Cosmology & Galaxies NAME:

Homework 25: Early Type Galaxies (ETGs)

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025 qmult 00130 1 1 3 easy memory: virial mass range for ellipticals

1. The range of virial mass (which is the fiducial total mass of galaxies determined in a tricky way) for elliptical galaxies (e.g., dwarf ellipticals (dEs), ellipticals (Es), and bright cluster ellipticals (BCEs)) is

   a) $\sim 10^5$–$10^6\,M_\odot$. b) $\sim 10^5$–$10^7\,M_\odot$. c) $\sim 10^8$–$10^{13}\,M_\odot$ or more. d) $\sim 10^5$–$10^{15}\,M_\odot$. e) $\sim 10^5$–$10^{20}\,M_\odot$.

   SUGGESTED ANSWER: (c) See Ci-126.

   Wrong answers:
   e) A bit extreme.

   Redaction: Jeffery, 2008jan01

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025 qmult 00230 1 1 2 easy memory: galaxy ellipticity for ellipticals

2. More so than for disk galaxies, the shape and orientation of isophotes is dependent on projected radius R (which could be the circularized radius) and position angle ϕ (measured counterclockwise from north on the sky) and therefore there is an ellipticity profile $\epsilon(R,\phi)$ (Ci-126–127). However, since a fiducial or characteristic ellipticity is useful is the galaxy ellipticity (Ci–127) defined by:

   a) $\epsilon(R_{\rm d})$. b) $\epsilon(R_{\rm e})$. c) $\epsilon(R_{\rm f})$. d) $\epsilon(R_{\rm g})$. e) $\epsilon(R_{\rm h})$.

   SUGGESTED ANSWER: (b)

   Wrong answers:
   a) $R_{\rm d}$ is the disc scale length or e-folding radius for face-on spiral galaxy surface brightness (Ci-55–56).
   a) $R_{\rm f}$? I've no idea.

   Redaction: Jeffery, 2008jan01

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025 qmult 00330 1 1 5 easy memory: ETG Sersic indices

3. When a Sérsic profile is fitted to the **CENTRAL** surface brightness of ETGs, the Sérsic index range is ~ 2 to ~ 10. However, when a single Sérsic profile is fitted to a galaxy the dividing line between later type galaxy Sérsic indices and ETG Sérsic indices is taken to be:

   a) 1. b) 1.25. c) 1.3. d) 1.5. e) 2.5.

   SUGGESTED ANSWER: (e) What Ci-32 says seems inconsistent until you read it carefully.

   Wrong answers:
   a) As Lurch would say AAAArrgh. **Redaction:** Jeffery, 2008jan01

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025 qmult 00430 1 1 1 easy memory: Hubble sequence E number

4. The Hubble sequence E number (E in range $[0,7]$) is nowadays determined by

$$E = 10 \times \epsilon = 10 \times \left(1 - \frac{b}{a}\right) ,$$

   where ϵ is the:

   a) ellipticity. b) eccentricity. c) effectiveness. d) e-folding. e) error.

   SUGGESTED ANSWER: (a)

   Wrong answers:
   b) Tempting, but eccentricity $e = \sqrt{1-(b/a)^2}$ (Wikipedia: Ellipse).

   Redaction: Jeffery, 2008jan01

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025 qmult 00510 1 4 5 easy deducto-memory: LOSVD defined

5. "Let's play *Jeopardy*! For \$100, the answer is: It is the distribution of velocity measured by the Doppler shift of some line along a line of sight (LOS) through an ETG using integral field spectroscopy (whereby a spectrum is obtained at each spatial pixel in the field of view)."

   What is a LOS ____________, Alex?

   a) Doppler distribution b) velocity disperson c) Doppler dispersion
   d) integral dispersion e) velocity distribution

   SUGGESTED ANSWER: (e)

   Wrong answers:
   a) As Lurch would say AAAARGH.

   Redaction: Jeffery, 2008jan01

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025 qmult 00630 1 1 3 easy memory: fast and slow rotators dividing line

6. The simple measure of ordered to random motions in ETGs is the ratio V/σ. The symbol V/σ seems also to be the name of the measure (Ci-132). Now V and σ have various definitions, but V is often the maximum line-of-slight (los) velocity $v_{\rm max}$ and σ is the central velocity dispersion defined by the surface brightness weighted los velocity dispersion formula

$$\sigma_0^2 = \frac{\int_{R_{\rm ap}} \sigma_{\rm los}(R)^2 I(R)\, d^2R}{\int_{R_{\rm ap}} I(R)\, d^2R} ,$$

   where $R_{\rm ap}$ stands for some aperature radius that is used for the determination (Ci-133). For low-redshift galaxies, R is usually in the range $0.1R_{\rm e}$ to $R_{\rm e}$. When $R_{\rm e}$ is used, one denotes σ_0 by $\sigma_{\rm e}$. For some darn good reason, the dividing line between fast rotators (above) and slow rotators (below) on a V/σ versus $\epsilon_{\rm e}$ plot is:

   a) $\sim (1/5)\epsilon_{\rm e}$. b) $\sim (1/5)\sqrt{\epsilon_{\rm e}}$. c) $\sim (1/3)\sqrt{\epsilon_{\rm e}}$. d) $\sim (1/3)\epsilon_{\rm e}$. e) $\epsilon_{\rm e}$.

   SUGGESTED ANSWER: (c) See Ci-134–135.

   Wrong answers:
   a) Why 1/5? But then why 1/3?

   Redaction: Jeffery, 2008jan01

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025 qfull 01000 1 3 0 easy math: proof of the virial theorem: On exams, do all parts.

7. The virial theorem is one most basic theorems of statistical mechanics taking the term statistical mechanics to include stellar systems formalism (which is about point-mass systems interacting by gravity) and other systems not ordinarily considered in conventional statistical mechanics. Here we consider only the classical virial theorem and not the quantum mechanical version. The general (non-quantum mechanical) virial theorem for a system of interacting particles isolated from all other forces.

$$\langle K\rangle = -\frac{1}{2}\left\langle \sum_i \vec{F}_i\cdot\vec{r}_i \right\rangle ,$$

   where the average is over time and the average is constant in time (i.e., the system is stationary), K is kinetic energy, the sum is over all particles in the system, $\vec{F}_i$ is the net force on particle i, and $\vec{r}_i$ position vector to particle i from a defined origin. The right-hand side of the equation is the virial itself (Wikipedia: Virial theorem).

   When all the forces in the system are interparticle forces derivable from potentials that depend only powers ℓ of interparticle of distances, the virial theorem specializes to

$$\langle K\rangle = \frac{1}{2}\sum_\ell \ell\,\langle U_\ell\rangle ,$$

   where sum is over all the potential energies.

   NOTE: There are parts a,b,c,d. All the parts can be done independently. So do not stop if you cannot do any part. On exams, do all parts with minimal words.

a) Prove the general virial theorem starting from the scalar moment of inertia

$$I = \sum_i m_i \vec{r}_i \cdot \vec{r}_i \ .$$

HINT: Take the first and second time derivatives of I and making use of the definitions of momentum and kinetic energy and Newton's 2nd law as needed.

b) Prove the special case virial theorem specified in the preamble: i.e., the important special case of the virial theorem where all the forces are derivable from potentials depending on power-law interparticle forces: i.e., the force of particle j on particle i is given by

$$\vec{F}_{ji} = -\sum_\ell \nabla U_{\ell,ji} r_{ji}^\ell = -\sum_\ell \ell U_{\ell,ji} r_{ji}^{\ell-1} \hat{r}_{ji} \ .$$

HINT: Just start from $\sum_i \vec{F}_i \cdot \vec{r}_i$ and march forward. You will need to do some trickery with indices.

c) Why must a stationary system have negative energy? What does this imply about a system to which the virial theorem applies: i.e., to a virialized system? What does the last implication imply about the kinds of potential energies of the special case virial theorem and what does it imply if there is only one kind of potential energy?

d) Specialize the special case virial theorem to the case where only the inverse-square force and linear force are present. This case actually the case for the large-scale structure of the universe where there is only the gravitation force and the cosmological constant force. Of course, this version of the virial theorem cannot apply to the universe as whole since one needs general relativistic physics for that.

SUGGESTED ANSWER:

a) To prove the virial theorem, we first specify the scalar moment of inertia and take its first and second time derivatives and we make use of the definitions of momentum and kinetic energy and Newton's 2nd law as needed:

$$\begin{aligned}
I &= \sum_i m_i \vec{r}_i \cdot \vec{r}_i \\
\frac{dI}{dt} &= 2\sum_i m_i \vec{v}_i \cdot \vec{r}_i = 2\sum_i \vec{p}_i \cdot \vec{r}_i \\
\frac{d^2 I}{dt^2} &= 2\left(\sum_i \vec{F}_i \cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \vec{v}_i\right) = 2\left(\sum_i \vec{F}_i \cdot \vec{r}_i + 2\sum_i K_i\right) \\
&= 2\left(\sum_i \vec{F}_i \cdot \vec{r}_i + 2K\right) \ ,
\end{aligned}$$

where the sum is over all particles of a system and K is the total kinetic energy. If the system is, in fact, stationary (i.e., in equilibrium) at the macroscopic level on average in time, then $\langle I \rangle$ is constant and all time-averaged derivatives of I are zero. Thus follows the general virial theorem:

$$\langle K \rangle = -\frac{1}{2}\left\langle \sum_i \vec{F}_i \cdot \vec{r}_i \right\rangle \ .$$

b) We find

$$\begin{aligned}
\sum_i \vec{F}_i \cdot \vec{r}_i &= \sum_{j,i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_i = -\sum_{j,i,j\neq i} \vec{F}_{ij} \cdot \vec{r}_i = -\sum_{j,i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_j \\
&= \frac{1}{2}\left[\left(\sum_{j,i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_i\right) - \left(\sum_{j,i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_j\right)\right]
\end{aligned}$$

$$= \frac{1}{2}\sum_{j,i,j\neq i} \vec{F}_{ji}\cdot\vec{r}_{ji} = -\frac{1}{2}\sum_{\ell,i,j,j\neq i}(\ell U_{\ell,ji} r_{ji}^{\ell-1}\hat{r}_{ji})\cdot\vec{r}_{ji} = -\frac{1}{2}\sum_{\ell,i,j,j\neq i}\ell U_{\ell,ji} r_{ji}^{\ell}$$
$$= -\sum_{\ell}\ell U_\ell \ ,$$

where the 1/2 was introduced to avoid double counting on the indexes ij and disappeared when we counted over all particles. Now the virial becomes

$$\langle K\rangle = \frac{1}{2}\sum_{\ell}\ell\,\langle U_\ell\rangle \ .$$

c) To be stationary particles cannot travel to infinity. Thus, the total energy energy must be negative and therefore the total energy of a virialized system must be negative. This means that at least one of the kinds of potential energy in a system must be negative and if there is only one kind of potential energy its power $\ell < 0$ in order for the kinetic energy to be positive as physically required.

d The inverse-square law force and the linear force have, respectively, $\ell = -1$ and $\ell = 2$, and so the virial theorem becomes

$$\langle K\rangle = -\frac{1}{2}\langle U_{-1}\rangle + \langle U_2\rangle \ .$$

Redaction: Jeffery, 2018jan01

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025 qfull 01230 1 3 0 easy math: mass determination using the virial theorem: On exams do all parts.

8. The crudest way of determining a galaxy mass is by a simple use of the virial theorem.

NOTE: There are parts a,b,c,d. On exams, do all parts with minimal words.

a) What is called the virial velocity dispersion $\sigma_{\rm vir}$ is defined by

$$K = \frac{1}{2}M_{\rm vir}\sigma_{\rm vir}^2 \ ,$$

where K is the total kinetic energy and $M_{\rm vir}$ is the mass out to some cutoff radius $r_{\rm cutoff}$. If you actually knew everything about self-gravitating system that was virialized within the shell defined by the cutoff radius $r_{\rm cutoff}$, then you would know K and $M_{\rm vir}$. What is the formula for $\sigma_{\rm vir}$ in this case?

b) What is called the gravitational radius $r_{\rm g}$ (which is not the cutoff radius $r_{\rm cutoff}$) is defined by

$$U = -\frac{GM_{\rm vir}^2}{r_{\rm g}} \ ,$$

where U is total gravitational potential energy out to the cutoff radius. The gravitational radius is just a characteristic radius since it is not the radius of anything in general. If you actually knew everything about self-gravitating system that was virialized within the shell defined by the cutoff radius $r_{\rm cutoff}$, then you would know U and $M_{\rm vir}$. What is the formula for $r_{\rm g}$ in this case?

c) Since for actual galaxies, we do not know a priori $M_{\rm vir}$, K, or U, we do not know $\sigma_{\rm vir}$ and $r_{\rm g}$ exactly and they are actually what we want in order to estimate $M_{\rm vir}$. However, we can guess that $\sigma_{\rm vir}$ and $r_{\rm g}$ will be of order, respectively, the central velocity dispersion σ_0 (however specified exactly) and the effective radius $R_{\rm e}$, but maybe only to within a factor of 10 for each one. So we parameterize

$$\sigma_0^2 = a\sigma_{\rm vir}^2 \qquad \text{and} \qquad R_{\rm e} = br_{\rm g} \ ,$$

where a and b are fudge factors. Use the virial theorem for gravity to solve for $M_{\rm vir}$ eliminating $\sigma_{\rm vir}$ and $r_{\rm g}$ via the fudge factors and then eliminate the fudge factors via the virial coefficient $k_{\rm vir} = 1/(ab)$.

d) In fact, $k_{\rm vir}$ can only be known accurately from detailed modeling. However, the fiducial value is 5, but deviations from this can be large. Write the virial mass formula in terms of fiducial values $k_{\rm vir} = 5$, $R_{\rm e} = 1\,{\rm kpc} = (3.08567758\ldots) \times 10^{19}\,{\rm m}$, $\sigma_0 = 200\,{\rm km/s}$, and solar masses $M_\odot = 1.98847 \times 10^{30}\,{\rm kg}$. Note the gravitational constant $G = 6.67430 \times 10^{-11}$ MKS.

In fact, the fiducial formula with $k_{\rm vir}$ actually set to 5 is called the dynamical mass $M_{\rm dyn}$ (Ci-147). When resolved kinematic information is not available for a galaxy, the virial mass from the formula (with $k_{\rm vir}$ set to 5 or some other good value) given in the answer to this question may be the best estimate of total mass one can get from observations of stellar light.

SUGGESTED ANSWER:

a) Behold:

$$\sigma_{\rm vir} = \sqrt{\frac{2K}{M_{\rm vir}}} \ .$$

b) Behold:

$$r_{\rm g} = -\frac{GM_{\rm vir}^2}{U} \ .$$

c) Behold:

$$1) \quad K = -\frac{1}{2}U \qquad 2) \quad \frac{1}{2}M_{\rm vir}(\sigma_0^2/a) = \frac{1}{2}\frac{GM_{\rm vir}^2}{(R_{\rm e}/b)} \qquad 3) \quad M_{\rm vir} = \frac{k_{\rm vir}R_{\rm e}\sigma_0^2}{G} \ .$$

d) Behold:

$$M_{\rm vir} = (4.6589 \times 10^{10}\,M_\odot)\left(\frac{k_{\rm vir}}{5}\right)\left(\frac{R_{\rm e}}{1\,{\rm kpc}}\right)\left(\frac{\sigma_0}{200\,{\rm km/s}}\right)^2 \ .$$

Fortran-95 Code

```
          print*
          gravcon=6.67430e-11_np   !
http://en.wikipedia.org/wiki/Gravitational_constant
          xmsun=1.9847e+30_np    ! https://en.wikipedia.org/wiki/Solar_mass
          xkpc=3.0856775814913673e+16_np*1.e+3_np  !
https://en.wikipedia.org/wiki/Parsec
          v=200.0_np*1.e+3_np
          xkvir=5.0_np
          con=xkvir*xkpc*v**2/(gravcon*xmsun)
          print*,'con'
          write(*,'(1p,e20.7)'),con   ! 4.6588631E+10 = 4.6589e10
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Redaction: Jeffery, 2018jan01

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027 qfull 01020 1 3 0 easy math: standard dark matter halo profiles: On exams, omit part d.

9. There several standard dark matter parameterized density profiles (i.e., profiles of density as a function of radius from the center of dark matter halos) that can be fitted to observed galaxy rotation curves with varying goodness. Here we study the behavior of some of them.

   NOTE: There are parts a,b,c,d. On exams, omit part d.

a) The NFW profile (i.e., Navarro-Frenck-White profile, 1996) is

$$\rho(r) = \frac{4\rho_{\rm s}}{(r/r_{\rm s})(1 + r/r_{\rm s})^2}$$

where the parameters are $r_{\rm s}$ the scale radius and $\rho_{\rm s}$ the density at the scale radius (e.g., Lin & Li 2019, p. 4). The NFW profile was suggeted by N-body simulations with dark matter particles, and so is a true theoretical dark matter halo density profile. It is a cusp profile in that $\rho(r \to 0)$ diverges. Show the limiting behaviors of $\rho(r)$ for $r/r_{\rm s} << 1$, $r/r_{\rm s} = 1$, and $r/r_{\rm s} >> 1$. Find the outer shell mass $M(r)$ from radius $r_{\rm outer} >> r_{\rm s}$ to general r. Discuss the converge/divergence properties of $M(r)$.

b) The Burkert profile (1995) is

$$\rho(r) = \frac{4\rho_{\rm s}}{(1 + r/r_{\rm s})[1 + (r/r_{\rm s})^2]}$$

where the parameters are $r_{\rm s}$ the scale radius and $\rho_{\rm s}$ the density at the scale radius (e.g., Lin & Li 2019, p. 4). The Burkert profile is a phenomenological profile chosen to fit galaxy rotation curves. If dark matter exists, $\rho_{\rm s}$ is true density parameter. If dark matter does not exist and MOND is true, then $\rho_{\rm s}$ is parameter with dimensions of density, but whose meaning is vague. The Burkert profile is a core profile in that $\rho(r \to 0)$ does not diverge. Show the limiting behaviors of $\rho(r)$ for $r/r_{\rm s} = 0$, $r/r_{\rm s} << 1$, $r/r_{\rm s} = 1$, and $r/r_{\rm s} >> 1$. Find the outer shell mass $M(r)$ from radius $r_{\rm outer} >> r_{\rm s}$ to general r. Discuss the converge/divergence properties of $M(r)$.

c) The Einasto profile (in the version of Wang 2020 September, Nature, p. 40) is

$$\rho(r) = \rho_{-2} \exp\left\{ -\left(\frac{2}{\alpha}\right)\left[\left(\frac{r}{r_{-2}}\right)^{\alpha} - 1\right]\right\} ,$$

where the parameters are r_{-2} the scale radius where the logarithmic slope is -2, ρ_{-2} the density at the scale radius, and $\alpha = 0.16 \approx 1/6$. The Einasto profile (in this version) is a fit to a huge number of high accuracy N-body simulation that span 20 orders of magnitude in dark matter halo mass. Almost everywhere the fit is accurate to within a few percent. The NFW profile is accurate to within 10 % almost everywhere, but has distinct shape relative to the Einasto profile. The Einasto profile is a core profile in that $\rho(r \to 0)$ does not diverge. Show the limiting behaviors of $\rho(r)$ for $r/r_{-2} = 0$, $r/r_{-2} << 1$, $r/r_{-2} = 1$, and $r/r_{-2} >> 1$.

d) For the Einasto profile of part (c), find the interior $M(r)$ from radius $r = 0$ to general r in terms of the incomplete factorial function

$$z(y')! = \int_0^{y'} e^{-y} y^z \, dy$$

(e.g., Ar-543). making the approximation $\alpha = 1/6$. You will find it convenient to make two transformations of the variable of integration. Determine the total mass $M(r = \infty)$ for general r_{-2} and ρ_{-2} by evaluating the factorial function (i.e., $z(y' = \infty)!$) making the approximation $\alpha = 1/6$.

SUGGESTED ANSWER:

a) Behold:

$$\rho(r) = \begin{cases} \dfrac{4\rho_{\rm s}}{(r/r_{\rm s})(1 + r/r_{\rm s})^2} = \dfrac{4\rho_{\rm s}}{(r/r_{\rm s}) + 2(r/r_{\rm s})^2 + (r/r_{\rm s})^3} & \text{in general.} \\ \dfrac{4\rho_{\rm s}}{(r/r_{\rm s})} & \text{for } r/r_{\rm s} << 1. \\ \rho_{\rm s} & \text{for } r/r_{\rm s} = 1. \\ \dfrac{4\rho_{\rm s}}{(r/r_{\rm s})^3} & \text{for } r/r_{\rm s} >> 1. \end{cases}$$

Note that there is inverse linear divergence as $r/r_{\rm s} \to 0$ which agrees with the description of the NFW profile as a cusp profile. For the outer shell mass, we find

$$M(r') = \int_{\rm outer}^{r'} \frac{4\rho_{\rm s}}{(r/r_{\rm s})^3} 4\pi r^2 \, dr = 4\pi r_{\rm s}^3 (4\rho_{\rm s}) \ln\left(\frac{r'}{r_{\rm outer}}\right) .$$

The outer shell mass diverges logarithmically with r. To keep the total mass of the halo finite, a cutoff radius must be defined. However, logarithmic divergence is rather gentle, and so it is likely that the total halo mass is not very dependent on a reasonably chosen cutoff radius. The cutoff radius could be the zero-force radius imposed by the cosmological constant force per unit mass:

$$\frac{\vec{F}}{m} = \frac{1}{3}\Lambda r \hat{r} ,$$

where $\Lambda = 9.9366 \times 10^{-36}\, s^{-2}$ (Wikipedia: Cosmological constant: Equation with value based on Planck 2018, arXiv 1807.06207).

b) Behold:

$$\rho(r)=\begin{cases}\dfrac{4\rho_{\rm s}}{(1+r/r_{\rm s})[1+(r/r_{\rm s})^2]} & \text{in general.}\\ 4\rho_{\rm s} & \text{for } r/r_{\rm s}=0.\\ \dfrac{4\rho_{\rm s}}{(1+r/r_{\rm s})}=4\rho_{\rm s}\left(1-\dfrac{r}{r_{\rm s}}\right) & \text{for } r/r_{\rm s}<<1.\\ \rho_{\rm s} & \text{for } r/r_{\rm s}=1.\\ \dfrac{4\rho_{\rm s}}{(r/r_{\rm s})^3} & \text{for } r/r_{\rm s}>>1.\end{cases}$$

Note that there is no divergence as $r/r_{\rm s}\to 0$ which agrees with the description of the Burkert profile as a core profile. The rest of part (b) is answered as in part (a).

c) Behold:

$$\rho(r)=\begin{cases}\rho_{-2}\exp\left\{-\left(\dfrac{2}{\alpha}\right)\left[\left(\dfrac{r}{r_{-2}}\right)^{\alpha}-1\right]\right\} & \text{in general.}\\ \rho_{-2}\exp\left(\dfrac{2}{\alpha}\right) & \text{for } r/r_{-2}=0.\\ \rho_{-2}\exp\left(\dfrac{2}{\alpha}\right)\left[1-\left(\dfrac{2}{\alpha}\right)\left(\dfrac{r}{r_{-2}}\right)^{\alpha}\right] & \text{for } r/r_{-2}<<1.\\ \rho_{-2} & \text{for } r/r_{-2}=1.\\ \rho_{-2}\exp\left[-\left(\dfrac{2}{\alpha}\right)\left(\dfrac{r}{r_{-2}}\right)^{\alpha}\right] & \text{for } r/r_{-2}>>1.\end{cases}$$

Note that there is no divergence as $r\to 0$ which agrees with the description of the Burkert profile as a core profile.

d) Behold:

$$M(r')=\rho_{-2}\exp\left(\frac{2}{\alpha}\right)\int_0^{r}\exp\left[-\left(\frac{r}{r_{-2}}\right)^{\alpha}\right]4\pi r^2\,dr$$

$$M\left(x'=\frac{r'}{r_{-2}}\right)=4\pi r_{-2}^3\rho_{-2}\exp\left(\frac{2}{\alpha}\right)\int_0^{x'}\exp\left(-x^{\alpha}\right)x^2\,dx$$

$$M\left[y'=(x')^{\alpha}=\left(\frac{r'}{r_{-2}}\right)^{\alpha}\right]=4\pi r_{-2}^3\rho_{-2}\frac{e^{2/\alpha}}{\alpha}\int_0^{y'}e^{-y}y^{3/\alpha-1}\,dy$$

The total mass with the approximation $\alpha=1/6$ is

$$\begin{aligned}M(r'=\infty)=M(y'=\infty)&=4\pi r_{-2}^3\rho_{-2}\frac{e^{2/\alpha}}{\alpha}\int_0^{\infty}e^{-y}y^{3/\alpha-1}\,dy\\ &=4\pi r_{-2}^3\rho_{-2}\left(6e^{12}\right)\int_0^{\infty}e^{-y}y^{17}\,dy=4\pi r_{-2}^3\rho_{-2}\left(6e^{12}\right)(17!)\\ &=4\pi r_{-2}^3\rho_{-2}\left(6e^{12}\right)\times\left(3.55687428096000\times 10^{14}\right)\ .\end{aligned}$$

Redaction: Jeffery, 2018jan01

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027 qfull 01030 1 3 0 easy math: the NFW profile explored: On exams, do only parts b,c.

10. The Navarro-Frenck-White (NFW) profile for the density profile of a quasi-equilibrium spherically symmetric dark matter halo derived from N-body simulations with scale radius $r_{\rm s}$, scale density, $\rho_{\rm s}$, and $x=r/r_{\rm s}$ is

$$\rho(r)=\begin{cases}\dfrac{4\rho_{\rm s}}{x(1+x)^2}=\dfrac{4\rho_{\rm s}}{x+2x^2+x^3} & \text{in general;}\\ \dfrac{4\rho_{\rm s}}{x} & \text{for } x<<1;\\ \rho_{\rm s} & \text{for } x=1;\\ \dfrac{4\rho_{\rm s}}{x^3} & \text{for } x>>1.\end{cases}$$

(Wikipedia: Navarro-Frenk-White profile). The logarithmic slope is

$$\frac{d\ln(\rho)}{d\ln(r)} = \frac{d\ln(\rho)}{d\ln(x)} = \frac{x}{\rho}\frac{d\rho}{dx} = -\frac{x}{\rho}(4\rho_{\rm s})\left[\frac{1+4x+3x^2}{(x+2x^2+x^3)^2}\right]$$
$$= \begin{cases} -\dfrac{1+4x+3x^2}{1+2x+x^2} & \text{in general;} \\ -2 & \text{for } x=1. \end{cases}$$

The scale radius $r_{\rm s}$ and scale density, $\rho_{\rm s}$ were chosen to yield logarithmic slope -2 when $x=1$ and density is $\rho_{\rm s}$.

The logarithmic slope -2 gives a flat (circular) velocity profile everywhere if it applies everywhere and gives an asymptotically flat velocity profile as radius $r \to \infty$ if it applies in the outer region of a mass distribution. However, the NFW profile actually only has logarithmic slope -2 at one point and does not yield an exactly flat density profile anywhere as we shall see.

Note an approximately flat velocity profile over some extended range of radius is characteristic of galaxy rotation curves for disk galaxies. However, the approximate flatness is a combination of dark matter and baryonic matter in actual galaxies and not of dark matter alone.

NOTE: There are parts a,b,c,d,e,f,g. On exams, do only parts b,c.

a) In fact, there is a semi-analytic argument for the NFW profile. Given that a dark matter halo density profile is approximately $\propto 1/r^2$ in its most characteristic region (which we center on $x=1$), one might be tempted to Taylor expand around the point where the logarithmic slope is exactly -2: i.e., where $x=1$. Argue that it is better to expand the specific volume $V_{\rm sp}$ (i.e., $1/\rho$) around $x=1$? Do the expansion for $V_{\rm sp}$ to 3rd order, collect like terms, and take the inverse using general symbols for the coefficients: i.e., ρ_0, $\rho_1 = c$, $\rho_2 = b$, and $\rho_3 = a$, where c, b, and a are chosen to conform to the conventions of tables of integrals. Why set the zeroth coefficent to zero? Why choose the 1st, 2nd, and 3rd order coefficients to be, respectively 1, 2, and 1 (given overall coefficient is set to be $\rho_{\rm s}$ times the sum of the coefficients in order to yield $\rho_{\rm s}$) other than the fact that that choice turns out to be good fitting parameters? **HINT:** To answer the last question, look at a table of integrals for the integrals needed to integrate density to get mass interior to radius x?

b) Determine the formula for $M(x)$ as a function of $r_{\rm s}$ and $\rho_{\rm s}$. You will have to use the table integrals:

$$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\ln(ax^2+bx+c) - \frac{b}{2a}\int\frac{dx}{ax^2+bx+c}\,,$$
$$\int \frac{dx}{ax^2+bx+c} = -\frac{2}{2ax+b} \qquad \text{for } b^2-4ac=0.$$

c) Rewrite the formula using the coefficient $M_{\rm s} = M(x=1)$ parameterized by $r_{\rm s}v_{\rm s}^2$ where $v_{\rm s}$ is the circular velocity Do not forget to normalize the function of x (i.e., the dimensionless mass function $f(x)$) that is required in the rewritten formula) to 1 at $x=1$ using a normalization constant A. In fact, a vast set of N-body simulations purely for dark matter particles shows that the NFW profile can be expected to hold usually to within 10 % for $x \in [0,30]$, but with some systematic deviations (Jie Wang et al. 2020, Nature, Sep02). For $x > 30$, large deviations from the NFW profile can be expected.

d) Compute $f(x)$ for x values 0, 0.1, 0.3, 1, 3, 10, and 30. What is $f(x)$ for $x \to \infty$ and what does this mean? **HINT:** Write a small computer program for the calculation.

e) Write the dimensionless circular velocity formula $g(x)$ normalized to 1 at $x=1$.

f) Compute a list of $g(x)$ values from $x=0$ to $x=30$. Describe the behavior. **HINT:** Extend your write small computer program to do the calculation.

g) The machine precision maximum characteristics of $g(x)$ can be determined by numerical methods. Setting the derivative of $g(x)$ to zero gives you a non-anaytically solvable equation for the maximizing x. An iteration formula that always converges can be obtained by isolating x on the left-hand side on on the right-hand side having a function where the expression under the square-root sign is never negative for $x > 0$. Convergence to machine precision however is slow. Convergence to machine precision is faster using the Newton-Raphson method (Wikipedia: Newton's method).

If you feel ambitious, use one or other some combination of both approaches to solve for $x_{\rm max}$ and $g(x_{\rm max})$. **HINT:** Extend your write small computer program to do the calculation.

SUGGESTED ANSWER:

a) Since the dark matter halo density profile goes approximately $\propto 1/r^2$ at the point we designate $x = 1$, the specific volume goes as $\propto r^2$ at $x = 1$. Since a Taylor series contains positive powers of the variable and not inverse powers, it seems likely a priori that a Taylor expansion for specific volume will capture more of the density behavior than that for density itself: i.e., the Taylor expansion for specific volume will likely have a larger region of convergence. The inverse of the specific volume expansion with like terms collected is

$$\rho(x) = \frac{1}{\rho_0 + cx + bx^2 + ax^3} \ .$$

N-body simulations show that $\rho(x << 1) \propto 1/x$, and so the 0th coefficient must be set to zero. It is possible that the specific volume expansion has a large enough region of convergence this may be the correct expansion coefficient, but on the other hand it could just be interpolation formula correction to the specific volume expansion or maybe some combination of the two. You can't know unless you actually have the exact $V_{\rm sp}(x)$—which we don't. Thus, we get

$$\rho(x) = \frac{(c + b + a)\rho_{\rm s}}{cx + bx^2 + ax^3} \ .$$

Choosing the 1st, 2nd, and 3rd order coefficients to be, respectively 1, 2, and 1 gives an integral for the mass that contains only one logarithm function and no arctangent function. It is a much simpler integral than any other case, and so to be preferred all other things be roughly equal. Of course, 1, 2, and 1 would be rejected if they did not make the NFW profile give a good fit the N-body simulations no matter how simple they made integrals.

b) Behold:

$$M(x) = 4\pi r_{\rm s}^3(4\rho_{\rm s})\left[\frac{1}{2a}\ln(ax'^2 + bx' + c) - \frac{2}{2ax' + b}\right]\Bigg|_0^x = 4\pi r_{\rm s}^3(4\rho_{\rm s})\left[\frac{1}{2}\ln(x^2 + 2x + 1) - \frac{x}{x+1}\right]$$
$$= 4\pi r_{\rm s}^3(4\rho_{\rm s})\left[\ln(x+1) - \frac{x}{x+1}\right] \ .$$

c) Behold:

$$M(x) = M_{\rm s}A\left[\ln(x+1) - \frac{x}{x+1}\right] = r_{\rm s}v_{\rm s}^2 A\left[\ln(x+1) - \frac{x}{x+1}\right] \ ,$$

where the normalization constant is

$$A = \frac{1}{\ln(2) - 1/2} = \frac{1}{0.19314718\ldots} = 5.177398899\ldots \ .$$

d) Behold:

$$f(x) = \begin{cases} A\left[\ln(x+1) - \dfrac{x}{x+1}\right] & \text{in general;} \\ 0 & \text{for } x = 0; \\ 0.02786\ldots & \text{for } x = 0.1; \\ 0.163580\ldots & \text{for } x = 0.3; \\ 1 & \text{for } x = 1.0; \\ 3.2943\ldots & \text{for } x = 3.0; \\ 7.70813\ldots & \text{for } x = 10.0; \\ 12.7687\ldots & \text{for } x = 30.0. \end{cases}$$

The mass diverges to infinity as $x \to \infty$ for the NFW profile. Clearly, the whole density profile of a dark matter halo cannot be described just by the NWF profile.

e) Since $v_{\rm cir} = \sqrt{GM(r)/r} \propto g(x)$, we must have

$$
g(x) = \sqrt{\frac{f(x)}{x}} = \begin{cases}
\sqrt{A\left[\frac{\ln(x+1)}{x} - \frac{1}{(x+1)}\right]} & \text{in general;} \\
\sqrt{A\left[1 - \frac{x}{2} + \frac{x^2}{3} - \ldots - (1 - x + x^2 - \ldots)\right]} & \\
\quad = \sqrt{A\left[\sum_{\ell=1}^{\infty}\left(\frac{\ell}{\ell+1}\right)(-1)^{\ell+1}x^{\ell}\right]} & \text{for } |x| < 1; \\
1 & \text{for } x = 1; \\
1.058035709947341977\ldots & \text{the maximum value at } x = 2.16258\ldots\ ; \\
1.0017\ldots & \text{at } x = 4.8 \text{, approximately the last point above 1;} \\
0.6524\ldots & \text{at } x = 30 \text{ of order the outermost point of NFW profile accuracy;} \\
\sqrt{A\left[\frac{\ln(x+1)}{x} - \frac{1}{(x+1)}\right]} & \\
\quad = \sqrt{A\left\{\frac{\ln[x(1+1/x)]}{x} - \frac{1/x}{(1+1/x)}\right\}} & \\
\quad \to \sqrt{A\left[\frac{\ln(x)}{x} - \frac{1}{x^2} - \frac{1}{x}\left(1 - \frac{1}{x}\right)\right]} & \\
\quad \to \sqrt{A\left[\frac{\ln(x)-1}{x}\right]} \to \sqrt{\left[\frac{\ln(x)}{x}\right]} & \text{for } x \to \infty.
\end{cases}
$$

f) From my computer list of $g(x)$ values and from the equation in the part (e) answer, the dimensionless circular velocity profile initially rises initially linearly from 0, reaches 1 at $x = 1$, reaches a maximum of $1.058035709947341977\ldots$ at $x = 2.162581587064609834\ldots$ (with values at 18-digit machine precision), falls slowly to ~ 1 at $x \approx 5$, and then declines asymptotically as $\sqrt{\ln(x)/x}$ as $x \to \infty$. As expected the NFW profile never gives an exactly flat circular velocity profile, but it gives a very flattish one for $x \in [1, 5]$ and for $x > 5$ still varies very slowly.

g) The derivative of $g(x)^2$ is

$$
\frac{dg(x)^2}{x} = A\left[-\frac{\ln(x+1)}{x^2} + \frac{1}{x(x+1)} + \frac{1}{(x+1)^2}\right] = \frac{A}{x^2(x+1)^2}\left[-(x+1)^2\ln(x+1) + x(x+1) + x^2\right]
$$

which is set to zero to get an equation for the stationary point which is obviously a maximum by parts (e) and (f). The equation for the stationary point is

$$
0 = (x+1)^2\ln(x+1) + x(x+1) + x^2 \ .
$$

Initially, it is not clear that an always-converging iteration equation can be obtained from the stationary point equation. However, after some fumbling, we find that the stationary point equation rearranges to an iteration equation

$$
x = -1 + \sqrt{\frac{2x^2 + x}{\ln(x+1)}} \ ,
$$

where the x on the left/right-hand side is the output/input and which converges we find experimentally for $x \in [0.01, 30]$, and so probably converges for all $x > 0$, but we are too weary to try to prove this. However, the iteration equation only very slowly grinds to 18-digit machine precision maximum value characteristics. The iteration can be vastly accelerated by using a Newton-Rapson method for finding a zero of generic function $f(x)$: i.e.,

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} \ ,$$

where $f'(x)$ is its derivative of $f(x)$. (Wikipedia: Newton's Method). Our functions $f(x)$ and $f'(x)$ are, respectively,

$$f(x) = -(x+1)^2 \ln(x+1) + x(x+1) + x^2 \qquad \text{and} \qquad f'(x) = -2(x+1) + 3x \ .$$

The Newton-Raphson method does have a limited convergence region in general. For our case, we experimentally find convergence for $x \gtrsim 1.4$ to probably $x \to \infty$, but we have not proven the upper limit. Experimentally, it seems nearly optimum convergence is obtained by using the iteration equation for iteration x values $x < 1.55$ and the Newton-Raphson method for iteration x values for $x \geq 1.55$. For an iteration starting from initial $x = 0.01$ with $x = 1.55$ as the dividing line between the iteration equation and the Newton-Raphson method, the iteration converges in 43 iterations to the 18-digit machine precision characteristics $x_{\text{max}} = 2.162581587064609834\ldots$ and $g(x_{\text{max}}) = 1.058035709947341977\ldots$.

For finding roots (i.e., zeros) in general, you should consult Pr-340–386 (i.e., *Numerical Recipes*). However, *Numerical Recipes* does not even discuss the iteration equation method. I guess *Numerical Recipes* considers it beneath contempt.

Fortran-95 Code: See
`/homes/jeffery/jef/aalib/dark_matter_halo.f`

Redaction: Jeffery, 2018jan01