Intro Physics Semester I **Name:**

Homework 8: Potential Energy and Mechanical Energy: One or two or no full answer questions will be marked. There will also be a mark for completeness. Homeworks are due usually the day after the chapter they are for is finished. Solutions will be posted soon thereafter. The solutions are intended to be (but not necessarily are) super-perfect and often go beyond a fully correct answer.

Answer Table

Name:

	a	b	c	d	e
1.	O	O	O	O	O
2.	O	O	O	O	O
3.	O	O	O	O	O
4.	O	O	O	O	O
5.	O	O	O	O	O
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12.	O	O	O	O	O
13.	O	O	O	O	O
14.	O	O	O	O	O
15.	O	O	O	O	O
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25.	O	O	O	O	O
26.	O	O	O	O	O
27.	O	O	O	O	O
28.	O	O	O	O	O
29.	O	O	O	O	O
30.	O	O	O	O	O

	a	b	c	d	e
31.	O	O	O	O	O
32.	O	O	O	O	O
33.	O	O	O	O	O
34.	O	O	O	O	O
35.	O	O	O	O	O
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38.	O	O	O	O	O
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40.	O	O	O	O	O
41.	O	O	O	O	O
42.	O	O	O	O	O
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44.	O	O	O	O	O
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46.	O	O	O	O	O
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48.	O	O	O	O	O
49.	O	O	O	O	O
50.	O	O	O	O	O
51.	O	O	O	O	O
52.	O	O	O	O	O
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54.	O	O	O	O	O
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58.	O	O	O	O	O
59.	O	O	O	O	O
60.	O	O	O	O	O

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008 qmult 00100 1 1 2 easy memory: potential energy definition
Extra keywords: physci

1. Potential energy is:

   a) the energy of position: it exists for nonconservative forces.
   b) the energy of position: it exists for conservative forces.
   c) the energy of motion: its formula is $PE = (1/2)mv^2$.
   d) the energy of position: its formula is $PE = (1/2)mv^2$.
   e) heat energy.

   SUGGESTED ANSWER: (b) There are plenty of clues.

   Wrong answers:
   e) Nah.

   Redaction: Jeffery, 2001jan01

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008 qmult 00110 2 1 3 moderate memory: paths and a conservative force

2. The work done by a conservative force on an object while the object moves on a path between two endpoints is:

   a) **INDEPENDENT** of the path and endpoints.
   b) **DEPENDENT** on the path.
   c) **INDEPENDENT** of the path between the endpoints.
   d) **DEPENDENT** on the path, but **NOT** on the endpoints.
   e) equal to the path length.

   SUGGESTED ANSWER: (c)

   Wrong answers:
   a) This seems to say that in general potential energy is independent of position altogether. Which is wrong.
   d) Exactly wrong.
   e) Not a dimensionally correct answer to say the least.

   Redaction: Jeffery, 2001jan01

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008 qmult 00120 1 4 1 easy deducto-memory: general potential energy formula

3. "Let's play *Jeopardy*! For \$100, the answer is: $\Delta PE = -W$."

   a) What is the formula relating **POTENTIAL** energy change in a conservative force field to work done by the conservative force (i.e., what is the general potential energy formula), Alex?
   b) What is Faraday's law, Alex?
   c) What are capacitors, Alex?
   d) What is ... no, no wait ... what is unicorn circular motion, Alex?
   e) What is the formula relating **KINETIC** energy change in a conservative force field to work done by the conservative force (i.e., what is the work-kinetic-energy theorem), Alex?

   SUGGESTED ANSWER: (a)

   Wrong answers:
   d) A rhinoceros chasing its tail?
   e) U is pretty much common for potential energy and never used for kinetic energy to my knowledge.

   Redaction: Jeffery, 2001jan01

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008 qmult 00130 1 4 5 easy deducto-memory: potential energy

4. "Let's play *Jeopardy*! For \$100, the answer is: Energy X for a force is an energy type defined, not by its particular intrinsic nature, but because its value for a body is set by the body's location in space. So energy X is a position energy—and probably should have been called that—but it's too late to change centuries of tradition. It is always true that in any real physical case of energy X, the energy is by its nature some kind of field energy: e.g., electric field energy, magnetic field energy, gravitational field

energy. A particular field energy may be a potential energy or not depending on the actual system considered. Energy X for some unreal imagined kind of force does not have any more fundamental explanation—unless one imagines one.”

What is ____________, Alex?

a) polecat b) pole energy c) potentate energy d) potent energy e) potential energy

SUGGESTED ANSWER: (e)

Wrong answers:
a) As Lurch would say AAAARGH.

Redaction: Jeffery, 2008jan01

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008 qmult 00140 1 1 5 easy memory: conservative force and system

5. Whether a force is conservative or not depends not only on the fundamental nature of the force, but also on the ____________ which is being considered. For example, the electric force is usually mentioned as a conservative force, but there are ____________ in which it is not: for example, those in which it is generated by the Maxwell-Faraday’s law. For another example, the magnetic force is often mentioned as a non-conservative force, but there ____________ in which it is: for example, magnetic dipoles are subject to a conservative magnetic force.

a) force/forces b) horse/horses c) law/laws d) motion/motions e) system/systems

SUGGESTED ANSWER: (c)

Wrong answers:
d) Not a best answer even if you argue till you are blue in the face.

Redaction: Jeffery, 2008jan01

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008 qmult 00142 1 1 4 easy memory: conservative force examples

6. Two forces that are conservative in ordinarily-thought-of systems are:

a) power and might. b) gravity and kinetic friction. c) gravity and work.
d) gravity and the linear (or spring) force. e) work and the linear (or spring) force.

SUGGESTED ANSWER: (d)

If you don’t know about the linear force, you can still solve the problem by deduction.

Wrong answers:
a) I don’t think might has ever found a home in physics-speak.

Redaction: Jeffery, 2001jan01

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008 qmult 00180 1 4 2 easy deducto-memory: energy and heat
Extra keywords: physci KB-73

7. British American Benjamin Thompson (1753–1814)—ennobled as Count Rumford—while employed as director of the Bavarian arsenal, noticed that in boring cannon—but not causing cannon ennui—that the boring motion and friction seemed to produce unlimited amounts of heat. He concluded:

a) heat was a substance of which there could only be so much of in any object.
b) that heat was somehow generated by motion and friction. This conclusion eventually led to the recognition of heat as another form of energy that could be converted from or converted into, e.g., mechanical or chemical energy and to the concept of conservation of energy.
c) that heat had no relation to motion and friction and was somehow spontaneously generated by cannon.
d) that cannon could be the plural of cannon.
e) that the biergartens in Munich were much better than the taverns in Boston and that Sam Adams, patriot-founding-father notwithstanding, could have learnt a thing or two about brewing beer.

SUGGESTED ANSWER: (b)

Putting a thing or two together the answer should be obvious. This question exemplifies my belief that some questions should be easy, but should drive in an idea like a spike. Anyway I used to stroll by Thompson/Rumford's statue in the Englischer Gartens sometimes in my Munich days. Hm—should they be called the Amerikaner Gartens if they were named after Thompson/Rumford as I vaguely seem to recall. There's a very pleasant biergarten, Der Chinisischer Turm in the Englischer Gartens.

Yes, Thompson/Rumford was loyal—when other Americans we re/denouncing their King, he stood up for the Union Jack—prefering perpetual exile—from Woburn, Massachusetts—to life among rebels.

Wrong answers:

a) In the 18th century, one theory of heat held that it was a substance that was conserved independently of anything else: a subtle fluid perhaps.

d) Probably English speakers (or as we call them in Canada Anglophones or Anglos or darned Anglos) already knew this in the 18th century and it's not even relevant either.

e) Personal experience suggests this was true in the late 20th century, but for the 18th century I'm just guessing. It certainly isn't the best answer in the context of the question.

Redaction: Jeffery, 2001jan01

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008 qmult 00200 1 4 1 easy deducto-memory: work-energy theorem

8. "Let's play *Jeopardy*! For \$100, the answer is: $\Delta E = W_{\text{nonconservative}}$."

What is the ____________, Alex?

a) work-energy theorem   b) work-kinetic-energy theorem   c) potential-energy-work formula
d) work-potential-energy theorem   e) kinetic energy formula

SUGGESTED ANSWER: (a)

Wrong answers:

e) As Lurch would say AAAARGH.

Redaction: Jeffery, 2008jan01

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008 qmult 00210 1 1 4 easy memory: mechanical energy conservation

Extra keywords: physci

9. Mechanical energy is the sum of kinetic energy and potential energy. It is a conserved quantity:

a) always.
b) whenever it has both kinetic and potential energy components.
c) if all the forces that do net work are **NONCONSERVATIVE**.
d) if all the forces that do net work are **CONSERVATIVE**.
e) whenever it is positive.

SUGGESTED ANSWER: (d)

Wrong answers:

a) Nah.

Redaction: Jeffery, 2001jan01

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008 qmult 00220 1 1 4 easy memory: workless constraint forces

10. Frequently, in conservation-of-mechanical energy problems, one encounters non-conservative forces that guide the motion and cause accelerations. Mechanical energy is conserved because these ____________ do work because they are always ____________ to the direction of motion. Actually, conservative forces can also be ____________ when they are ____________.

a) work-doing constraint forces; parallel   b) work-doing constraint forces; perpendicular
c) workless constraint forces; parallel   d) workless constraint forces; perpendicular
e) worthless unconstrained forces; peculiar

SUGGESTED ANSWER: (d)

Wrong answers:

a) Exaclty wrong.

Redaction: Jeffery, 2008jan01

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008 qmult 00270 1 3 5 easy math: dog drops brick mech. energy conserved
Extra keywords: physci

11. A brick has mass 1 kg. A dog (from a joke that I'll tell you someday) drops the brick (which it was holding in its mouth or, one might say, with its jowl) 1 m. What is the kinetic energy of the brick just before it hits the ground? **HINT:** The calculator is superfluous.

a) 9.8 watts. b) 9.8 gems. c) 9.8 newtons. d) 9.8 jowls. e) 9.8 joules.

SUGGESTED ANSWER: (e)

The potential energy at 1 meter of a 1 kilogram brick is 9.8 joules. If it drops 1 meter its potential energy becomes zero and its kinetic energy becomes 9.8 joules by the conservation of mechanical energy.

Wrong answers:

d) James Prescott Joule (1818–1889) British physicist and brewer proved that mechanical, heat, and chemical energies were all different forms of the same thing within experimental uncertainty. He was one of the last of the great gentleman scientists. He actually pronounced his name jowl (rhymes with bowel), but in the interests of euphony we usually pronounce the unit named after him jool (rhymes with drool).

Redaction: Jeffery, 2001jan01

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008 qmult 00274 2 5 5 moderate thinking: girl on a swing KE/PE
Extra keywords: physci KB-95-19

12. A girl on a swing oscillates between being 2 m off the ground where she is ________ and 1 m off the ground where her speed is a ____________. No nonconservative forces do work. What is her maximum speed?

a) moving; minimum; 0 m/s. b) at rest; minimum; 0 m/s. c) at rest; maximum; 1 m/s.
d) at rest; maximum; 2.4 m/s. e) at rest; maximum; 4.4 m/s.

SUGGESTED ANSWER: (e)

Because nonconservative forces do not do any net work, mechanical energy (KE plus PE) is conserved. In this system the energy will oscillate back and forth between KE and PE perpetually.

At the highest point of the cycle the swing is momentarily at rest and all the mechanical energy PE. At the lowest point of the cycle the PE is at minimum and KE must be at a maximum by conservation of energy. Let us take the lowest point as the zero point of PE for convenience and let the height coordinate be y. Recall we are always free to choose the zero point of PE since only changes in PE affect any other quantity. The lowest point is then zero PE.

In general,

$$\Delta E_{\text{mechanical}} = W_{\text{nonconservative}} \ .$$

In this case, $W_{\text{nonconservative}} = 0$, and so

$$0 = \Delta E_{\text{mechanical}} = \left(\frac{1}{2}mv^2 + mgy\right) - \left(\frac{1}{2}mv_0^2 + mgy_0\right) = \frac{1}{2}mv^2 - mgy_0$$

since the initial KE is zero and the final PE is zero. Algebra then gives

$$mgy_0 = \frac{1}{2}mv^2$$

which leads to

$$v = \sqrt{2gy_0} = \sqrt{2 \times 9.8 \times 1} = 4.43\,\text{m/s} \ .$$

Note that we did not have to solve for the motion with Newton's laws. We found the solution more easily using energy. But on the other hand we do not know everything. We do not know the position of the girl as a function of time. To do that we do have to solve for the whole motion. That is a harder calculation, but it can be done.

The situation is typical: energy methods give some information easily, but not complete information.

Fortran Code

```
print*
gg=9.80
hh=1.
vmax=sqrt(2.*gg*hh)
print*,'vmax=',vmax   ! 4.42719
```

Wrong answers:

a) There is an instant of rest at the highest point of the oscillation as you well know from all those days in the playground.

Redaction: Jeffery, 2001jan01

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008 qmult 00360 1 1 1 easy memory: turning points in a potential well

13. An object is trapped and moving around in some kind of potential well: it's a bound particle we'd say. What are those special points called where the kinetic energy of the particle momentarily goes to zero? Why are they so called?

a) Turning points—so called because when a particle reaches one, it must come to rest and reverse direction.
b) Stable static equilibrium points—so called because when a particle reaches one it stops.
c) Stable static equilibrium points—so called because when a particle reaches one it accelerates.
d) Unstable static equilibrium points—so called because when a particle reaches one it accelerates.
e) Turning points—so called because of the film "The Turning Point" starring Shirley Maclaine, Anne Bancroft, and Mikhail Barishnykov.

SUGGESTED ANSWER: (a)

Equilibrium points are stationary points for smooth potential energies. But if the potential is unsmooth, one could have a potential energy minimum or maximum at some kind of cusp. Such a minimum would be a stable static equilibrium too. Of course, at the microscopic level I think potential energies are almost always smooth.

Wrong answers:

e) Had great dancing as recall.

Redaction: Jeffery, 2001jan01

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008 qmult 00370 1 1 1 easy memory: SHO for small perturbations

Extra keywords: maybe should be moved to oscillations chapter

14. If one makes a sufficiently small displacement from stable static equilibrium of almost any system a with smooth potential energy function and then lets the system evolve in isolation, the system will approximate a/an:

a) simple harmonic oscillator. b) anharmonic oscillator. c) traveling wave.
d) unstable equilibrium system. e) a vector field.

SUGGESTED ANSWER: (a)

Wrong answers:

b) This will often happen if the perturbation is not sufficiently small.
d) No.
e) A nonsense answer.

Redaction: Jeffery, 2001jan01

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008 qmult 00380 1 5 3 easy thinking: perfect balance scale

Extra keywords: seems a miscellaneous question in chapter 8

15. Why can't a practical balance scale ever be in perfect unstable static equilibrium when balancing?

a) Nothing is perfect.
b) Everything is perfect.

c) There is always some frictional force that makes the balance position, however, slightly metastable or the scale never really balances: the perturbations just grow so slowly that one can make a measurement before they become obvious. Of course, the less the stabilizing friction and the less the perturbations, the **MORE** exactly a mass can be determined.
d) There is always some frictional force that makes the balance position, however, slightly metastable or the scale never really balances: the perturbations just grow so slowly that one can make a measurement before they become obvious. Of course, the less the stabilizing friction and the less the perturbations, the **LESS** exactly a mass can be determined.
e) No one wants to.

SUGGESTED ANSWER: (c)

I think this discussion is broadly unimpeachable (unlike certain presidents one could name), but I suppose a refined discussion might be more exactly true.

Wrong answers:
a) Some things are defined to be perfect: e.g, 6 is a perfect number in that it is the sum of its factors (excluding itself): $1+2+3=6$. There are 35 known perfect numbers: all of them even; it is not known if there can be odd perfect numbers. See

http://www.maths.uts.edu.au/numericon/perfect.html

for more on perfect numbers.
b) Well 1 isn't a perfect number.
d) Exactly wrong.
e) That no one wants to do something doesn't make the thing impossible to do.

Redaction: Jeffery, 2001jan01

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008 qmult 00400 1 1 1 easy memory: power definition
Extra keywords: physci
16. Work per unit time or energy transformed per unit time is:

a) power. b) might. c) oomph. d) strength. e) pay.

SUGGESTED ANSWER: (a)

Wrong answers:
e) Not the best answer in this context.

Redaction: Jeffery, 2001jan01

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008 qmult 00470 1 5 3 easy thinking: sunlight power
Extra keywords: physci KB-99-17
17. If you could capture it all for useful work, the energy sunlight delivers to a square meter of ground would run one or two ordinary incandescent light bulbs. The power delivered by the Sun to a square meter of ground on average is to order or magnitude:

a) 1 W. b) 10 W. c) 100 W. d) 10^6 W. e) 1 MW.

SUGGESTED ANSWER: (c)

The solar constant is the solar flux above the atmosphere in the direction perpendicular to the cross sectional area of the Earth: it is about $1370\,\mathrm{W/m^2}$. The Earth captures solar power over an area of πR^2, where R is the radius of the Earth. But this must be spread on average over the Earth's spherical surface which has an area of $4\pi R^2$. This reduces the average capture per square meter by a factor of 4. Only about half of this captured power reaches the ground. Thus at the ground, the Sun delivers about $170\,\mathrm{W/m^2}$ (Smil2006–27).

Rounding $170\,\mathrm{W/m^2}$ to order of magnitude gives $100\,\mathrm{W/m^2}$.

Wrong answers:
a) C'mon, ordinary light bulbs require tens to hundreds of watts.
e) A megawatt is the same as 10^6 W.

Redaction: Jeffery, 2001jan01

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008 qmult 00472 2 5 2 moderate thinking: boy running up stairs
Extra keywords: physci KB-93-23

18. A 50 kg boy runs up a flight of stairs of 5 m in height in 5 s at a constant rate. His power output just to work against gravity is:

    a) 50 W. b) 490 W. c) 980 W. d) 10^6 W. e) 1 MW.

SUGGESTED ANSWER: (b)

Note we are just counting his power to do work against gravity. A real boy—unlike Pinocchio—would have use power to work against friction internal and external and to make motions that sustain his balance as well as all the usual life sustaining bodily functions. But the question only concerns his work against gravity. Also he is an ideal boy.

The calculation is

$$P = \frac{W}{\Delta t} = \frac{mg\Delta y}{\Delta t} = \frac{50 \times 9.8 \times 5}{5} = 490\,\mathrm{W} \ ,$$

where P is power, W is work done, $g = 9.8\,\mathrm{m/s^2}$ m is mass, Δy is the change in height, and $mg\Delta y$ is the gravitiatonal potential energy change which must equal the work done against gravity.

Wrong answers:

e) Superboy flies again.

Redaction: Jeffery, 2001jan01

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008 qmult 00474 2 5 2 moderate thinking: mountain climber power output
Extra keywords: physci KB-95-7

19. A 100 kg mountain climber climbs 4000 m in 10 hours. What is his power output going into gravitational potential energy? What is his total power output?

    a) 3.92×10^6 W and 3.92×10^6 W.
    b) The power going into gravitational potential energy is 109 W. His total power output cannot be exactly calculated since a lot of power must go into waste heat due to frictional forces and into the body heat which is lost to the environment. All one can easily say is that 109 W is a **LOWER BOUND** on the total power output.
    c) The power going into gravitational potential energy is 3.92×10^6 W. His total power output cannot be exactly calculated since a lot of power must go into waste heat due to frictional forces and into the body heat which is lost to the environment. All one can easily say is that 3.92×10^6 W is a **LOWER BOUND** on the total power output.
    d) The power going into gravitational potential energy is 3.92×10^6 W. His total power output cannot be exactly calculated since a lot of power must go into waste heat due to frictional forces and into the body heat which is lost to the environment. All one can easily say is that 3.92×10^6 W is an **UPPER BOUND** on the total power output.
    e) The power going into gravitational potential energy is 109 W. His total power output cannot be exactly calculated since a lot of power must go into waste heat due to frictional forces and and into the body heat which is lost to the environment. All one can easily say is that 109 W is an **UPPER BOUND** on the total power output.

SUGGESTED ANSWER: (b)

Fortran Code

```
      print*
      xmass=100.
      gg=9.8
      hh=4.e+3
      tt=3600.*10.
      energy=xmass*gg*hh
      power=xmass*gg*hh/tt
      print*,'energy,power'
      print*,energy,power
*           3920000.       108.8889
```

Wrong answers:

e) All things are wrong.

Redaction: Jeffery, 2001jan01

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008 qfull 00200 1 3 0 easy math: work-energy theorem proven

20. We will now prove the work-kinetic energy theorem. Don't panic.

a) Write down the work-kinetic-energy theorem.

b) Separate the work done W in the work-kinetic-energy theorem in to that done by conservative forces and that done by non-conservative forces. Nothing forbids us from doing this.

c) Use the general formula for the potential energy of conservative forces to eliminate the work done by the conservative forces.

d) Show that the sum of the changes in the kinetic and potential energies equals the work done by the non-conservative forces. This is the work-energy theorem.

e) Another form of the work-energy theorem formula is obtained by defining mechanical energy by

$$E = KE + PE \ .$$

Write it down.

f) If the non-conservative forces do no work—but they may be present as workless constraint forces—what is conserved? What is not necessarily conserved in this case.

SUGGESTED ANSWER:

a) Behold:

$$\Delta KE = W \ ,$$

where ΔKE is the change in the center-of-mass kinetic energy caused by the net work done on the body by a net external force during a center-of-mass motion.

b) Behold:

$$\Delta KE = W_{\rm con} + W_{\rm non} \ .$$

c) The general formula for potential energy is $\Delta PE = -W$, where W is the work done by a conservative force and PE is the force's potential energy. Now we have

$$\Delta KE = -\Delta PE + W_{\rm non} \ .$$

d) Well in the part (c) answer, the PE term is just begging to be moved to the other side of the equation. Following its desire, we get

$$\Delta KE + \Delta PE = W_{\rm non}$$

which is the work-energy theorem.
Actually, one can leave some of the conservative work in the formula on the right-hand side, and so treat some conservative forces as non-conservative forces. This is sometimes a useful trick.

e) Behold:

$$\Delta E = W_{\rm non}$$

which is the other form of the work-energy theorem.

f) If $W_{\rm non} = 0$, then from the work-energy theorem $\Delta E = 0$, and thus mechanical energy is conserved. Kinetic and potential energy are not necessarily conserved in this case.

Redaction: Jeffery, 2008jan01

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008 qfull 00210 1 3 0 easy math: gravity PE to KE, spanner

21. You drop a 2.0 kg spanner from rest to a friend standing on the ground which 10 m below the drop height. She will catch the spanner 1.5 m above the ground. Neglect air drag.

a) What's the work done by gravity in the drop?

b) What's the change in gravitational potential energy in the drop?

c) Using an energy conservation calculation find the speed of the spanner as it reaches your friend's hands.

SUGGESTED ANSWER:

a) To about 2-digit accuracy ,

$$W_g = mg|\Delta y|\cos\theta \approx 170\,\mathrm{J} \ .$$

Note $\Delta y < 0$ if y is height, but only the magnitude comes into the work formula. The $\cos\theta$ is the cosine of the angle between the force which is mg down and the displacement which is $|\Delta y|$ down.

b) Well

$$\Delta PE = mg\Delta y \approx -170\,\mathrm{J} \ .$$

One could also use the potential energy work relation:

$$\Delta PE = -W_g \approx -170\,\mathrm{J} \ .$$

c) Well the work-energy theorem is

$$\Delta E_{\mathrm{mech}} = W_{\mathrm{nonconservative}} \ .$$

In this case, $W_{\mathrm{nonconservative}} = 0$, and so we have conservation of the mechanical energy. Skipping the tedious description,

$$KE_{\mathrm{final}} = -\Delta PE \ ,$$

and thus

$$v_{\mathrm{final}} = \sqrt{\frac{2KE_{\mathrm{final}}}{m}} = \sqrt{\frac{-2\Delta PE}{m}} \approx 13\,\mathrm{m/s} \ .$$

Redaction: Jeffery, 2001jan01

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008 qfull 00220 1 5 0 easy thinking: ideal climbing

22. This gentleman of fortune Daniel Goodwin in 1981 scaled the Sears Building in Chicago using suction cups and metal clips. The building is 443 m high. Let's guess he had a mass of 70 kg. Let us assume Dan is an **IDEAL** climber: i.e., all the body chemical energy he puts out goes into his own macroscopic kinetic energy or his own gravitational potential energy and into no other forms ever even eventually.

a) How much body chemical energy did Dan expend getting to the top? He started from rest and ended at rest?

b) If he'd just climbed the stairs, how much body chemical energy again starting and ending at rest.

c) Why did he probably use a lot more body chemical energy than an ideal climber?

SUGGESTED ANSWER:

a) In going to the top his gravitational potential energy increased by $mg\Delta y \approx 3\times10^5\,\mathrm{J}$. If he was an ideal climber starting from rest and ending at rest, his body chemical energy decreased by exactly the same amount.

b) Climbing the stairs makes no difference: his body chemical energy ideally would decrease by $mg\Delta y \approx 3\times10^5\,\mathrm{J}$.

c) Just standing around doing nothing costs energy. You have to pump blood, maintain body temperature, digest food, grow hair, etc. In moving about there is internal and external dissipation of chemical energy to heat. Note that much of the kinetic energy one generates in doing various has to be dissipated to waste heat to complete the act. For example, throwing a hand out to catch something quickly: you must turn chemical energy into kinetic energy to get the hand moving, but then dissipate that energy in stopping the hand to make the catch. He probably used a lot more than $mg\Delta y \approx 3 \times 10^5$ J of body chemical energy.

Redaction: Jeffery, 2001jan01

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008 qfull 00230 3 3 0 tough math: steel dragon

Extra keywords: CJ-177

23. The Steel Dragon in Mie, Japan is one of the world's fastest and tallest roller coasters.

a) Assuming only gravity does work on a coaster find the formula for its speed v at any height y given that its initial speed and height were, respectively, v_0 and y_0. **NOTE:** We actually have to assume that the coaster is a point mass. Otherwise, we would have to worry about the kinetic energy of its internal parts: i.e., its spinning wheels. Note we are neglecting friction and air drag.

b) What does the normal force do in the roller coaster system? **NOTE:** We will assume that the tension in the chain or cable that pulls the coaster is negligible, but this might not be the actual case.

c) Say that $v_0 = 3.0\,\text{m/s}$ and $y_0 = 93.5\,\text{m}$. What is the speed when $y = 0\,\text{m}$?

d) Why can't we calculate the time it takes for the coaster to go from height y_0 to y in the part (c) case?

e) What is the **COMPONENT** of the force of gravity along the track direction and what is the **ACCELERATION** if only gravity is acting along the track direction? Take the angle of the track to the horizontal to be θ.

f) Assuming for the part (c) question that the motion was all downhill and the displacement in the horizontal direction was about the same as in the vertical direction, estimate the travel time between the two locations.

SUGGESTED ANSWER:

a) The work-energy theorem is

$$\Delta E = \Delta KE + \Delta PE = W_{\text{non}} ,$$

where ΔE is the change in mechanical energy, ΔKE is the change in kinetic energy, ΔPE is the change in potential energy, and W_{non} is the work done by nonconservative forces. In this case, $W_{\text{non}} = 0$, $KE = (1/2)mv^2$, and $PE = mgy$. Thus,

$$0 = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 + mg(y - y_0) .$$

Solving for v gives

$$v = \sqrt{v_0^2 + 2g(y_0 - y)} .$$

This is the required formula.

b) The normal force of the track on the coaster is very important to the motion since it helps to shape the path of the coaster. Among other things it prevents the coaster from falling straight down. But the normal force does no work since it is always perpendicular to the direction of motion. This is why we can ignore the normal force in our part (a) answer.

The normal force is in the jargon of physics is a workless constraint force.

c) Just plugging in the numbers into the part (a) answer gives $v = 42.9\,\text{m/s}$.

d) The essential missing ingredient is the shape of the track. With the shape we could in principle calculate the whole kinematics. Of course, the actual practice might be tough.

Say we had slope angle θ as functions of coordinate x: this function is one way of specifying the shape of the track. We could then find the acceleration as a function of x using

$$a = g \sin[\theta(x)]$$

and then we could numerically solve x as a function of time t. For example, we could solve iteratively for x as a function of t using

$$x_i = \frac{1}{2}a_{i-1}\Delta t_i^2 + v_{i-1}\Delta t_i + x_{i-1} \ ,$$
$$v_i = a_{i-1}\Delta t_i + v_{i-1} \ ,$$

where x_i is the x coordinate and v_i is the velocity at

$$t_i = \sum_{j=1}^{i} \Delta t_j$$

which is time in increments of Δt_j from time zero, a_{i-1} is the acceleration at x_{i-1} and v_{i-1} is the velocity at x_{i-1}. These solutions would approach exactness as the Δt_i were made to approach zero. Actually better numerical procedures for solution exist, but would take more words to explain. Analytic solutions are not possible in general although for special cases they are: e.g., for θ a constant.

e) From trigonometry, the component of gravity along the track and the acceleration if only gravity along the track acts are, respectively,

$$F_g = mg\sin\theta \qquad \text{and} \qquad a = g\sin\theta \ ,$$

where we have used Newton's 2nd law to get the acceleration. The directions of both quantities are downhill.

One could make some useful conventions and say that θ is measured from the horizontal direction in the forward direction of the coaster motion with $\theta > 0$ for a increasing height and $\theta < 0$ for decreasing height. Then

$$F_g = -mg\sin\theta \qquad \text{and} \qquad a = -g\sin\theta \ ,$$

with the components being for the forward direction when positive and for the backward direction when negative.

f) With the given conditions, the average angle of the track to the horizontal has to be about 45°. If we approximate the track angle by this value, then the approximate acceleration is constant and given by

$$a = g\sin\theta \ ,$$

where $\theta = 45°$. Making use of the constant-acceleration kinematic equation for velocity,

$$v = at + v_0 \ ,$$

we find

$$t \approx \frac{v - v_0}{a} \approx \frac{43 - 3}{10 \times 0.7} \approx 6\,\text{s} \ .$$

This number is rough, but it is unlikely to be wrong by a factor of 2 for the given conditions. For example, if the coaster just fell straight down from rest,

$$t = \sqrt{\frac{2(y_0 - y)}{g}} \approx 4\,\text{s} \ .$$

Fortran-95 Code

```
v1=3.d0
y1=93.5d0
y=0.d0
gg=9.8d0
v=sqrt(v1**2-2.d0*gg*(y-y1))
print*,'Steel Dragon drop speed v'
print*,v
```

`! 42.91386722261232`

Redaction: Jeffery, 2008jan01

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008 qfull 00260 2 3 0 moderate math: boll weevil and igloo 1

24. A boll weevil of mass m is sitting on top of a hemispherical igloo of radius R. An infinitesimal perturbation starts him sliding down starting from speed **ZERO**. The igloo is frictionless and there is no air drag.

a) Sketch a diagram of the system with the boll weevil at a general position on the igloo. Indicate angle θ and the forces that act on the boll weevil. Note, forces, not force components.

b) Find an explicit formula for the boll weevil's speed on the igloo as a function of angle θ on the igloo measured from the vertical using conservation of mechanical energy. **SHOW** how you found the formula.

c) At some height (measured from the ground) the boll weevil flies off the igloo. Find an explicit formulae for angle and the height at which the boll weevil flies off the igloo. **SHOW** how you found the formulae. **HINT:** This is purely a force analysis problem. Consider the normal force on the boll weevil and note that the radial (component of) acceleration is instantaneously given by the $a = -v^2/r$ just as for uniform circular motion even when v is not constant provided that r is constant. Recall v is just the tangential speed in the radial acceleration formula.

SUGGESTED ANSWER:

a) You will have to imagine the diagram.

b) Gravity is only force that does work on the boll weevil: the normal force is always perpendicular to the direction of motion. Thus conservation of mechanical energy gives

$$E = mgR = \frac{1}{2}mv^2 + mgR\cos\theta \ ,$$

and so

$$v = \sqrt{2gR(1-\cos\theta)} \ .$$

c) If one draws a free body diagram for any position of the boll weevil on the igloo, and analyzes the tangential and radial equations of motion one finds

$$ma_{\rm tan} = mg\sin\theta$$

and

$$ma_{\rm rad} = F_{\rm N} - mg\cos\theta \ .$$

Now the general expression for acceleration in a plane in polar coordinates is

$$\frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(2\frac{dr}{dt}\omega + r\alpha\right)\hat{\theta} \ .$$

If r is constant, then the radial acceleration is $-r\omega^2 = -v^2/r$ just as for uniform circular motion even though the tangential speed v is not constant in time. Thus the radial equation of motion becomes

$$-m\frac{v^2}{R} = F_{\rm N} - mg\cos\theta \ .$$

It's useful to reflect on this equation for a moment. Gravity and the normal force together combine to provide the centripetal acceleration. But for places where they do this gravity is too large or just large enough by itself to provide the centripetal acceleration. The normal force cancels part of gravity and prevents the boll weevil from accelerating into the igloo. But the normal force is a force of reaction (i.e., a constraint force). It would not turn on, if gravity were not trying to pull the boll weevil into the igloo. What if the required centripetal force magnitude exceeded $mg\cos\theta$? Then the boll weevil would fly off and follow a free-fall parabolic trajectory since there is no air drag. The trajectory would show no discontinuity: the slope of the igloo and the parabolic trajectory must be the same when the boll weevil flys off. We have no intrinsic expression for the ideal normal force, but we know it is never attractive. When gravity acting alone is too weak to hold the boll weevil on the circular path, the normal force can't turn into an attractive force and help out: the normal force just goes to zero when this happens.

Now to continue with the math. Substituting for v from the part (a) answer gives

$$F_{\rm N} = mg\,(3\cos\theta - 2) \ .$$

This expression for the normal force decreases monotonically from $\theta = 0$ to $\theta = 90°$ and decreases through zero. From the discussion above, we know the expression is in error for real normal forces when it goes below zero because the expression then gives an attractive force. The expression was derived to match a requirement that the normal force can't match if the requirement requires an attractive force. So when our normal force expression goes to zero and is heading below zero, the boll weevil must fly off the igloo and follow a parabolic trajectory. This happens for

$$\cos\theta = \frac{2}{3} \ , \qquad \theta = \cos^{-1}\left(\frac{2}{3}\right) \approx 48° \ , \qquad \text{height} = \frac{2}{3}R \ .$$

The results for the fly-off angle and height are independent of mass which is not surprising since gravity depends linearly on mass and mass often cancels out in problems where only gravity does work. More surprisingly, the result is independent of g. The height of fly-off is $2/3$ of the igloo height for any constant gravity-field environment: it would be the same on the Moon or Mars.

If one were paranoid, one would show explicitly for consistency that the parabolic path after the fly-off was higher than the the circular path of the igloo. It's not so hard to do, but I'm not that paranoid at the moment.

Redaction: Jeffery, 2001jan01

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008 qfull 00270 3 5 0 tough thinking: Dingo slides on a loop falls off not

25. Dingo the daredevil dog (who has mass m), starting from **REST**, slides down a frictionless track from an initial height y_0. The track becomes level at $y = 0$, then goes into a circular loop of radius R, and then goes level at $y = 0$ again. Dingo is actually a particle dog.

a) What is Dingo's speed at any height y assuming he stays on the track? Give the speed as function of y, y_0, g, and, if necessary, m.

b) Find a general formula for the normal force on Dingo when he is on the loop as a function of the angle θ between a general radial vector and the radial vector pointing to the top of the loop. Assume that the direction toward the center is the positive direction. The only variables in the formula should be y_0, R, m, g, and θ. Simplify as much as possible and take radially inward as positive. **HINT:** Note that the magnitude of the radial (component of) acceleration is instantaneously given by $a = v^2/r$ just as for uniform circular motion even when v is not constant provided that r is constant. Recall v is just the tangential speed in this expression and the direction of the radial acceleration is toward the center.

c) For what θ is the normal force formula smallest? What is the normal force at this angle?

d) What is the mathematical condition—sufficient, not just necessary—needed so that the normal force formula never specifies an attractive normal force (i.e., a force attracting Dingo to the track) anywhere on the loop? Explain why there is this condition. What happens to Dingo if the formula did specify an attractive normal force?

e) Say $y_0 = 50\,\text{m}$ and the loop radius $R = 10\,\text{m}$. Does Dingo stay on the loop? (A demonstration is needed, not just a yes or no answer. But it's not a long demonstration.)

SUGGESTED ANSWER:

a) From conservation of mechanical energy when nonconservative forces do no net work, we find

$$E = mgy_0 = \frac{1}{2}mv^2 + mgy \ ,$$

and thus

$$v = \sqrt{2g(y_0 - y)} \ .$$

There is no Dingo mass m in the formula. Mass often cancels out when gravity is the only force doing net work because gravity is a mass-dependent force. Maybe always cancels out—I'm not sure.

b) The centripetal force when Dingo is on the loop satisfies

$$m\frac{v^2}{R} = F_{\rm N} + mg\cos\theta \ ,$$

where we have taken radially inward as positive. This expression applies for any $\theta \in [-\pi, \pi]$. Thus

$$\begin{aligned} F_{\rm N} &= m\frac{v^2}{R} - mg\cos\theta \\ &= mg\left[\frac{2(y_0 - y)}{R} - \cos\theta\right] \\ &= mg\left[\frac{2(y_0 - R - R\cos\theta)}{R} - \cos\theta\right] \\ &= mg\left(2\frac{y_0}{R} - 2 - 3\cos\theta\right) \ , \end{aligned}$$

where we have used

$$y = R(1 + \cos\theta) = R + R\cos\theta$$

for Dingo's height while on the loop.

Remember, we have no intrinsic expression for the normal force. The formula just derived is what the force perpendicular to the surface must be to insure circular motion: that formula can go negative: i.e., give an attractive attractive force in this context. If that force formula is positive, then the normal force will supply the force. Dingo will try to push into the rigid loop and the normal force will repel enough with a force equal to what the formula gives. But though we have no intrinsic formula for the normal force, we do have this intrinsic behavior, it will **NOT** attract. So when our normal force formula goes to zero and is heading below zero, Dingo must leave the loop and go ballistic. The normal force will not be able to hold him on the loop.

c) Clearly, the normal force is smallest for $\theta = 0$. The expression then is

$$F_{\rm N}(\theta = 0) = mg\left(2\frac{y_0}{R} - 5\right) \ .$$

d) The smallest real normal force for any y_0 and R occurs when $\theta = 0$. Thus, it follows from the part (c) answer that for the normal force formula **NEVER** to become attractive, we must have

$$\frac{y_0}{R} \geq \frac{5}{2} \qquad \text{or} \qquad \frac{y_0}{D} \geq \frac{5}{4} \ ,$$

where D is diameter. As we discussed in the part (b) answer, if our normal force formula goes below zero, Dingo must fly off the loop since there is no attractive normal force to hold him on.

Note that the condition for Dingo is stricter than for bead on a frictionless wire. The bead can rise up on the wire to y_0 where all its is potential energy. But this is because the bead is constrained to the wire at all times: there is always a normal force since the bead is strung on the wire. Dingo isn't constrained to the track in the same way. Once the normal force goes to zero, he must leave the loop. This happens if $D > (4/5)y_0$. Dingo can't rise to a height of y_0 on the loop.

e) Yes, Dingo stays on the loop. The ratio $y_0/R = 5 > 5/2$. So Dingo has still has a positive normal force at the top of the loop. As he slides done the reverse side of the loop, the situation is symmetrical to the forward side. His speed is the same at every height by conservation of energy and this means the normal force expression gives a positive (or repulsive) normal force.

A more interesting case is if the ratio $y_0/R = 5/2$. In this case, the normal force formula just goes to zero at the top of the loop. It seems intuitively clear that Dingo should then make the loop since for just a slightly higher ratio Dingo will make the loop clearly. But I confess to doubt. Can I assuage my doubt?

In order to stay on the loop, the ballistic path must be higher than or equal to the track. The track then constrains Dingo from following the ballistic path. In this situation, the ballistic path height is

$$y = -\frac{1}{2}gt^2 + 2R = -\frac{1}{2}g\left(\frac{x}{v_{\mathrm{top}}}\right)^2 + 2R$$

where x is the horizontal component counted positive in the direction Dingo is moving at the top of the loop and v_{top} is the x direction velocity at the top of the loop speed. On the ballistic path the x direction velocity will stay constant since there is no x direction force.

Now

$$v_{\mathrm{top}} = \sqrt{2g(y_0 - 2R)} = \sqrt{gR}$$

since $y_0 = (5/2)R$. Thus,

$$y = -\frac{1}{2}\frac{x^2}{R} + 2R\ .$$

Now the track path height is

$$y_{\mathrm{tr}} = \sqrt{R^2 - x^2} + R\ .$$

Let's assume $y \geq y_{\mathrm{tr}}$ and see if that leads to a true result. Behold:

$$\begin{aligned}
-\frac{1}{2}\frac{x^2}{R} + 2R &\geq \sqrt{R^2 - x^2} + R\ , \\
-\frac{1}{2}\frac{x^2}{R} + R &\geq \sqrt{R^2 - x^2}\ , \\
-\frac{1}{2}\left(\frac{x}{R}\right)^2 + 1 &\geq \sqrt{1 - \left(\frac{x}{R}\right)^2}\ , \\
\frac{1}{4}\left(\frac{x}{R}\right)^4 - \left(\frac{x}{R}\right)^2 + 1 &\geq 1 - \left(\frac{x}{R}\right)^2\ , \\
\frac{1}{4}\left(\frac{x}{R}\right)^4 &\geq 0\ ,
\end{aligned}$$

which is an obviously true inequality with the equality holding only for $x = 0$ which is the top of the loop itself. The assumed inequality leads to a true result. The steps in reverse lead to the result we wished to prove, and so we have proven it. The ballistic path is always higher than or equal to the track path in this case, and so Dingo does stay on the track. My doubt is assuaged.

Redaction: Jeffery, 2001jan01

Equation Sheet for Introductory Physics Calculus-Based

This equation sheet is intended for students writing tests or reviewing material. Therefore it is neither intended to be complete nor completely explicit. There are fewer symbols than variables, and so some symbols must be used for different things: context must distinguish.

The equations are mnemonic. Students are expected to understand how to interpret and use them.

1 Constants

$$c = 2.99792458 \times 10^8\,\mathrm{m/s} \approx 2.998 \times 10^8\,\mathrm{m/s} \approx 3 \times 10^8\,\mathrm{m/s} \approx 1\,\mathrm{lyr/yr} \approx 1\,\mathrm{ft/ns} \qquad \text{exact by definition}$$

$$e = 1.602176487(40) \times 10^{-19}\,\mathrm{C}$$

$$G = 6.67428(67) \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \qquad \text{(2006, CODATA)}$$

$$g = 9.8\,\mathrm{m/s^2} \qquad \text{fiducial value}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551787\ldots \times 10^9 \approx 8.99 \times 10^9 \approx 10^{10}\,\mathrm{N\,m^2/C^2}\text{exact by definition}$$

$$k_{\mathrm{Boltzmann}} = 1.3806504(24) \times 10^{-23}\,\mathrm{J/K} = 0.8617343(15) \times 10^{-4}\,\mathrm{eV/K} \approx 10^{-4}\,\mathrm{eV/K}$$

$$m_e = 9.10938215(45) \times 10^{-31}\,\mathrm{kg} = 0.510998910(13)\,\mathrm{MeV}$$

$$m_p = 1.672621637(83) \times 10^{-27}\,\mathrm{kg} = 938.272013(23), \mathrm{MeV}$$

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.8541878176\ldots \times 10^{-12}\,\mathrm{C^2/(N\,m^2)} \approx 10^{-11} \qquad \text{vacuum permittivity (exact by definition)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2} \qquad \text{exact by definition}$$

2 Geometrical Formulae

$$C_{\mathrm{cir}} = 2\pi r \qquad A_{\mathrm{cir}} = \pi r^2 \qquad A_{\mathrm{sph}} = 4\pi r^2 \qquad V_{\mathrm{sph}} = \frac{4}{3}\pi r^3$$

$$\Omega_{\mathrm{sphere}} = 4\pi \qquad d\Omega = \sin\theta\, d\theta\, d\phi$$

3 Trigonometry Formulae

$$\frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta \qquad \frac{y}{x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$c^2 = a^2 + b^2 \qquad c = \sqrt{a^2 + b^2 - 2ab\cos\theta_c} \qquad \frac{\sin\theta_a}{a} = \frac{\sin\theta_b}{b} = \frac{\sin\theta_c}{c}$$

$$f(\theta) = f(\theta + 360^\circ)$$

$$\sin(\theta + 180^\circ) = -\sin(\theta) \qquad \cos(\theta + 180^\circ) = -\cos(\theta) \qquad \tan(\theta + 180^\circ) = \tan(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

$$\sin(\theta + 90^\circ) = \cos(\theta) \qquad \cos(\theta + 90^\circ) = -\sin(\theta) \qquad \tan(\theta + 90^\circ) = -\tan(\theta)$$

$$\sin(180^\circ - \theta) = \sin(\theta) \qquad \cos(180^\circ - \theta) = -\cos(\theta) \qquad \tan(180^\circ - \theta) = -\tan(\theta)$$

$$\sin(90^\circ - \theta) = \cos(\theta) \qquad \cos(90^\circ - \theta) = \sin(\theta) \qquad \tan(90^\circ - \theta) = \frac{1}{\tan(\theta)} = \cot(\theta)$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a)$$

$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a - b) - \cos(a + b)\right] \qquad \cos(a)\cos(b) = \frac{1}{2}\left[\cos(a - b) + \cos(a + b)\right]$$

$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a - b) + \sin(a + b)\right]$$

$$\sin^2\theta = \frac{1}{2}[1 - \cos(2\theta)] \qquad \cos^2\theta = \frac{1}{2}[1 + \cos(2\theta)] \qquad \sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

4 Approximation Formulae

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx} \qquad \frac{1}{1 - x} \approx 1 + x \ : \ (x << 1)$$

$$\sin\theta \approx \theta \qquad \tan\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{1}{2}\theta^2 \qquad \text{all for } \theta << 1$$

5 Quadratic Formula

$$\text{If} \qquad 0 = ax^2 + bx + c\ , \qquad \text{then} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

6 Vector Formulae

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2} \qquad \theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y)$$

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \phi = \tan^{-1}\left(\frac{a_y}{a_x}\right) + \pi? \qquad \theta = \cos^{-1}\left(\frac{a_z}{a}\right)$$

$$\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z)$$

$$\vec{a} \cdot \vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{c} = \vec{a} \times \vec{b} = ab\sin(\theta)\hat{c} = (a_y b_z - b_y a_z, a_z b_x - b_z a_x, a_x b_y - b_x a_y)$$

7 Differentiation and Integration Formulae

$$\frac{d(x^p)}{dx} = px^{p-1} \quad \text{except for } p = 0; \qquad \frac{d(x^0)}{dx} = 0 \qquad \frac{d(\ln|x|)}{dx} = \frac{1}{x}$$

Taylor's series
$$f(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} f^{(n)}(x_0)$$
$$= f(x_0) + (x - x_0)f^{(1)}(x_0) + \frac{(x - x_0)^2}{2!} f^{(2)}(x_0) + \frac{(x - x_0)^3}{3!} f^{(3)}(x_0) + \ldots$$

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a) \quad \text{where} \quad \frac{dF(x)}{dx} = f(x)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad \text{except for } n = -1; \qquad \int \frac{1}{x}\,dx = \ln|x|$$

8 One-Dimensional Kinematics

$$v_{\text{avg}} = \frac{\Delta x}{\Delta t} \qquad v = \frac{dx}{dt} \qquad a_{\text{avg}} = \frac{\Delta v}{\Delta t} \qquad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$v = at + v_0 \qquad x = \frac{1}{2}at^2 + v_0 t + x_0 \qquad v^2 = v_0^2 + 2a(x - x_0)$$

$$x = \frac{1}{2}(v_0 + v)t + x_0 \qquad x = -\frac{1}{2}at^2 + vt + x_0 \qquad g = 9.8\,\mathrm{m/s^2}$$

$$x_{\text{rel}} = x_2 - x_1 \qquad v_{\text{rel}} = v_2 - v_1 \qquad a_{\text{rel}} = a_2 - a_1$$

$$x' = x - v_{\text{frame}}t \qquad v' = v - v_{\text{frame}} \qquad a' = a$$

9 Two- and Three-Dimensional Kinematics: General

$$\vec{v}_{\text{avg}} = \frac{\Delta\vec{r}}{\Delta t} \qquad \vec{v} = \frac{d\vec{r}}{dt} \qquad \vec{a}_{\text{avg}} = \frac{\Delta\vec{v}}{\Delta t} \qquad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

10 Projectile Motion

$$x = v_{x,0}t \qquad y = -\frac{1}{2}gt^2 + v_{y,0}t + y_0 \qquad v_{x,0} = v_0\cos\theta \qquad v_{y,0} = v_0\sin\theta$$

$$t = \frac{x}{v_{x,0}} = \frac{x}{v_0\cos\theta} \qquad y = y_0 + x\tan\theta - \frac{x^2 g}{2v_0^2\cos^2\theta}$$

$$x_{\text{for } y \text{ max}} = \frac{v_0^2\sin\theta\cos\theta}{g} \qquad y_{\text{max}} = y_0 + \frac{v_0^2\sin^2\theta}{2g}$$

$$x(y = y_0) = \frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin(2\theta)}{g} \qquad \theta_{\text{for max}} = \frac{\pi}{4} \qquad x_{\text{max}}(y = y_0) = \frac{v_0^2}{g}$$

$$x(\theta = 0) = \pm v_0\sqrt{\frac{2(y_0 - y)}{g}} \qquad t(\theta = 0) = \sqrt{\frac{2(y_0 - y)}{g}}$$

11 Relative Motion

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad \vec{v} = \vec{v}_2 - \vec{v}_1 \qquad \vec{a} = \vec{a}_2 - \vec{a}_1$$

12 Polar Coordinate Motion and Uniform Circular Motion

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\vec{r} = r\hat{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\omega\hat{\theta} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\hat{\theta}$$

$$\vec{v} = r\omega\hat{\theta} \qquad v = r\omega \qquad a_{\tan} = r\alpha$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} = -r\omega^2\hat{r} \qquad a_{\text{centripetal}} = \frac{v^2}{r} = r\omega^2 = v\omega$$

13 Very Basic Newtonian Physics

$$\vec{r}_{\text{cm}} = \frac{\sum_i m_i \vec{r}_i}{m_{\text{total}}} = \frac{\sum_{\text{sub}} m_{\text{sub}} \vec{r}_{\text{cm sub}}}{m_{\text{total}}} \qquad \vec{v}_{\text{cm}} = \frac{\sum_i m_i \vec{v}_i}{m_{\text{total}}} \qquad \vec{a}_{\text{cm}} = \frac{\sum_i m_i \vec{a}_i}{m_{\text{total}}}$$

$$\vec{r}_{\text{cm}} = \frac{\int_V \rho(\vec{r})\vec{r}\,dV}{m_{\text{total}}}$$

$$\vec{F}_{\text{net}} = m\vec{a} \qquad \vec{F}_{21} = -\vec{F}_{12} \qquad F_g = mg \qquad g = 9.8\,\text{m/s}^2$$

$$\vec{F}_{\text{normal}} = -\vec{F}_{\text{applied}} \qquad F_{\text{linear}} = -kx$$

$$f_{\text{normal}} = \frac{T}{r} \qquad T = T_0 - F_{\text{parallel}}(s) \qquad T = T_0$$

$$F_{\text{f static}} = \min(F_{\text{applied}}, F_{\text{f static max}}) \qquad F_{\text{f static max}} = \mu_{\text{static}} F_{\text{N}} \qquad F_{\text{f kinetic}} = \mu_{\text{kinetic}} F_{\text{N}}$$

$$v_{\text{tangential}} = r\omega = r\frac{d\theta}{dt} \qquad a_{\text{tangential}} = r\alpha = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}$$

$$\vec{a}_{\text{centripetal}} = -\frac{v^2}{r}\hat{r} \qquad \vec{F}_{\text{centripetal}} = -m\frac{v^2}{r}\hat{r}$$

$$F_{\text{drag,lin}} = bv \qquad v_{\text{T}} = \frac{mg}{b} \qquad \tau = \frac{v_{\text{T}}}{g} = \frac{m}{b} \qquad v = v_{\text{T}}(1 - e^{-t/\tau})$$

$$F_{\text{drag,quad}} = bv^2 = \frac{1}{2}C\rho A v^2 \qquad v_{\text{T}} = \sqrt{\frac{mg}{b}}$$

14 Energy and Work

$$dW = \vec{F}\cdot d\vec{s} \qquad W = \int \vec{F}\cdot d\vec{s} \qquad KE = \frac{1}{2}mv^2 \qquad E_{\text{mechanical}} = KE + PE$$

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t} \qquad P = \frac{dW}{dt} \qquad P = \vec{F}\cdot\vec{v}$$

$$\Delta KE = W_{\text{net}} \quad \Delta PE_{\text{of a conservative force}} = -W_{\text{by a conservative force}} \quad \Delta E = W_{\text{nonconservative}}$$

$$F = -\frac{dPE}{dx} \qquad \vec{F} = -\nabla PE \qquad PE = \frac{1}{2}kx^2 \qquad PE = mgy$$