nd law there is no change in the total momentum of the system. This means that ____________ conserved through the collision.

    a) total kinetic energy **IS** b) total mass **IS** c) total momentum **IS**
    d) total momentum is **NOT** e) total mass is **NOT**

18. In elementary discussions, collisions are usually divided into three types:

a) inelastic, completely inelastic, elastic. b) inelastic, hardly inelastic, elastic.
c) inelasticated, completely inelasticated, elasticated.
d) inelasticated, hardly inelasticated, elasticated. e) good, bad, ugly.

19. In a completely (or perfectly) inelastic collision:

    a) total kinetic energy is conserved. b) the colliding objects bounce off each other.
    c) total momentum is **NOT** conserved. d) the colliding objects stick together.
    e) the colliding objects have no hair.

20. Two identical point masses elastically collide in a 1-dimensional setup. Mass 1 had velocity v before the collision and velocity zero after. Mass 2's velocity was initially zero and after the collision it was:

    a) v. b) $2v$. c) $-v$. d) $-2v$. e) still zero.

21. In a 1-dimensional elastic collision the initial velocity $= v_{\rm rel,i}$ and final relative $v_{\rm rel,f}$ are related by:

    a) $v_{\rm rel,f} = v_{\rm rel,i}^2$. b) $v_{\rm rel,f} = 2v_{\rm rel,i}$. c) $v_{\rm rel,f} = -2v_{\rm rel,i}$. d) $v_{\rm rel,f} = \frac{1}{2}v_{\rm rel,i}$.
    e) $v_{\rm rel,f} = -v_{\rm rel,i}$.

22. A ball hits a wall with an incident angle of $30°$ measured from a normal to the wall. Consider the normal as a pole: angles measured from the pole are polar angles; those around the pole are azimuthal angles. The ball rebounds at polar angle:

    a) $30°$ in a plane **PERPENDICULAR** to the one defined by the incident path and the normal.
    b) $30°$, but in a random azimuthal direction.
    c) $60°$, but in a random azimuthal direction.
    d) $60°$ in a plane **PERPENDICULAR** to the one defined by the incident path and the normal.
    e) $30°$ in the plane **DEFINED** by the incident path and the normal.

23. The operation of a rocket in space is based on:

    a) conservation of momentum. b) conservation of angular momentum. c) jet fuel pushing on the vacuum. d) starlight pressure. e) running an internal treadmill.

24. The force that accelerates a rocket in the vacuum of space is the:

    a) force of the vacuum. b) force exerted by the ejected fuel on the rocket: the **THRUST** force. c) force exerted by the ejected fuel on the rocket: the **THRUSH** force. d) force of light pressure. e) Force.

25. You have a 1-dimensional object of length a: i.e., a non-uniform rod. It's linear density (i.e., mass per unit length) is given by

$$\lambda = Cx^p ,$$

    where C is a constant, $p > -1$ (i.e., p is a real number greater than -1) and the origin is at one end of the rod. Find the center of mass of the rod.

26. A shell of mass m is at a certain instant is exactly in horizontal flight with velocity v in the positive x-direction. At that same instant the shell explodes instantaneously (virtually instantaneously to be more realistic) into 3 fragments: two fragments have zero x velocity, one going straight up, the other straight down; the 3rd fragment, which has mass m_3, just at the instant after explosion continues in the x direction with only its x velocity nonzero.

    a) What is the x-direction velocity v_3 of the 3rd fragment in terms of given variables? Derive the formula for v_3 from a basic principle.

    b) Just after the explosion, what must the total y-direction momentum be of the two fragments that went straight up and down? Explain why it is this value.

    c) How far does the 3rd fragment go in the x direction if it starts a height y_0 above a flat plain assuming no air drag? Derive the formula for the distance including a derivation of the the horizontal-launch projectile range formula. Give the answer in terms of given variables and g.

27. Your dog King—a she, but somehow that wasn't obvious before the puppies ("Ooh, arn't they sweethearts.")—is on a long flat-bottomed boat (10 m in length to be precise) on a placid lake. King now

muscles in at 90 kg—you must not have been listening when her grandsire Gross Hans the St. Bernard was being aired—"he could haul an SUV". You, also on the boat, however, are a petite 50 kg—which makes opening the door when you get home from work, King keening just inside, its own demonstration in macroscopic conservation of momentum. The boat is 60 kg: it's made of that light-weight stuff that the astronauts ate on the Moon. It is fore-aft symmetric: i.e., its center of mass is exactly **MIDSHIPS**. Assume there is no net external force acting on the boat-King-you system and that the system is **ONE-DIMENSIONAL** along the boat axis.

a) Act 1, Scene I: The boat is at rest (or even becalmed): King's motionless (thankfully) at the bow (bowwow?); you are at the stern likewise enjoying a breather. Give the formula in terms of **VARIABLES** (i.e., **SYMBOLS**) for the center-of-mass position of the total boat-King-you system using the stern as the **ORIGIN** and recall the system is one-dimensional. Now what is the center-of-mass position (i.e., what is the numerical value)? **HINT:** Draw a diagram.

b) Act 1, Scene II: King in a sudden attack of nautical anxiety needs some loving and, after a brief and inconsequential acceleration phase, bounds toward you at 6.0 m/s **RELATIVE** to the boat: this, of course, should be −6.0 m/s if the positive direction is aft-to-fore. (We are really treating King as free-flying object that is not constantly pounding her paws on the deck. She's an ideal dog.) Find the formula in terms of **VARIABLES** for the boat's velocity relative to the water now. Show how you got the formula.

   What is the boat's velocity relative to the water? What is your velocity relative to the water? What is King's velocity relative to the water? What is the boat-King-you system's center of mass velocity relative to the water? Take the aft-to-fore direction as positive. Numerical values are expected.

c) Act 1, Scene III: King makes a **COMPLETELY INELASTIC COLLISION** with you at the stern: i.e., she sticks to you with a deep, warm feeling—"Belay King, Belay!" Assuming you don't all tip backward into the drink, what are the velocities of you, King, the boat, and the boat-King-you system center of mass now? Briefly explain your answers.

28. Richthofen and Mannock are two goats with masses 40 kg and 50 kg, respectively. They charge each other head-on on a icy pond—amazingly they stay upright, but goats are sure-hooféd. Just before impact Richthofen is moving at 10.0 m/s—he's way out of control—and Mannock's moving at −5.0 m/s. On impact they lock horns literally (i.e., stick together). Treat the goats as point masses and their collision as an ideal collision (i.e., one in which only the collision forces are non-negligible). The problem is entirely 1-dimensional.

    a) What type of collision has occurred? It is one of the basic kinds of collisions.

    b) What is the center-of-mass velocity of the Richthofen-Mannock system just **BEFORE** and just **AFTER** the collision? Make the collision approximation. Be explicit: which velocity is before and which is after?

    c) Assuming that the coefficient of kinetic friction μ_{kin} between goat and ice is 0.13, what is the displacement of the unhappy Richthofen-Mannock system from the impact site to where it stops moving?

29. Two objects collide elastically in a 1-dimensional setup. The first has mass $m_1 = 1.00$ kg and initial velocity $v_1 = 6.00$ m/s and the second, mass $m_2 = 8.00$ kg and initial velocity $v_2 = 3.00$ m/s. Note initial velocities are given, not speeds. What are the **TWO** final velocities?

30. Now you can't (to good approximation) push or pull on space. But one can take a swing about a celestial body. This maneuver in NASAese is called a slingshot encounter. Say we have a space probe on Jupiter's orbital path opposite to Jupiter's motion. The path can be regarded as straight line over the distance scale of our problem. The probe mass has $m = 1000$ kg and Jupiter has $m_{\text{Ju}} = 1.8988 \times 10^{27}$ kg ≈ 318 Earth masses. The probe velocity $v_{\text{probe}\ i} = -10$ km/s and Jupiter's velocity $v_{\text{Ju}\ i} = 13.1$ km/s: both velocities are relative to the Sun's rest frame. The encounter is planned such that, if all goes well, the probe does 180° swing around Jupiter (i.e., a U-turn). Regarding the encounter as a 1-dimensional elastic collision what is the outgoing probe speed? Now why does NASA uses slingshot shot encounters? Give two reasons.

31. Teenagers on ice—thin ice—each of mass 50 kg, slide together and cling. They are both moving at 3 m/s prior to the collision. An old codger of 75 kg moving at 2 m/s then collides with them and the whole

conglomerate then comes to a dead stop. What is the angle between the initial velocity vectors of the two teenagers. **HINT:** A diagram might help.

32. Lost deep in the void of eternal space there is a rocket in an inertial frame. At time zero, it's initial velocity is v_0. Suddenly its engine fires ejecting fuel at a constant exhaust speed v_{ex} (i.e., a constant relative velocity with respect to the rocket).

    a) What is the general formula for the ratio of mass to initial mass (i.e., m/m_0) in terms of velocity v, v_0, and v_{ex}?

    b) What is the mass ratio when v is equal to the exhaust speed? After the time of this event what is the direction of motion of ejected fuel relative to the inertial frame?

    c) What is the mass ratio when v is equal to the exhaust speed if $v_0 = 0$?

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67428(67) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2006, CODATA)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\mathrm{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\mathrm{sphere}} = 4\pi \qquad d\Omega = \sin\theta\, d\theta\, d\phi$$

3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1-x} \approx 1 + x \ : \ (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

5 Quadratic Formula

$$\text{If} \qquad 0 = ax^2 + bx + c\ , \qquad \text{then} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series
$$f(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0)$$
$$= f(x_0) + (x - x_0) f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0 \qquad v^2 = v_0^2 + 2a(x - x_0)$$

$$x = \frac{1}{2}(v_0 + v)t + x_0 \qquad x = -\frac{1}{2}at^2 + vt + x_0 \qquad g = 9.8\,\mathrm{m/s^2}$$

$$x_{\text{rel}} = x_2 - x_1 \qquad v_{\text{rel}} = v_2 - v_1 \qquad a_{\text{rel}} = a_2 - a_1$$

$$x' = x - v_{\text{frame}} t \qquad v' = v - v_{\text{frame}} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\text{avg}} = \frac{\Delta \vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0} t \qquad y = -\frac{1}{2} g t^2 + v_{y,0} t + y_0 \qquad v_{x,0} = v_0 \cos\theta \qquad v_{y,0} = v_0 \sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0 \cos\theta} \qquad y = y_0 + x \tan\theta - \frac{x^2 g}{2 v_0^2 \cos^2\theta}$$

$$x_{\text{for } y \text{ max}} = \frac{v_0^2 \sin\theta \cos\theta}{g} \qquad y_{\text{max}} = y_0 + \frac{v_0^2 \sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2 v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g} \qquad \theta_{\text{for max}} = \frac{\pi}{4} \qquad x_{\text{max}}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0 \sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2 r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\tan} = r\alpha$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = v\omega$$

13 **Very Basic Newtonian Physics**

$$\vec{r}_{\mathrm{cm}} = \frac{\sum_i m_i \vec{r}_i}{m_{\mathrm{total}}} = \frac{\sum_{\mathrm{sub}} m_{\mathrm{sub}} \vec{r}_{\mathrm{cm\ sub}}}{m_{\mathrm{total}}} \qquad \vec{v}_{\mathrm{cm}} = \frac{\sum_i m_i \vec{v}_i}{m_{\mathrm{total}}} \qquad \vec{a}_{\mathrm{cm}} = \frac{\sum_i m_i \vec{a}_i}{m_{\mathrm{total}}}$$

$$\vec{r}_{\mathrm{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\mathrm{total}}}$$

$$\vec{F}_{\mathrm{net}} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,\mathrm{m/s^2}$$

$$\vec{F}_{\mathrm{normal}} = -\vec{F}_{\mathrm{applied}} \qquad F_{\mathrm{linear}} = -kx$$

$$f_{\mathrm{normal}} = \frac{T}{r} \qquad T = T_0 - F_{\mathrm{parallel}}(s) \qquad T = T_0$$

$$F_{\mathrm{f\ static}} = \min(F_{\mathrm{applied}}, F_{\mathrm{f\ static\ max}}) \qquad F_{\mathrm{f\ static\ max}} = \mu_{\mathrm{static}} F_{\mathrm{N}} \qquad F_{\mathrm{f\ kinetic}} = \mu_{\mathrm{kinetic}} F_{\mathrm{N}}$$

$$v_{\mathrm{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\mathrm{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\mathrm{centripetal}} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\mathrm{centripetal}} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\mathrm{drag,lin}} = bv \qquad v_{\mathrm{T}} = \frac{mg}{b} \qquad \tau = \frac{v_{\mathrm{T}}}{g} = \frac{m}{b} \qquad v = v_{\mathrm{T}}(1 - e^{-t/\tau})$$

$$F_{\mathrm{drag,quad}} = bv^2 = \frac{1}{2}C\rho A v^2 \qquad v_{\mathrm{T}} = \sqrt{\frac{mg}{b}}$$

14 **Energy and Work**

$$dW = \vec{F}\cdot d\vec{s} \qquad W = \int \vec{F}\cdot d\vec{s} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\mathrm{mechanical}} = KE + PE$$

$$P_{\mathrm{avg}} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F}\cdot\vec{v}$$

$$\Delta KE = W_{\mathrm{net}} \quad \Delta PE_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \quad \Delta E = W_{\mathrm{nonconservative}}$$

$$F = -\frac{dPE}{dx} \qquad \vec{F} = -\nabla PE \qquad PE = \frac{1}{2}kx^2 \qquad PE = mgy$$

15 **Momentum**

$$\vec{F}_{\text{net}} = m\vec{a}_{\text{cm}} \qquad \Delta KE_{\text{cm}} = W_{\text{net,external}} \qquad \Delta E_{\text{cm}} = W_{\text{not}}$$

$$\vec{p} = m\vec{v} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \qquad \vec{F}_{\text{net}} = \frac{d\vec{p}_{\text{total}}}{dt}$$

$$m\vec{a}_{\text{cm}} = \vec{F}_{\text{net non-flux}} + (\vec{v}_{\text{flux}} - \vec{v}_{\text{cm}})\frac{dm}{dt} = \vec{F}_{\text{net non-flux}} + \vec{v}_{\text{rel}}\frac{dm}{dt}$$

$$v = v_0 + v_{\text{ex}} \ln\left(\frac{m_0}{m}\right) \qquad \text{rocket in free space}$$

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16 Collisions

$$\vec{I} = \int_{\Delta t} \vec{F}(t)\,dt \qquad \vec{F}_{\text{avg}} = \frac{\vec{I}}{\Delta t} \qquad \Delta p = \vec{I}_{\text{net}}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \qquad \vec{v}_{\text{cm}} = \frac{\vec{p}_1 + \vec{p}_2}{m_{\text{total}}}$$

$$KE_{\text{total } f} = KE_{\text{total } i} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{1'} = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2} \qquad \text{1-d Elastic Collision Expression}$$

$$v_{2'} - v_{1'} = -(v_2 - v_1) \qquad v_{\text{rel}'} = -v_{\text{rel}} \qquad \text{1-d Elastic Collision Expressions}$$