Cosmology & Galaxies **NAME:**

Homework 23: Cosmic Present Galaxies as a Benchmark for Evolutionary Studies

1. "Let's play *Jeopardy*! For \$100, the answer is: It and surface brightness are the same physical quantities though in some conventions surface brightness has an extra factor of 4π. The name used just depends on context."

   What is ____________, Alex?

   a) radiant flux b) absolute magnitude c) apparent magnitude d) mean intensity
   e) specific intensity

2. The hybrid representation (AKA logarithmic representation of specific intensity satisfies equation:

   a) $I_E = I_\nu = I_\lambda$.
   b) $I_E/E = I_\nu/\nu = I_\lambda/\lambda$.
   c) $EI_E = \nu I_\nu = \lambda I_\lambda$.
   d) $I_E = I_\nu = 1/I_\lambda$.
   e) $I_E = 1/I_\nu = 1/I_\lambda$.

3. "Let's play *Jeopardy*! For \$100, the answer is:

$$I_\lambda = I_{\lambda,0} \exp\left(bx^{1/n}\right) = I_{\lambda,\mathrm{e}} \exp\left[b(x^{1/n} - 1)\right] ,$$

   where I_λ is the surface brightess as a function x, $x = R/R_{\mathrm{e}}$ is the radius (elliptical or circularized radius) in units of the effective radius R_{e}, $I_{\lambda,0} = I_\lambda(x = R/R_{\mathrm{e}} = 0)$, $I_{\lambda,\mathrm{e}} = I_\lambda(x = R/R_{\mathrm{e}} = 1)$, n is an index parameter typically in the range $[1, 2.5]$ for star forming galaxies (SFGs) and in the range $[2.5, 10]$ for early type galaxies (ETGs), and b is a function of n (i.e., $b = b(n)$).

   What is the ____________, Alex?

   a) de Vaucouleurs profile b) Sérsic profile c) Navarro-Frenk-White profile (NFW profile)
   d) Burkert profile e) Brownstein profile

4. Specific intensity and related quantities (e.g., energy density per unit wavelength) are conventionally given in three representations: photon energy representation I_E, frequency representation I_ν, and wavelength representation I_λ. These representations are related by differential expression

$$I_E\,dE = I_\nu\,d\nu = I_\lambda\,(-d\lambda) ,$$

   where the minus sign is occasionally omitted if one knows what one means—which is that a differential increase in photon energy/frequency corresponds to a differential decrease in wavelength.

   There are parts a,b,c,d. On exams, omit part d.

   a) As well as the three conventional representations, there is a hybrid representation (AKA logarithmic representation)

$$EI_E = \nu I_\nu = \lambda I_\lambda$$

   which has the same value whichever of E, ν, or λ is used as the independent variable. Prove the hybrid representation equality. **Hint:** You will have to use differentials of the logarithm of the independent variables (e.g., $d[\ln(E)]$) and make use of the de Broglie relations $E = h\nu = hc/\lambda$.

   b) Suggest two or three reasons why people might want to use the hybrid representation for graphing.

   c) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency representation is

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^x - 1} , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda} .$$

   Derive the explicit energy representation B_E, wavelength representation B_λ, and hybrid representation $EB_E = \nu B_\nu = \lambda B_\lambda$ in all three of the E, ν and λ forms.

   d) Derive the Rayleigh-Jeans law (small x, small E, small ν, large λ approximation) and the Wien approximation (large x, large E, large ν, small λ approximation) for B_E, B_ν, and B_λ **Hint:** This pretty easy albeit tedious.

5. Say you need to find a root to equation

$$g(y) = 0$$

and no analytic solution is available. The equation my be transcendental: i.e., no finite number operations results in a solution. There are many sophisticated of doing this (e.g., Pr-340ff), but a simple one is by an iteration function suitable if you can constrain the root you are looking for to some interval $y \in [a, b]$. First reaarange the equation as iteration equation

$$y = f(y)$$

and then iterate by feeding the output of function $f(y)$ back into function $f(y)$ as an argument or input. The iteration starts with an initial estimate solution y_0 and proceeds via iterates y_1, y_2, ..., y_{i-1}, y_i, etc. using equation

$$y_i = f(y_{i-1}) \ .$$

But how do you know you will get convergence and not divergence or just wandering. We will investigate convergence in this question.

Note the iteration equation approach (assuming it converges) may be very slow both in computer time and iterations especially if you are trying to converge to high machine precision and, of course, for transcendental equations you will never find exact numerical solution. Faster methods are available (e.g., the Brent method (Pr-352) and Newton-Raphson method (Pr-355)), but if you are just solving a simple one-off problem, the iteration equation method may be fine. In vast multiple variable problems like astrophysical atmosphere problems, a multivariable iteration "equation" may be all you have.

HINT: Drawing diagrams as needed helps.

NOTE: There are parts a,b,c,d,e,f,g,h,i. On exams, only do parts f,g,h.

a) First, without loss of generality adjust the variables such that root is zero. Of course, course you cannot do this in an actual problem unless you already know the answer, but for the proof you can assume you do know the answer. Define two functions

$$y = \pm x \qquad \text{and} \qquad y = f(x) \ .$$

The first function divides the Cartesian plane into 4 quadrants. Show that if $f(x)$ is confined to the side quadrants and never tounches the lines defined by $y = \pm x$ in interval $[-a, a]$ (except at the origin itself which is in the interval $[-a, a]$) that convergence is guaranteed for zeroth iterate $y_0 \in [-a, a]$.

b) In a real problem the interval surrounding the root may not be symmetric about the root. This can lead to divergence with some easily imagined bad behavior in the side-quadrant-confined iteration function $f(x)$. How is divergence easily prevented?

c) In terms of sufficient and necessary conditions for convergence how would you describe the side-quadrant-confined iteration function $f(x)$ condition?

d) What is a simple sufficient, but not necessary, condition side-quadrant-confined iteration function $f(x)$ to give convergence?

e) How would iterates behave if side-quadrant-confined iteration function $f(x)$ were monotonically increasing/decreasing?

f) What makes an iteration function to solve for a root (AKA a zero) better thinking in the simplest sense? Think of the ideal limit.

g) Consider the transcendental equation

$$\frac{1}{2} = (x + 1)e^{-x} \ .$$

Find an iteration function to solve for a that is probably divergent at a first guess. Note this is a real problem, and so the solution is not the origin.

h) For the transcendental equation of part (g)

$$\frac{1}{2} = (x + 1)e^{-x}$$

find an iteration function guaranteed to converge for some interval about the solution. Find the interval of convergence and prove convergence in the interval.

i) Try to solve your convergent iteration equation from part (h) by series expansion in small x. You may have to consult Wikipedia (Wikipedia: Natural logarithm) to see where the series expansion is covergent and where a truncated version is a valid approximation. Then just use the Wikipedia plot to estimate the solution: i.e., the point where $y = x$ and the y value from the iteration function.

j) If you know how to code, iterate to function you found in part (h) to convergence to within machine precision and give the number iteration needed and the result.