Cosmology **NAME:**

Homework 9: The Density of the Universe and Dark Matter

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009 qfull 00410 1 3 0 easy math: Virial theorem derivation

1. The virial theorem was first derived by Rudofl Clausius (1822–1888) in 1870 (before going off to serve in the Franco-Prussian War) in the context of thermodynamics. A quantum mechanics version was later dervived. Here we are only interested in the classical mechanics version for a system of interacting particles that have arrived at a steady-state when averaged over time:

$$\langle T\rangle = -\frac{1}{2}\sum_i \langle \vec{r}_i \cdot \vec{F}_i\rangle \ ,$$

where averaging is over time, T is the total kinetic energy, the sum is over all particles i, $\vec{r}_i$ is the position vector of particle i relative to a general origin, $\vec{F}_i$ is the net force on particle i, and the right-hand side is the virial or virial of Clausius itself. The term virial seems to have no meaning outside of the context of the virial theorem.

If the only internal forces act and they are derivable from a single power-law potential energy with power p that just depends on inter-particle separation, the virial theorem specializes to the power-law-potential-energy form

$$\langle T\rangle = \frac{p}{2}\langle V\rangle \ ,$$

where V is the total potential energy. One can drop the angle brackets for the virial theorem if one knows what one means.

a) The derivation of the virial theorem starts from the scalar rotational inertia formula

$$I = \sum_i m_i r_i^2 \ ,$$

where m_i is the mass of particle i and note the position vectors are **NOT** from and perpendicular to an axis as for the ordinary scalar rotational inertia, but from a general origin. If the system has reached steady-state when averaged over time,

$$\left\langle \frac{dI}{dt}\right\rangle = 0 \qquad \text{and} \qquad \left\langle \frac{d^2I}{dt^2}\right\rangle = 0 \ .$$

As a first step in the derivation, take the 2nd derivative of $I/2$ and substitute using $\vec{p}_i = m_i v_i$, Newton's second law, and the kinetic energy formula. Then take the time average assuming time-average steady-state to get the classical mechanics virial theorem.

b) Assume the system has only internal forces, where F_{ji} is the force of paricle j on particle i. Use some relabeling trickery and Newton's third to obtain

$$\sum_i \vec{r}_i \cdot \vec{F}_i = \frac{1}{2}\sum_{ij} \vec{r}_{ji} \cdot \vec{F}_{ji} \ ,$$

where $\vec{r}_{ji} = \vec{r}_i - \vec{r}_j$.

c) Assume all the internal forces can be derived from the same power-law potential energy dependent only on inter-particle separation: i.e.,

$$\vec{F}_{ji} = -\frac{\partial V}{\partial r_{ji}}\hat{r}_{ji} \qquad \text{with} \qquad V_{ji} \propto r_{ji}^p \ .$$

Complete the the power of the power-law-potential-energy virial theorem.

d) Special the power-law-potential-energy virial theorem for the case of gravity.

SUGGESTED ANSWER:

a) Behold:

$$\frac{1}{2}\frac{dI}{dt} = \sum_i m_i \vec{r}_i \cdot \vec{v}_i = \sum_i \vec{r}_i \cdot \vec{p}_i$$

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \sum_i \left(\vec{v}_i \cdot \vec{p}_i + \vec{r}_i \cdot \frac{d\vec{p}_i}{dt} \right) = 2T + \sum_i \vec{r}_i \cdot \vec{F}_i$$

where $\vec{F}_i$ is the net force on particle i. Assuming the time average steady state gives I constant, and so the general classical mechanics virial theorem follows:

$$\langle T \rangle = -\frac{1}{2} \sum_i \langle \vec{r}_i \cdot \vec{F}_i \rangle \qquad \text{QED.}$$

b) Behold:

$$\sum_i \vec{r}_i \cdot \vec{F}_i = \sum_{ij} \vec{r}_i \cdot \vec{F}_{ji} = \sum_{ij} \vec{r}_j \cdot \vec{F}_{ij} = -\sum_{ij} \vec{r}_j \cdot \vec{F}_{ji}$$

$$= \frac{1}{2} \sum_{ij} (\vec{r}_i - \vec{r}_j) \cdot \vec{F}_{ji} = \frac{1}{2} \sum_{ij} \vec{r}_{ji} \cdot \vec{F}_{ji} \ .$$

c) Behold:

$$\sum_i \vec{r}_i \cdot \vec{F}_i = \frac{1}{2} \sum_{ij} \vec{r}_{ji} \cdot \vec{F}_{ji} \ . = -\frac{1}{2} \sum_{ij} r_{ji} p \frac{V_{ij}}{r_{ji}} = -\frac{p}{2} \sum_{ij} V_{ij} = -pV \ ,$$

where the 1/2 vanished to cancel double counting in the sum over ij—a very tricky point. Taking the time average of the last expression and the power-law-potential-energy virial theorem follows from the general virial theorem:

$$\langle T \rangle = \frac{p}{2} \langle V \rangle \qquad \text{QED.}$$

d) Since $p = -1$ for gravity, we find the gravity virial theorem

$$\langle T \rangle = -\frac{1}{2} \langle V \rangle \qquad \text{QED.}$$

Redaction: Jeffery, 2018jan01

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009 qfull 00420 1 3 0 easy math: spherical symmetrical system potential energy

2. The general differential equation for gravitational potential energy and the special case for a spherically symmetric system are, respectively,

$$dU = V\,dm \qquad \text{and} \qquad dU = -\frac{Gm(r)}{r}\,dm \ ,$$

where V is graviational potential and $m(r)$ is the enclosed mass to radius r. In this problem, we investigate the potential energy of spherical symmetric systems with power-law density profiles.

a) Given

$$\rho = \rho_R \left(\frac{r}{R} \right)^p \ , \qquad \text{derive} \qquad m(r) = \frac{4\pi R^3 \rho_R}{p+3} \left(\frac{r}{R} \right)^{p+3} \ ,$$

where R is the maximum radius and $m(R) = M$ the total mass. Write out the total mass formula $M = m(R)$ explicitly. When will it equal the constant-density total mass formula? Under what condition will $m(r)$ diverge?

b) Now derive the formula for the total potential energy

$$U = -\left(\frac{p+3}{2p+5} \right) \frac{GM^2}{R} \ .$$

Under what condition will U diverge? **Hint:** Write dm in terms of dr.

c) How does $f(p) = (p + 3)/(2p + 5)$ behave as a function of p?

d) Determine the special cases of $f(p)$ for p with values $-5/2$, $-9/4$, -2, -1, 0, 1, 2, and ∞.

SUGGESTED ANSWER:

a) Behold:

$$m(r) = \int_0^r \rho(4\pi r'^2)\, dr' = 4\pi R^3 \rho_R \int_0^x x'^{p+2}\, dx' = \frac{4\pi R^3 \rho_R}{p+3} x^{p+3} = M x^{p+3}$$

where $x = r/R$. The total mass formula is

$$M = \frac{4\pi R^3 \rho_R}{p+3}$$

which equals the constant density formula for $p = 0$, of course. The quantity $m(r)$ will diverge logarithmically for $p = -3$ and like a power for $p < -3$.

b) Behold:

$$U(r) = \int_0^M \left[-\frac{Gm(r)}{r}\right] dm = \int_0^r \left[-\frac{Gm(r)}{r}\right] \rho(4\pi r'^2)\, dr'$$
$$= -G\left(\frac{4\pi R^3 \rho_R}{p+3}\right)(4\pi R^2 \rho_R) \int_0^x x'^{2p+4}\, dx' = -\frac{GM^2}{R}\left(\frac{p+3}{2p+5}\right) x^{2p+5}$$
$$U = U(r = R) = -\left(\frac{p+3}{2p+5}\right)\frac{GM^2}{R}$$

The potential energy will diverge logarithmically for $p = -5/2$ and like a power for $p < -5/2$.

c) The function $f(p)$ is a rectangular hyperbola with asymptotes at $x = -5/2$ and $f = 1/2$. The asymptote at $x = -5/2$ gives a $\pm\infty$ in the function. The slope is

$$f' = -\frac{1}{(2p+5)^2} \ ,$$

and so is negative everywhere except at $\pm\infty$ where it is zero and $x = -5/2$ where it is undefined. The function f starts from a local maximum of 1/2 at $p = -\infty$, declines to zero at $p = -3$, is less than zero for $p \in (-3, -5/2)$, switches from $-\infty$ to ∞ at $p = -5/2$, declines from ∞ to 1 at $p = -2$, and declines to a local minimum of 1/2 at $p = \infty$

d) Behold:

$$f(p) = \begin{cases} (p+3)/(2p+5) & \text{in general;} \\ \pm\infty & \text{for } p = -5/2; \\ 3/2 & \text{for } p = 9/4; \\ 1 & \text{for } p = -2\text{: the ideal galaxy rotation curve case;} \\ 2/3 & \text{for } p = -1; \\ 3/5 & \text{for } p = 0\text{: the uniform density case;} \\ 4/7 & \text{for } p = 1; \\ 5/9 & \text{for } p = 2 \\ 1/2 & \text{for } p = \infty\text{: the hollow thin shell case.} \end{cases}$$

Redaction: Jeffery, 2018jan01

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009 qfull 00430 1 3 0 easy math: astrophysical virial theorem

3. The (most common) astrophysical virial theorem formula is

$$\sigma^2 = \frac{GM}{R} \ ,$$

where σ is the dispersion (i.e., standard deviation) of line-of-sight velocities of a gravitationally-bound, virialized (i.e., evolved to time-averaged steady state) astronomical system (most commonly a galaxy cluster) relative to its center of mass within the virial radius, R is the virial radius defined to be where $\sigma(R)$ is a maximum, and M is the virial mass which is just a characteristic mass for the astronomical system. Virial mass is a simple characteristic mass for comparison between astronomical systems. If one wants a better determination of an astronomical system's mass, one must do a more elaborate calculation. The astrophyical virial theorem can be written in the fiducial-value form for galaxy clusters

$$M = \frac{R\sigma^2}{G} = 2.3251 \times 10^{14} M_\odot \times \left(\frac{R}{1\,\text{Mpc}}\right)\left(\frac{\sigma}{1000\,km/s}\right)^2 ,$$

where the fiducial values have been chosen to be typical for galaxy clusters: diameters of order 2 to 10 Mpc and velocituy dispersions of order 1000 km/s.

Given the gravitational virial theorem and the power-law gravitational potential energy formula (with power p),

$$T = -\frac{1}{2}U \qquad \text{and} \qquad U = -\left(\frac{p+3}{2p+5}\right)\frac{GM^2}{R} ,$$

derive the astrophysical virial theorem formula making reasonable assumptions as needed.

SUGGESTED ANSWER:

Well the total kinetic energy out to virial radius for a set i of astronomical objects is

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \frac{3}{2} \sum_i m_i v_{z,i}^2 = \frac{3}{2} M \frac{\sum_i m_i v_{z,i}^2}{M} = \frac{3}{2} M\sigma^2 ,$$

where we've assumed that all directions have equal dispersions and that one can at least approximately weight the calculation of σ by mass. Recall, we can only measure velocities along the line-of-sight (here labeled by z) by the Doppler effect. Yours truly did wonder if people as a matter of course multiplied the observed line-of-sight dispersion (i.e., σ) by $\sqrt{3}$ to get a total dispersion, but apparently not: no source hints that they do that (e.g., Bi-22).

Now from the gravitational virial theorem and the power-law gravitational potential energy formula, we find

$$\sigma^2 = \frac{1}{3}\left(\frac{p+3}{2p+5}\right)\frac{GM}{R} ,$$

A priori, don't know that the astronomical system will have a power-law density profile at all. In fact, it will probably not have exactly a power-law density profile in any case. However, we can assume that to order of magnitude a power-law density profile is an adequate approximation with a characteristic p value. But what characteristic p should we use? Well very likely the characteristic p value will be smaller than the uniform density case $p = 0$ (which gives p-coefficient 3/5) and it may be smaller than ideal galaxy rotation curve case $p = -2$ (which gives p-coefficient 1). So in any case, a reasonable p-coefficient will be of order 1. So to get a clean characteristic formula for virial mass, we choose the overall coefficent to be 1 and get most common astrophysical virial theorem formula

$$\sigma^2 = \frac{GM}{R} \qquad \text{QED.}$$

Actually, multiplying the σ^2 by 3 to give a total dispersion squared might give a truer estimate of the mass enclosed by R, but apparently that is not the standard rule.

Also note that mass exterior to R is not measured by the astrophysical virial theorem formula. However, since σ reaches its maximum at R, beyond R it may just scale with inverse square-root radius in Keplerian-orbit behavior since there may be no significant mass beyond R in which case the astrophysical virial theorem formula extended to larger radius would not give a different virial mass. To explicate by formula, say that beyond R one has

$$\sigma^2\left(\frac{R}{r}\right) = \frac{GM}{r} .$$

Clearly, M stays constant for $r > R$ with the extended formula.

Fortran-95 Code

```
          grav=6.67384e-11_np
    ! http://en.wikipedia.org/wiki/Gravitational_constant MKS error (31):
    !                    ! so 4 digit accurate, but there is controversy
          pc_m=(1.49597870700e11_np/(pi/(180.0_np*3600.0_np)))
    !      ! http://en.wikipedia.org/wiki/Astronomical_unit
    !      !   r_pc * pc_m = b_AU *au_m /(theta_arcs * pi/(180*3600) defines pc_m
          xmpc_m=pc_m*1.e+6_np
          xmsun=1.98855e30_np ! https://en.wikipedia.org/wiki/Sun
          r=xmpc_m ! https://en.wikipedia.org/wiki/Galaxy_cluster#Basic_properties
diameter 2--10 Mpc
          sigma=1000.0e+3_np !
https://en.wikipedia.org/wiki/Galaxy_cluster#Basic_properties diameter 1000 km/s
          xm=(r*sigma**2/grav)/xmsun
          print*,'Fiducial mass',xm ! 232 5081 8142 0926.805679
```

Redaction: Jeffery, 2018jan01