Cosmology **NAME:**

Homework 8 All: The Age of the Universe

1. "Let's play *Jeopardy*! For \$100, the answer is: Characteristic time time and length scales can be derived from this parameter of the Friedmann equation models for the universe."

   What is the ____________ parameter, Alex?

   a) Lemaître b) de Sitter c) Einstein d) Eddington e) Hubble parameter

2. The exact solution $t(a)$ in scaled parameters for matter-Λ universe (which is the Λ-CDM universe not counting the comparatively brief radiation era) is

$$w = \ln\left(z + \sqrt{z^2+1}\right) ,$$

where the scalings are

$$w = \frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t \qquad \text{and} \qquad z = \left[\frac{a/a_0}{(\Omega_{\mathrm{m},0}/\Omega_{\Lambda,0})^{1/3}}\right]^{3/2} ,$$

where 0 indicates cosmic present, a_0 is the cosmic present scale factor (conventionally set to 1), $\Omega_{\mathrm{m},0}$ is the cosmic present matter density parameter (fiducial value 0.3), $\Omega_{\Lambda,0}$ is the cosmic present Λ or constant dark energy density parameter (fiducial value 0.7), and H_0 is the Hubble constant (fiducial value 70 (km/s)/Mpc).

There are parts a,b,c,d,e,f. The parts c and f can be done independently of part a, but the other parts cannot.

   a) Undo the scalings, replace $\Omega_{\mathrm{m},0}$ by $(1-x)$, $\Omega_{\Lambda,0}$ by x, set $a = a_0$, and scale time to τ using $\tau = H_0 t$ for a simplified age of the universe formula. Simplify the formula as much as you reasonably can.

   b) Starting from the part (a) result, derive the Taylor expansion formula for τ to all orders small x **Hint:** You will need the Taylor expansion

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} .$$

   The Taylor expansion formula for τ is remarkably simple.

   c) Why might you want a small-x Taylor expansion even if you have the exact formula?

   d) Write a pseudocode fragment to sum the Taylor expansion of part (b) to the Kth term. Make it numerically accurate (by adding from smallest terms up) and efficient.

   e) Derive the 2-term asymptotic formula for τ as $x \to 1$.

   f) The exact formula for τ can be replaced by an interpolation formula accurate to within 3 % for all $x \leq 0.99$ and also at $x = 1$:

$$\tau_{\mathrm{interp}} = -\frac{1}{3}\left[\ln(1-x) + \sum_{k=1}^{2}\frac{x^k}{k}\right] + \frac{2}{3}\left[\sum_{k=0}^{2}\frac{x^k}{2k+1}\right] .$$

   Why in general might one want a simple interpolation formula to complement a complex exact formula or procedure of evaluation?

3. The quadratic formula (which is the solution of the quadratic equation) is an infamous example of case where the standard analytic form (which is what everyone remembers) is numerically rotten. The equation and formula in standard form are, respectively,

$$ax^2 + bx + c = 0 \qquad \text{and} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

The numerical rottenness occurs if $|4ac| << b^2$: in this case, one of the roots can become affected by severe round-off error. We'll see how to fix the problem in this problem.

There are parts a,b,c,d,e,f. The parts cannot be done independently, but parts (a) and (b) are not so hard and the later parts are just intricate.

a) Solve the quadratic equation for the standard analytic quadratic formula using completing the square. Note we assume that a, b, and c are pure real numbers.

b) The robust numerical form of the quadratic formula can be derived starting from the steps in part (a)

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a} \qquad x+\frac{b}{2a}=\pm\frac{1}{2a}\sqrt{b^2-4ac}$$

when you realize that an equally valid second step to the first step is

$$x+\frac{b}{2a}=\pm\frac{\mathrm{sgn}(b)}{(-2a)}\sqrt{b^2-4ac}\ ,$$

where the sign function is given by

$$\mathrm{sgn}(b)=\begin{cases}1 & \text{for } b>0;\\ 1 & \text{for } b=0 \text{ which is unlike the usual definition of 0;}\\ -1 & \text{for } b<0.\end{cases}$$

From the equally valid second step, solve for both x_+ (i.e., the upper case solution) and x_- (the lower case solution) in terms of

$$q=-\frac{1}{2}\mathrm{sgn}(b)\left(|b|+\sqrt{b^2-4ac}\right)$$

and explain why these formulae are numerically robust. **Hint:** You will have to use difference of squares: i.e.,

$$(a+b)(a-b)=a^2-ab+ab-b^2=a^2-b^2\ .$$

c) What can you say about the robust solutions when the discriminant $(b^2-4ac)<0$ and what can you say about q, a, b, and c in this case.

d) What can you say about the robust solutions when $a=0$ and $q\neq 0$, and what can you say about q, b, and c in this case.

e) What can you say about the robust solutions when $a\neq 0$ and $q=0$, and what can you say about a, b, and c in this case.

f) What can you say about the robust solutions when $a=0$ and $q=0$, and what can you say about b and c in this case.