Cosmology **NAME:**

Homework 8 All: The Age of the Universe

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008 qmult 00100 1 4 5 easy deducto-memory: Hubble parameter

1. "Let's play *Jeopardy*! For \$100, the answer is: Characteristic time time and length scales can be derived from this parameter of the Friedmann equation models for the universe."

   What is the ____________ parameter, Alex?

   a) Lemaître b) de Sitter c) Einstein d) Eddington e) Hubble

   SUGGESTED ANSWER: (e)

   Wrong answers:

   a) The IAU renamed Hubble's law as the Hubble-Lemaître law in 2018, but the Hubble constant and the Hubble parameter were not changed.

   Redaction: Jeffery, 2008jan01

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008 qfull 00320 1 3 0 easy math: exact age of the universe formula for the Lambda-CDM model

2. The exact solution $t(a)$ in scaled parameters for matter-Λ universe (which is the Λ-CDM universe not counting the comparatively brief radiation era) is

$$w = \ln\left(z + \sqrt{z^2+1}\right) ,$$

where the scalings are

$$w = \frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t \qquad \text{and} \qquad z = \left[\frac{a/a_0}{(\Omega_{\mathrm{m},0}/\Omega_{\Lambda,0})^{1/3}}\right]^{3/2} ,$$

where 0 indicates cosmic present, a_0 is the cosmic present scale factor (conventionally set to 1), $\Omega_{\mathrm{m},0}$ is the cosmic present matter density parameter (fiducial value 0.3), $\Omega_{\Lambda,0}$ is the cosmic present Λ or constant dark energy density parameter (fiducial value 0.7), and H_0 is the Hubble constant (fiducial value 70 (km/s)/Mpc).

There are parts a,b,c,d,e,f. The parts c and f can be done independently of part a, but the other parts cannot.

   a) Undo the scalings, replace $\Omega_{\mathrm{m},0}$ by $(1-x)$, $\Omega_{\Lambda,0}$ by x, set $a = a_0$, and scale time to τ using $\tau = H_0 t$ for a simplified age of the universe formula. Simplify the formula as much as you reasonably can.

   b) Starting from the part (a) result, derive the Taylor expansion formula for τ to all orders small x **Hint:** You will need the Taylor expansion

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} .$$

   The Taylor expansion formula for τ is remarkably simple.

   c) Why might you want a small-x Taylor expansion even if you have the exact formula?

   d) Write a pseudocode fragment to sum the Taylor expansion of part (b) to the Kth term. Make it numerically accurate (by adding from smallest terms up) and efficient.

   e) Derive the 2-term asymptotic formula for τ as $x \to 1$.

   f) The exact formula for τ can be replaced by an interpolation formula accurate to within 3 % for all $x \leq 0.99$ and also at $x = 1$:

$$\tau_{\mathrm{interp}} = -\frac{1}{3}\left[\ln(1-x) + \sum_{k=1}^{2}\frac{x^k}{k}\right] + \frac{2}{3}\left[\sum_{k=0}^{2}\frac{x^k}{2k+1}\right] .$$

   Why in general might one want a simple interpolation formula to complement a complex exact formula or procedure of evaluation?

SUGGESTED ANSWER:

a) Behold:

$$w = \ln\left(z + \sqrt{z^2+1}\right)$$
$$\tau = \frac{2}{3}\frac{1}{\sqrt{x}}\ln\left(\frac{\sqrt{x}}{\sqrt{1-x}} + \sqrt{\frac{x}{1-x}+1}\right) = \frac{2}{3}\frac{1}{\sqrt{x}}\ln\left(\frac{1+\sqrt{x}}{\sqrt{1-x}}\right) .$$

b) Behold:

$$\begin{aligned}\tau &= \frac{2}{3}\frac{1}{\sqrt{x}}\ln\left(\frac{1+\sqrt{x}}{\sqrt{1-x}}\right) = \frac{2}{3}\frac{1}{\sqrt{x}}\left[\ln(1+\sqrt{x}) - \frac{1}{2}\ln(1-x)\right] \\ &= \frac{2}{3}\frac{1}{\sqrt{x}}\left[\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{k/2}}{k} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{x^k}{k}\right] = \frac{2}{3}\frac{1}{\sqrt{x}}\left[\sum_{k=0}^{\infty}\frac{x^{(2k+1)/2}}{2k+1} - \sum_{k=1}^{\infty}\frac{x^k}{2k} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{x^k}{k}\right] \\ &= \frac{2}{3}\left[\sum_{k=0}^{\infty}\frac{x^k}{2k+1}\right] .\end{aligned}$$

c) The exact formula may become numerically inaccurate for small x due to subraction of nearly equal values. However, the Taylor expansion taken to sufficiently many terms can be as numerically accurate as you desire for small x to within numerical precision.

d) Behold:

```
print*'Fortran-95 Code'
tau=0.0_np
do410 : do i=K,1,-1
 tau=x*(1.0_np/real(2*i+1,np)+tau)
end do do410
tau=(2.0_np/3.0_np)*(1.0_np+tau)
```

The code is actually in fortran 95, but that passes for pseudocode. Note that by adding terms in order of increasing size you minimize round-off error. This is important when you are using the Taylor expansion to achieve high accuracy for small x.

If you want to use a fixed-length accurate Taylor expansion, you can implement a more compact and maybe more efficient summation: e.g., for $k = 3$ implementing the second formula,

$$\tau_{k=3} = \frac{2}{3}\left(1 + \frac{x}{3} + \frac{x^2}{5} + \frac{x^3}{7}\right) = \frac{2}{3}(C_0 + x(C_1 + x(C_2 + x(C_3)))) ,$$

where $C_k = 1/(2k+1)$. The compact, accurate formula may well be numerically more efficient than the conventional summation if your program executable does powers inefficiently.

In some sense, the compact, accurate formula is obvious. But can we make it more obvious? Behold:

$$S = \sum_{\ell=0}^{L} x^\ell C_\ell = C_0 + \left\{\prod_{\ell=1}^{L} x(C_\ell + \right\} 0\times)^L ,$$

where the last expression is somewhat symbolic. Maybe this helps.

e) By inspection,

$$t_{\text{asy}} = \frac{2}{3}\ln\left(\frac{2}{\sqrt{1-x}}\right) = -\frac{1}{3}\ln(1-x) + \frac{2}{3}\ln(2) .$$

f) A simple interpolation formula allows mental and visual understanding of a complex exact formula/procedure. Also, in some cases, the interpolation formula may be preferred for calculations if it is sufficiently accurate and more efficient than the exact formula/procedure. It can even be more accurate numerically in some cases than the exact formula/procedure.

Redaction: Jeffery, 2018jan01

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008 qfull 00510 1 3 0 easy math: quadratic formula made numerically robust

3. The quadratic formula (which is the solution of the quadratic equation) is an infamous example of case where the standard analytic form (which is what everyone remembers) is numerically rotten. The equation and formula in standard form are, respectively,

$$ax^2 + bx + c = 0 \qquad \text{and} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

The numerical rottenness occurs if $|4ac| << b^2$: in this case, one of the roots can become affected by severe round-off error. We'll see how to fix the problem in this problem.

There are parts a,b,c,d,e,f. The parts cannot be done independently, but parts (a) and (b) are not so hard and the later parts are just intricate.

a) Solve the quadratic equation for the standard analytic quadratic formula using completing the square. Note we assume that a, b, and c are pure real numbers.

b) The robust numerical form of the quadratic formula can be derived starting from the steps in part (a)

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \qquad x + \frac{b}{2a} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac}$$

when you realize that an equally valid second step to the first step is

$$x + \frac{b}{2a} = \pm\frac{\mathrm{sgn}(b)}{(-2a)}\sqrt{b^2 - 4ac} ,$$

where the sign function is given by

$$\mathrm{sgn}(b) = \begin{cases} 1 & \text{for } b > 0; \\ 1 & \text{for } b = 0 \text{ which is unlike the usual definition of 0;} \\ -1 & \text{for } b < 0. \end{cases}$$

From the equally valid second step, solve for both x_+ (i.e., the upper case solution) and x_- (the lower case solution) in terms of

$$q = -\frac{1}{2}\mathrm{sgn}(b)\left(|b| + \sqrt{b^2 - 4ac}\right)$$

and explain why these formulae are numerically robust. **Hint:** You will have to use difference of squares: i.e.,

$$(a + b)(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2 .$$

c) What can you say about the robust solutions when the discriminant $(b^2 - 4ac) < 0$ and what can you say about q, a, b, and c in this case.

d) What can you say about the robust solutions when $a = 0$ and $q \neq 0$, and what can you say about q, b, and c in this case.

e) What can you say about the robust solutions when $a \neq 0$ and $q = 0$, and what can you say about a, b, and c in this case.

f) What can you say about the robust solutions when $a = 0$ and $q = 0$, and what can you say about b and c in this case.

SUGGESTED ANSWER:

a) Assuming a is nonzero, we proceed as follows:

$$0 = ax^2 + bx + c \qquad 0 = x^2 + \frac{b}{a}x + \frac{c}{a} \qquad 0 = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} \qquad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \qquad x + \frac{b}{2a} = \pm\frac{1}{|2a|}\sqrt{b^2 - 4ac} \qquad x + \frac{b}{2a} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

b) Behold:

$$
\begin{aligned}
x + \frac{b}{2a} &= \pm\frac{\mathrm{sgn}(b)}{-2a}\sqrt{b^2 - 4ac} \\
x_\pm &= \frac{-(1/2)\mathrm{sgn}(b)\left(|b| \pm \sqrt{b^2-4ac}\right)}{a} \\
x_\pm &= \frac{-(1/2)\mathrm{sgn}(b)^2\left[|b|^2 - (b^2-4ac)\right]}{a[\mathrm{sgn}(b)]\left(|b| \mp \sqrt{b^2-4ac}\right)} \\
x_\pm &= \frac{-(1/2)(4ac)}{a[\mathrm{sgn}(b)]\left(|b| \mp \sqrt{b^2-4ac}\right)} \\
x_\pm &= \frac{c}{-(1/2)\mathrm{sgn}(b)\left(|b| \mp \sqrt{b^2-4ac}\right)} \ .
\end{aligned}
$$

We now see that

$$x_+ = \frac{q}{a} \qquad \text{and} \qquad x_- = \frac{c}{q} \ ,$$

and that both these formulae are numerically robust because q is not subject to round-off error when $|4ac| << b^2$ since it involves only an addition of $|b|$ and $\sqrt{b^2-4ac}$. Recall

$$q = -\frac{1}{2}\mathrm{sgn}(b)\left(|b| + \sqrt{b^2-4ac}\right) \ .$$

c) If discriminant $(b^2 - 4ac) < 0$, there are two complex solutions. All you can say about about q, a, b, and c in this case is that q is complex, $|b| < 2\sqrt{ac}$, and neither a and c can be zero and both must be positive or both must be negative.

d) In this case, x_+ is indeterminate, $q = -b$, and $x_- = -c/b$ is the only solution. Note $b \neq 0$ since $q \neq 0$ and c is unconstrained. Note also that the $x_- = -c/b$ solution is what you get directly from the quadratic equation with $a = 0$, and so is exactly correct.

e) In this case, x_- is indeterminate and $x_+ = 0$ is the only solution. Note since $q = 0$, we must have $b = 0$ (since $\sqrt{b^2-4ac}$ can only contribute a positive value or an imaginary value to q and neither of them can cancel $|b| \neq 0$) and then $c = 0$ for $q = (1/2)\mathrm{sgn}(b)\sqrt{-4ac} = 0$ with $a \neq 0$.

f) In this case, the both x_+ and x_- are indeterminate and there are no solutions. Since $q = 0$, we have $b = 0$ (since $\sqrt{b^2-4ac}$ can only contribute a positive value or an imaginary value to q and neither of them can cancel $|b| \neq 0$). Also, $c = 0$ for a consistent quadratic equation.

Redaction: Jeffery, 2018jan01