Cosmology & Galaxies NAME:

Homework 26: Cosmic Present Galaxy Environments: Interactions, Galaxy Groups, Galaxy Clusters, Galaxy Superclusters, Large-Scale Structure

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026 qmult 00210 1 1 3 easy memory: groups and clusters differentiated

1. There is no sharp distinction, in fact, between groups and clusters and one can regard groups as just very poor clusters. However, by Ci-165,167,174's account, fiducially groups have 3 to x non-dwarf galaxies and clusters have x to a few thousand non-dwarf galaxies, and groups have virial mass $\lesssim y\,M_\odot$ and clusters have virial mass y to $10^{15}\,M_\odot$. Now x and y are, respectively:

   a) 10 and 10^{10}. b) 20 and 10^{11}. c) 50 and 10^{14}. d) 500 and 3×10^{14}. e) 1000 and 3×10^{14}.

   SUGGESTED ANSWER: (c) See Ci-165,167,174.

   Wrong answers:
   a) A nonsense answer.

   Redaction: Jeffery, 2008jan01

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026 qmult 00320 1 1 5 easy memory: local group non-dwarf galaxies

2. The Local Group has only 3 non-dwarf galaxies (all spiral galaxies): the Milky Way, the Andromeda Galaxy (M31), and the:

   a) Aries Galaxy (M33). b) Boötes Galaxy (M33). c) Monoceros Galaxy (M33). d) Pegasus Galaxy (M33). e) Triangulum Galaxy (M33).

   SUGGESTED ANSWER: (e)

   Wrong answers:
   c) Yes, it is a constellation.

   Redaction: Jeffery, 2008jan01

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026 qmult 00620 1 1 4 easy memory: cluster mass components and the cosmic baryonic mass fraction

3. Galaxy clusters have $(1 - x)$ to $90\,\%$ of their mass as dark matter. Baryonic matter mostly in the form of intracluster gas is the rest of the mass. Stars make up only 1 to $5\,\%$ of the mass. The value x is, in fact, the cosmic baryonic mass fraction set by Big Bang nucleosynthesis and other information. Among dark matter halo structures in the observable universe, only the largest clusters have baryonic mass fraction as large as the cosmic mass fraction x whose value is:

   a) $50\,\%$. b) $40\,\%$. c) $33\,\%$. d) $16\,\%$. e) $12\,\%$.

   SUGGESTED ANSWER: (d) See Ci-27,174–175.

   Wrong answers:
   a) Way to high for a dark matter dominated observable universe.

   Redaction: Jeffery, 2008jan01

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026 qmult 00810 1 1 1 easy memory: intracluster medium (ICM) temperature

4. The intracluster medium (ICM) temperature is

   a) $(2\text{–}10) \times 10^7$ K. b) $(5\text{–}10) \times 10^7$ K. c) $(2\text{–}10) \times 10^8$ K. d) $(5\text{–}10) \times 10^8$ K. e) $(2\text{–}10) \times 10^9$ K.

   SUGGESTED ANSWER: (a) See Ci-165.

   Wrong answers:
   e) Way too high.

   Redaction: Jeffery, 2008jan01

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026 qfull 01020 1 3 0 easy math: classical pressure force deriviation: On exams, do all parts

5. In this problem, we derive the general classical pressure for formula and some special cases. A remarkable fact is that the same formula follows from a quantum mechanical derivation with box quantization (Wikipedia: Particle in a box). This suggests that the formula is really very general.

   NOTE: There are parts a,b,c,d. On exams, do all parts with minimal words.

   a) Draw a diagram with a horizontal differential surface area vector $d\vec{A}$ with the vector pointing up. Now consider a flow of particles in a general direction through $d\vec{A}$. Write down the formula for the differential $dP\,dt\,dA$ (where capital P is pressure) for the flow of particles of momentum p through dA with velocity v in differential time dt, in differential particle momentum range dp, in differential angle $d\Omega = d\mu\,d\phi$ (where $\mu = \cos(\theta)$ and $d\mu = d\cos(\theta) = -\sin(\theta)\,d\theta$), and given the isotropic directional distribution of particles per volume per momentum $N(p)/(4\pi)$ (where the angle-integrated distribution is $N(p)$). **HINT:** You will need two factors of $\cos(\theta)$: one to account for the fact that it is only the component of mometum in the direction of $d\vec{A}$ that contributes to pressure and the other to account for the fact that there is reduced area for the beam of particles going through dA obliquely.

   b) Now write down the momentum integral for pressure after having integrated over all angle.

   c) Let ε be kinetic energy density. Write for formula for pressure as a function of ε in two limits: the non-relativistic (NR) limit where $p = mv$ and the extreme relativistic (ER) limit where $v = c$ and $p = K/c$, where K is kinetic energy per particle.

   d) We note that the pressure formulae of parts (b) and (c) are independent of the nature of the distribution $N(p)$. It could be a thermodynamic equilibrium distribution, but also anything else. For the (thermodynamic equilibrium) Maxwell-Boltzmann distribution for NR classical particles $N(p)\,dp = nf(v)\,dv$ (where n is particle density),

   $$\langle v^2 \rangle = \frac{3kT}{m} \qquad \text{and} \qquad E_{\text{energy per particle}} = \frac{3}{2}kT$$

   (Wikipedia: Maxwell-Boltzmann distribution; Wikipedia: Ideal gas law: Energy associated with a gas). On the other hand for a photon gas (which is made of ER particles),

   $$\varepsilon = aT^4 \ ,$$

   where a is the radiation density constant (Wikipedia: Photon gas; Wikipedia: Stefan-Boltzmann law). Write down the pressure formulae for the cases of the Maxwell-Boltzmann distribution and the photon gas.

SUGGESTED ANSWER:

a) You will have to imagine the diagram. Now

$$dP\,dt\,dA = p\cos(\theta)[dA\cos(\theta)](v\,dt)(d\Omega)\frac{N(p)}{(4\pi)}\,dp = pv\mu^2\frac{N(p)}{(4\pi)}\,dA\,dt\,d\Omega\,dp \ ,$$

where $\mu = \cos(\theta)$ and $d\Omega = \sin(\theta)\,d\theta\,d\phi = -d[\cos(\theta)]\,d\phi = -d\mu\,d\phi$.

b) Behold:

$$P = \left(2\pi\int_{-1}^{1}\mu^2\,d\mu\right)\left[\int_0^\infty pv\frac{N(p)}{(4\pi)}\,dp\right] = \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\int_0^\infty pvN(p)\,dp \ ,$$

where the 2π comes from the integral over ϕ and the minus sign in $-d\mu\,d\phi$ disappears in changing the limits of integration from those for θ to those for μ.

c) Behold:

$$P = \begin{cases} \dfrac{2}{3}\varepsilon & \text{NR limit;} \\[2ex] \dfrac{1}{3}\varepsilon & \text{ER limit.} \end{cases}$$

d) Behold:

$$P = \begin{cases} \left(\dfrac{2}{3}\right)\left(\dfrac{1}{2}\right)nm\langle v^2\rangle = nkT & \text{for the Maxwell-Boltzmann distribution:} \\ & \text{this is just the ideal gas law;} \\[2ex] \dfrac{1}{3}aT^4 & \text{for the photon gas.} \end{cases}$$

Redaction: Jeffery, 2018jan01

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026 qfull 01040 1 3 0 easy math: mean atomic weight with electrons: On exams, do all parts

6. The mean atomic mass is defined

$$\frac{1}{\mu} = \sum_i \frac{X_i}{A_i} ,$$

where the sum is over all species present, X_i is the mass fraction of species i and A_i is the atomic mass (i.e., the mass in a standard microscopic unit). Cimatti (2020) uses the proton mass $m_{\rm p}$ as the standard microscopic mass probably since the universe is made of hydrogen and not of daltons (i.e., 1/12 of an unperturbed carbon-12 atom). Written it as we have, the quantity $1/\mu = n/(\rho/m_{\rm p})$ (where n is the number of free particles) is the mean number of free particles per proton mass in the substance and $\mu = (\rho/m_{\rm p})/n$ is the mean mass in units of the proton mass of the free particles.

NOTE: There are parts a,b,c. On exams, do all parts with minimal words.

a) What is the formula for the number density of a substance with mass density ρ?

b) Say you have a gas of completely ionized hydrogen. What is the exact formula for $1/\mu$ and what is the approximate value of $1/\mu$. Assume $m_{\rm p}$ is the exactly the proton mass and not just the hydrogen atom mass.

c) In this part, assume that the sum is only over nuclides and does not include free electrons. Say you have a completely ionized gas with the hydrogen mass fraction X_1 and everything else collective mass fraction $(1 - X_1)$. Assume the atomic mass of hydrogen is $A_1 = 1$ and for everything not hydrogen approximate $(Z_i + 1)/A_i = 1/2$. What is the formula for the approximate mean atomic mass in terms of X_1? Give the special cases where X_1 equals 1, 3/4, 1/2, 1/3, and 0.

SUGGESTED ANSWER:

a) Behold:

$$\frac{\rho}{\mu m_{\rm p}} = \sum_i \frac{X_i \rho}{A_i m_{\rm p}} = \sum_i n_i = n , \qquad \text{and thus} \qquad n = \frac{\rho}{\mu m_{\rm p}} .$$

b) Behold:

$$\frac{1}{\mu} = \sum_i \frac{X_i}{A_i} = \frac{m_{\rm p}}{(m_{\rm p} + m_{\rm e}) \times 1} + \frac{m_{\rm e}}{(m_{\rm p} + m_{\rm e})(m_{\rm e}/m_{\rm p})} = \frac{2m_{\rm p}}{(m_{\rm p} + m_{\rm e})} \approx 2 .$$

This mean atomic mass means there are approximately 2 particles for every proton mass.

c) Behold:

$$\frac{1}{\mu} = \frac{2X_1}{A_1} + \sum_{i,i\neq 1} \frac{(Z_i + 1)X_i}{A_i} = 2X_1 + \frac{1}{2}(1 - X_1) = \frac{3}{2}X_1 + \frac{1}{2} ,$$

and thus

$$\frac{1}{\mu} = \begin{cases} \frac{3}{2}X_1 + \frac{1}{2} & \text{in general;} \\ 2 & \text{for } X_1 = 1; \\ \frac{13}{8} = 1.625 & \text{for } X_1 = 3/4 \text{ (which the fiducial primordial} \\ & \text{mass fraction of hydrogen);} \\ \frac{5}{4} & \text{for } X_1 = 1/2; \\ 1 & \text{for } X_1 = 1/3; \\ \frac{1}{2} & \text{for } X_1 = 0. \end{cases}$$

Redaction: Jeffery, 2018jan01

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026 qfull 01050 1 3 0 easy math: the beta-model: On exams, do only parts a,b,c

7. In this problem we investigate the β-model of (galaxy cluster) gas particle density. The β-model is probably only order of magnitude accurate, but it is a standard fiducial model for the gas particle density.

 NOTE: There are parts a,b,c,d. On exams, do only parts a,b,c with minimal words.

a) The hydrostatic equilibrium equation for a spherically symmetric mass distribution is

$$\frac{dp}{dr} = -\frac{Gm(r)}{r^2}\rho \ ,$$

where r is radius coordinate, p is pressure, ρ is density, and $m(r)$ is interior mass (i.e., the mass interior to a shell of radius r). In fact, the equation can be written separately for each species in the distribution if they are decoupled: i.e., the pressure of one species is felt only by that species. In galaxy clusters, there are 3 decoupled species:

 1) Gas with particle density $n = \rho/(\mu m_{\rm p})$ and $p = nkT$ (with T approximated as constant for the β-model).
 2) Galaxies with galaxy number density $n_{\rm gal}$, galaxy mass density $n_{\rm gal}m_{\rm gal}$ (with $m_{\rm gal}$ approximated as constant for the β-model), and pressure approximated $n_{\rm gal}m_{\rm gal}\sigma_{\rm los}^2$ (where $\sigma_{\rm los}$ the mean line-of-sight dispersion for the galaxies for the β-model). Note $d\ln(n_{\rm gal}m_{\rm gal}\sigma_{\rm los}^2) = d\ln(n_{\rm gal})$ since $m_{\rm gal}$ and $\sigma_{\rm los}^2$ are constants.
 3) Dark matter with density $\rho_{\rm DM}$ and effective pressure $p_{\rm DM}$ (whatever that may be).

Write the hydrostatic equilibrium equation in terms of the $(p/\rho)d\ln(p)/d\ln(r)$ specialized for each species compactly on one line.

b) Using the results of part (a), solve for the proportionality between n and $n_{\rm gal}$ in terms of the constant

$$\beta = \frac{\sigma_{\rm los}^2}{kT/(\mu m_{\rm p})} = \frac{(\mu m_{\rm p})\sigma_{\rm los}^2}{kT} \ .$$

c) Given fiducial (but probably only order a magnitude accurate) King profile

$$n_{\rm gal}(r) = n_{\rm gal,0}\left[1+\left(\frac{r}{r_{\rm c}}\right)^2\right]^{-3/2} = \frac{n_{\rm gal,0}}{\left[1+(r/r_{\rm c})^2\right]^{3/2}}$$

(where $n_{\rm gal,0}$ is central galaxy density and $r_{\rm c}$ a core radius), determine a β-model density profile. Given that fiducial range for β is $[1/2, 1]$, what can one say about the gas density profile relative to the galaxy density profile.

d) The X-ray emissivity from galaxy clusters is approximately given $j_{\rm X} \propto n^2$ and the line-of-sight surface brightness at project radius R is for optically thin gas

$$I_{\rm X}(R) = 2\int_R^\infty \frac{j_{\rm X} r\,dr}{\sqrt{r^2-R^2}} \ ,$$

where r radial coordinate to the line-of-sight coordinate and spherical symmetry is assumed. Solve the integral approximately to within an unspecified factor for part (c) gas density profile. You will have to make an order of magnitude approximation whose chief virtue is that it makes the integral analytically tractable.

SUGGESTED ANSWER:

a) Behold:

$$-\frac{Gm(r)}{r} = \frac{p}{\rho}\frac{d\ln(p)}{d\ln(r)} = \frac{kT}{\mu m_{\rm p}}\frac{d\ln(n)}{d\ln(r)} = \sigma_{\rm los}^2\frac{d\ln(n_{\rm gal})}{d\ln(r)} = \frac{p_{\rm DM}}{\rho_{\rm DM}}\frac{d\ln(p_{\rm DM})}{d\ln(r)} \ .$$

b) Behold:

$$1)\quad \frac{kT}{\mu m_{\rm p}}\frac{d\ln(n)}{d\ln(r)} = \sigma_{\rm los}^2\frac{d\ln(n_{\rm gal})}{d\ln(r)} \qquad 2)\quad \frac{kT}{\mu m_{\rm p}}\,d\ln(n) = \sigma_{\rm los}^2\,d\ln(n_{\rm gal})$$

$$3)\quad d\ln(n) = \frac{(\mu m_{\rm p})\sigma_{\rm los}^2}{kT}\,d\ln(n_{\rm gal}) = \beta\,d\ln(n_{\rm gal})$$

$$4)\quad n \propto n_{\rm gal}^\beta$$

c) Behold:

$$n(r) = n_0 \left[1 + \left(\frac{r}{r_{\rm c}}\right)^2\right]^{-(3/2)\beta} = \frac{n_0}{\left[1 + (r/r_{\rm c})^2\right]^{(3/2)\beta}} \ .$$

Since typically $\beta \lesssim 1$, the gas density profile relative to the galaxy density profile is broader.

d) Behold:

$$\begin{aligned} I_{\rm X}(R) &= 2\int_R^\infty \frac{j_{\rm X} r\,dr}{\sqrt{r^2 - R^2}} \approx C\int_R^\infty \frac{n^2 r\,dr}{\sqrt{r^2 - R^2}} \approx C n_0^2 r_{\rm c} \int_{R/r_{\rm c}}^\infty \frac{x\,dx}{\sqrt{x^2 - (R/r_c)^2}\,(1+x^2)^{3\beta}} \\ &\approx C n_0^2 r_{\rm c} \int_{R/r_{\rm c}}^\infty \frac{x\,dx}{(x^2+1)^{1/2}\,(1+x^2)^{3\beta}} \approx C n_0^2 r_{\rm c} \int_{R/r_{\rm c}}^\infty \frac{x\,dx}{(1+x^2)^{3\beta+1/2}} \\ &\approx \frac{C n_0^2 r_{\rm c}}{[(3/2)\beta - 1/2]} \left[1 + \left(\frac{R}{r_c}\right)^2\right]^{-3\beta+1/2} \ , \end{aligned}$$

where $\sqrt{x^2 - (R/r_c)^2}$ has been approximated by $(x^2+1)^{1/2}$ which makes the result an underestimate, perhaps a bad underestimate.

Redaction: Jeffery, 2018jan01

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026 qfull 01550 1 3 0 easy math: 2-point correlation function. On exams, do all parts.

8. There are many statistical measures for the distribution of galaxies. All of them are trying to capture aspects of large-scale structure. The ideal statistical measure would capture all aspects and would exactly specify large-scale strurcture completely. But the ideal has not been reached, and so multiple statistical measures are used. Comparing a statistical measure's values for large-scale structure simulations and those for observed large-scale structure is a test of the simulations.

Probably the simplest statistical measure is the 2-point correlation function $\xi(r)$ which appears in the following equation

$$dN = n[1 + \xi(r)]\,dV \ ,$$

where n is the mean number of galaxies per unit volume in the observable universe and dN is the mean number of galaxies in volume dV located at a distance r from a reference galaxy at $r = 0$ (Ci-188–190). There must be some probability distribution from which this mean is derived, but yours truly cannot located it at the moment. However, if the $\xi(r) = 0$ everywhere, the distribution is the Poisson distribution and the mean number of galaxies in dV is just $n\,dV$. Note if $\xi(r) > 0$ for small r, galaxies tend to cluster and if $\xi(r) < 0$ for small r, galaxies tend to avoid each other.

NOTE: There are parts a,b,c. On exams, do all parts with minimal words.

a) Prove

$$\lim_{V\to\infty} \int_V \xi(r)\,dV = 0 \ .$$

b) For $r \leq 10\,$Mpc, the fiducial version of the 2-point correlation function is

$$\xi(r) = \left(\frac{r_{\rm s}}{r}\right)^\alpha = x^{-\alpha} \ ,$$

where scale radius $r_{\rm s} = 5\,$Mpc, $\alpha = 1.8$, and $x = r/r_{\rm s}$ (Ci-189). For $r \gtrsim 10\,$Mpc, $\xi(r)$ oscillates around zero, but there is a positive feature at the baryon acoustic oscillation (BAO) scale $\sim (140/h_{70})\,$Mpc (Ci-189). Determine the function $N(x)$ for $x \leq 2$ and give expressions for $N(0)$, $N(1)$, and $N(2)$: numerical evaluation is not required.

c) As mentioned above, the probability distribution from which the mean given by the 2-point correlation function is derived has not been located at the moment by yours truly. However, the Poisson distribution is

$$P(k) = e^{-\mu}\frac{\mu^k}{k!} \ ,$$

where k is the number of events and μ is the mean of the distribution (i.e., the mean number of events) (Be-36–43,53). The Poisson distribution can be viewed as the extreme limit of the binomial distribution where total possible number of events is infinity, and so the number of events observed is always small. The ℓth moment of the Poisson distribution is given by

$$\langle k^\ell \rangle = e^{-\mu} \sum_{k=0}^{\infty} \frac{k^\ell \mu^k}{k!} = e^{-\mu} \left(\mu \frac{d}{d\mu} \right)^\ell e^\mu \ ,$$

where the last formula is a trick where you treat μ as variable. Evaluate the moments for $\ell \in [0, 2]$ and the standard deviation for the Poisson distribution.

SUGGESTED ANSWER:

a) Since

$$n = \lim_{V\to\infty} \frac{N}{V} = \lim_{V\to\infty} \frac{n \int_V dV}{V} = \lim_{V\to\infty} \frac{n \int_V [1 + \xi(r)]\, dV}{V} \ ,$$

we must have

$$\lim_{V\to\infty} \int_V \xi(r)\, dV = 0 \ .$$

b) Behold:

$$N(x) = 4\pi r_{\rm s}^3 \int_0^x n(1 + x^{-\alpha}) x'^2\, dx' = 4\pi r_{\rm s}^3 \left(\frac{x^3}{3} + \frac{x^{3-\alpha}}{3-\alpha} \right)$$

for $x \leq 2$. Therefore

$$N = 4\pi r_{\rm s}^3 \times \begin{cases} \left(\dfrac{x^3}{3} + \dfrac{x^{3-\alpha}}{3-\alpha} \right) & \text{in general;} \\ 0 & \text{for } x = 0; \\ \left(\dfrac{1}{3} + \dfrac{1}{3-\alpha} \right) = 1.16666\ldots & \text{for } x = 1; \\ \left(\dfrac{8}{3} + \dfrac{2^{3-\alpha}}{3-\alpha} \right) = 4.58116\ldots & \text{for } x = 2. \end{cases}$$

c) Behold:

$$\langle k^\ell \rangle = \begin{cases} 1 & \text{for } \ell = 0; \\ \mu & \text{for } \ell = 1; \\ e^{-\mu} \left(\mu \dfrac{d}{d\mu} \right) (\mu e^\mu) = e^{-\mu} (\mu^2 e^\mu + \mu e^\mu) = \mu^2 + \mu & \text{for } \ell = 2. \end{cases}$$

The standard deviation is then

$$\sigma = \sqrt{\langle (k-\mu)^2 \rangle} = \sqrt{\langle k^2 \rangle - \mu^2} = \sqrt{\mu}$$

which is a well known result.

Fortran-95 Code

```
          print*
          bao_scale=100.0_np/0.7_np
          bao_scale_wik=490.0_np*0.306601_np !
https://en.wikipedia.org/wiki/Light-year
          print*,'bao_scale,bao_scale_wik'
          print*,bao_scale,bao_scale_wik
    !   142.85714285714285715         150.23449000000000000
          alpha=1.8_np
          x=1.0_np
          f=x**3/3.0_np + x**(3.0_np-alpha)/(3.0_np-alpha)
          print*,'x,f'
          print*,x,f
    !   1.0000000000000000000         1.1666666666666666666
          x=2.0_np
```

```
      f=x**3/3.0_np + x**(3.0_np-alpha)/(3.0_np-alpha)
      print*,'x,f'
      print*,x,f
!   2.0000000000000000000         4.5811639249950583449
```

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