Cosmology & Galaxies **Name:**

Homework 6: The Cosmic Background Radiation, the Cosmic Temperature, and Recombination

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006 qfull 00110 1 3 0 easy math: nu,lambda,log represenations: On exams, omit part d,e

1. Specific intensity and related quantities (e.g., energy density per unit wavelength) are conventionally given in three representations: photon energy representation I_E, frequency representation I_ν, and wavelength representation I_λ. These representations are related by differential expression

$$I_E\,dE = I_\nu\,d\nu = I_\lambda\,(-d\lambda)\ ,$$

where the minus sign is occasionally omitted if one knows what one means—which is that a differential increase in photon energy/frequency corresponds to a differential decrease in wavelength.

There are parts a,b,c,d,e. On exams, omit parts d,e and use mimimal words. Parts a,b,c can done independently, and so do not stop if you can't do a part.

a) As well as the three conventional representations, there is a logarithmic representation

$$EI_E = \nu I_\nu = \lambda I_\lambda$$

which has the same value whichever of E, ν, or λ is used as the independent variable. Prove the logarithmic representation equality. **HINT:** You will have to use differentials of the logarithm of the independent variables (e.g., $d[\ln(E)]$) and make use of the de Broglie relations $E = h\nu = hc/\lambda$.

b) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency representation is

$$B_\nu = \frac{2hv^3}{c^2}\frac{1}{e^x - 1}\ , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda}\ .$$

Derive the explicit energy representation B_E, wavelength representation B_λ, and logarithmic representation $EB_E = \nu B_\nu = \lambda B_\lambda$ in all three of the E, ν and λ forms.

c) Write the Planck's law in the dimensionless frequency representation expression $B_x\,dx$ and derive for $B_x\,dx$ the Rayleigh-Jeans law form (small x) and the Wien approximation form (large x).

d) Suggest one or two reasons why people might want to use the logarithmic representation for plots.

e) Derive the Rayleigh-Jeans law (small x, small E, small ν, large λ approximation) and the Wien approximation (large x, large E, large ν, small λ approximation) for B_E, B_ν, and B_λ **HINT:** This pretty easy albeit tedious.

SUGGESTED ANSWER:

a) First,

$$I_E\,dE = I_\nu\,d\nu = I_\lambda\,(-d\lambda)$$
$$EI_E\,d[\ln(E)] = \nu I_\nu\,d[\ln(\nu)] = \lambda I_\lambda\,\{-d[\ln(\lambda)]\}\ .$$

Second,

$$E = h\nu = hc/\lambda$$
$$\ln(E) = \ln(h\nu) = \ln(hc/\lambda)$$
$$d[\ln(E)] = d[\ln(\nu)] = -d[\ln(\lambda)]\ .$$

Dividing the first result by the second gives the required result:

$$EI_E = \nu I_\nu = \lambda I_\lambda \qquad \text{QED.}$$

b) Behold:

$$B_E = B_\nu\,\frac{d\nu}{dE} = \frac{2hv^3}{c^2}\frac{1}{e^x - 1}\left(\frac{1}{h}\right) = \frac{2v^3}{c^2}\frac{1}{e^x - 1} = \frac{2E^3}{h^3c^2}\frac{1}{e^x - 1}$$

and

$$B_\lambda = -B_\nu \frac{d\nu}{d\lambda} = -\frac{2hv^3}{c^2}\frac{1}{e^x-1}\left(-\frac{c}{\lambda^2}\right) = -\frac{2hc^3}{c^2\lambda^3}\frac{1}{e^x-1}\left(-\frac{c}{\lambda^2}\right) = \frac{2hc^2}{\lambda^5}\frac{1}{e^x-1} ,$$

and so

$$EB_E = \frac{2E^4}{h^3c^2}\frac{1}{e^x-1} = \nu B_\nu = \frac{2hv^4}{c^2}\frac{1}{e^x-1} = \lambda B_\lambda = \frac{2hc^2}{\lambda^4}\frac{1}{e^x-1} .$$

c) Behold:

$$B_x\,dx = B_\nu\,d\nu = \frac{2hv^3}{c^2}\frac{d\nu}{e^x-1} = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \begin{cases} \dfrac{x^3\,dx}{e^x-1} & \text{in general.} \\ x^2\,dx & \text{Rayleigh-Jeans law form.} \\ x^3e^{-x}\,dx & \text{Wien approximation form.} \end{cases}$$

d) First, since $EI_E = \nu I_\nu = \lambda I_\lambda$, there is no wondering about how the values on graphs of them would differ no matter which independent variable is used to evaluate them. The logarithmic represention is neutral. Of course, a graph using log wavelength would have a mirror inversion relative to those using log energy and log frequency. Second, for the logarithmic representation for blackbody or partially blackbody spectra, the maximum location is a good representative location for where the bulk of the energy is located. To explicate, in the dimensionless frequency $x = h\nu/kT$, the peak in energy/frequency representation occurs for $x = 2.82\ldots$ where the integrated energy (integrating from zero frequency) is about 37 %. The peak in wavelength representation occurs for $x = 4.95\ldots$ where the integrated energy (integrating from zero frequency) is about 76 %. Now the peak in the logarithmic representation oocurs for $x = 3.92\ldots$ where the integrated energy (integrating from zero frequency) is about 59 % which is closest to the middle of the energy accumulation. Also about 54 % of the energy is within a band of width $\Delta x = 3$ centered on the maximum location.

e) Behold:

$$B_E = \begin{cases} \dfrac{2E^3}{h^3c^2}\dfrac{1}{e^x-1} & \text{in general;} \\ \dfrac{2E^3}{h^3c^2x} = \dfrac{2E^2}{h^3c^2}kT & \text{for } x << 1\text{: Rayleigh-Jeans law;} \\ \dfrac{2E^3}{h^3c^2}e^{-x} & \text{for } x >> 1\text{: Wien approximation;} \end{cases}$$

$$B_\nu = \begin{cases} \dfrac{2hv^3}{c^2}\dfrac{1}{e^x-1} & \text{in general;} \\ \dfrac{2hv^3}{c^2x} = \dfrac{2v^2}{c^2}kT & \text{for } x << 1\text{: Rayleigh-Jeans law;} \\ \dfrac{2hv^3}{c^2}e^{-x} & \text{for } x >> 1\text{: Wien approximation;} \end{cases}$$

$$B_\lambda = \begin{cases} \dfrac{2hc^2}{\lambda^5}\dfrac{1}{e^x-1} & \text{in general;} \\ \dfrac{2hc^2}{\lambda^5x} = \dfrac{2c}{\lambda^4}kT & \text{for } x << 1\text{; Rayleigh-Jeans law;} \\ \dfrac{2hc^2}{\lambda^5}e^{-x} & \text{for } x >> 1\text{: Wien approximation.} \end{cases}$$

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006 qfull 00220 1 3 0 easy math: Debye function and blackbody radiation results

2. The total Debye function (i.e., the sum of the first and second Debye functions) is

$$D_z = \int_0^\infty \frac{x^z}{e^x-1}\,dx = z!\zeta(z+1) ,$$

(e.g., Wolfram Mathworld: Debye functions; Wikipedia: Debye function) where the factorial function

$$z! = \begin{cases} \int_0^\infty x^z e^{-x}\,dx = z(z-1)! & \text{for } z \text{ not a negative integer and also} \\ & \quad z \neq 0 \text{ for the second form (Ar-543);} \\ n! & \text{for integer } n \geq 0; \\ \sqrt{\pi} & \text{for } z = -1/2 \text{ (Ar-543,544);} \\ \dfrac{(2z)!!}{2^{(z+1/2)}}\sqrt{\pi} & \text{for half-integer } z \geq -1/2 \text{ with } (-1)!! = 1; \end{cases}$$

and Riemann zeta function (without anayltic continuation considered)

$$\zeta(s) = \begin{cases} \displaystyle\sum_{\ell=1}^{\infty} \frac{1}{\ell^s} & \text{in general;} \\ \zeta(1) = \displaystyle\sum_{\ell=1}^{\infty} \frac{1}{\ell} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots & \text{the divergent} \\ & \text{harmonic series} \\ & \text{(Ar-279);} \\ \zeta(2) = \dfrac{\pi^2}{6} = \dfrac{\pi^2}{2\cdot 3} = 1.644934066848226436472415166646\ldots & \\ \zeta(3) = 1.2020569031595942853997381615114\ldots & \\ \zeta(4) = \dfrac{\pi^4}{90} = \dfrac{\pi^4}{2\cdot 3^2\cdot 5} = 1.082323233711138191516003696541\ldots & \\ \zeta(5) = 1.036927755143369926331365486457\ldots & \\ \zeta(6) = \dfrac{\pi^6}{945} = \dfrac{\pi^6}{3^3\cdot 5\cdot 7} = 1.0173430619844491397145179297909\ldots & \\ \zeta(7) = 1.008349277381922826839797549849\ldots & \\ \zeta(8) = \dfrac{\pi^8}{9450} = \dfrac{\pi^8}{2\cdot 3^3\cdot 5^2\cdot 7} = 1.004077356197944339378685238508\ldots & \\ \zeta(9) = 1.002008392826082214417852769232\ldots & \\ \approx \displaystyle\sum_{\ell=1}^{k-1} \frac{1}{\ell^s} + \int_{k-1/2}^{\infty} \frac{1}{x^s}\,dx = \sum_{\ell=1}^{k-1} \frac{1}{\ell^s} + \frac{1/(k-1/2)^{s-1}}{s-1} & \text{integral} \\ & \quad\text{approximation} \\ & \quad\text{for } s > 1; \\ 1 + \dfrac{1}{2^s} & \text{2nd simplest} \\ & \text{asymptotic form} \\ & \quad\text{as } s \to \infty; \\ 1 & \text{asymptotic form as} \\ & \quad s \to \infty \end{cases}$$

(e.g., Wikipedia: Riemann zeta function; OEIS: Riemann zeta function).

There are parts a,b,c,d,e,f. On exams, do only parts a,b,c. Parts a,b,c can be done independently, so don't stop if you can't do one.

a) Prove $D_z = z!\zeta(z+1)$.

b) Determine the general moment formula M_n (where n is the moment power) for the distribution $f(x) = Ax^z/(e^x - 1)$, where A is the normalization constant which you must determine too. Specialize for $n = 0$ (the normalization), $n = 1$ (the mean), and $n = 2$. Determine the general formula for the variance σ^2.

c) From the Planck's law specific intensity,

$$B_\nu = \frac{2hv^3}{c^2}\frac{1}{e^x - 1}\ , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda}\ ,$$

show the total energy density of a blackbody radiation field is

$$\epsilon = a_{\mathrm{R}} T^4\ ,$$

where the radiation constant

$$a_{\rm R} = \frac{8\pi^5 k^4}{15h^3c^3} = (7.56573325028000\ldots)\times 10^{-16}\,{\rm J/m^3/K^4} = 1\,{\rm J/m^3}\times\left(\frac{1}{6029.61649612301\ldots\quad {\rm K}}\right)^4$$

and $T = 6029.61649612301$ is the temperature that gives $1\,{\rm J/m^3}$. The numerical values are **NOT** required for the answer. **HINT:** Remember to change an isotropic specific intensity into a density you must multiply by $4\pi/c$.

d) Show that the mean photon energy of blackbody radiation field is

$$E = \frac{\zeta(4)}{\zeta(3)}(3kT) = (2.70117803291906\ldots)\times kT$$
$$= 2.327695131004933\times 10^{-4}\,{\rm eV}\times T = 1\,{\rm eV}\times\left(\frac{T}{4296.09525182222\ldots\,{\rm K}}\right) ,$$

where $k = (0.8617333262\ldots)\times 10^4\,{\rm eV/K}$. The numerical values are **NOT** required for the answer.

e) Prove by induction that

$$z! = \frac{(2z)!!}{2^{(z+1/2)}}\sqrt{\pi}$$

for half-integer $z \geq -1/2$ with $(-1)!! = 1$.

f) For $s > 1$ and $k \geq 2$,

$$\zeta(s) = \sum_{\ell=1}^{\infty}\frac{1}{\ell^s} \approx \sum_{\ell=1}^{k-1}\frac{1}{\ell^s} + \int_{k-1/2}^{\infty}\frac{1}{x^s}\,dx = \sum_{\ell=1}^{k-1}\frac{1}{\ell^s} + \frac{1/(k-1/2)^{s-1}}{s-1} ,$$

where the summation-to-integral approximation is just the reverse of the Riemann integral-to-midpoint-summation rule which remarkably is more accurate than the trapezoid rule (Wikipedia: Riemann sum: Midpoint rule). The series truncated at term k is always a lower limit on the Riemann zeta function since all the terms are positive. Prove that the integral approximation is always larger (except in the limit that $s \to \infty$) than the term k which means the integral approximation never underestimates the Riemann zeta function. **HINT:** You will need to use L'Hôpital's rule.

SUGGESTED ANSWER:

a) Behold:

$$D_z = \int_0^\infty \frac{x^z}{e^x - 1}\,dx = \int_0^\infty \frac{x^z e^{-x}}{1-e^{-x}}\,dx = \int_0^\infty x^z e^{-x}\left(\sum_{\ell=0}^{\infty} e^{-\ell x}\right)dx = \sum_{\ell=0}^{\infty}\int_0^\infty x^z e^{-(\ell+1)x}\,dx$$
$$= \sum_{\ell=0}^{\infty}\frac{1}{(\ell+1)^{z+1}}\int_0^\infty t^z e^{-t}\,dt = z!\sum_{\ell=1}^{\infty}\frac{1}{\ell^{z+1}} = z!\zeta(z+1) \qquad \text{QED},$$

where we have used the geometric series (Ar-279) given by

$$\frac{1}{1-y} = \sum_{\ell=0} y^\ell$$

which is absolutely convergent for $|y| < 1$. Note we can integrate to where the series is divergent since that point gives zero contribution.

b) Behold:

$$M_n = A\int_0^\infty \frac{x^{z+n}}{e^x - 1}\,dx = \begin{cases} \dfrac{(z+n)!\zeta(z+n+1)}{z!\zeta(z+1)} & \text{in general;} \\ 1 & \text{for normalization } n = 0; \\ (z+1)\dfrac{\zeta(z+2)}{\zeta(z+1)} & \text{for the mean } n = 1; \\ (z+2)(z+1)\dfrac{\zeta(z+3)}{\zeta(z+1)} & \text{for } n = 2. \end{cases}$$

For the variance,

$$\sigma^2 = \langle (x-\bar{x})^2 \rangle = \langle x^2 \rangle - \bar{x} = M_2 - M_1^2 = \left[\frac{(z+1)}{\zeta(z+1)}\right]^2 \left[\left(\frac{z+2}{z+1}\right)\zeta(z+3)\zeta(z+1) - \zeta(z+2)^2\right] .$$

c) Behold:

$$\begin{aligned}
\epsilon &= \frac{4\pi}{c}\int_0^\infty B_\nu\,d\nu = \frac{4\pi}{c}\int_0^\infty \frac{2hv^3}{c^2}\frac{1}{e^x-1}\,d\nu \\
&= \frac{4\pi}{c}\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4\int_0^\infty \frac{x^3}{e^x-1}\,dx = \frac{4\pi}{c}\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4[3!\zeta(4)] = \frac{4\pi}{c}\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4\frac{\pi^4}{15} \\
&= \left(\frac{8\pi^5k^4}{15h^3c^3}\right)T^4 \qquad \text{QED.}
\end{aligned}$$

d) Behold:

$$E = \frac{\int_0^\infty B_\nu\,d\nu}{\int_0^\infty B_\nu/(h\nu)\,d\nu} = kT\frac{\int_0^\infty B_\nu\,d\nu}{\int_0^\infty (B_\nu/x)\,d\nu} = kT\left[\frac{3!\zeta(4)}{2!\zeta(3)}\right] = \frac{\zeta(4)}{\zeta(3)}(3kT) \qquad \text{QED.}$$

e) Prove by induction that

$$z! = \frac{(2z)!!}{2^{(z+1/2)}}\sqrt{\pi}$$

for half-integer $z \geq -1/2$ with $(-1)!! = 1$.

Proof:

Step 1 $\quad \frac{(-1)!!}{2^0}\sqrt{\pi} = \sqrt{\pi} = \left(-\frac{1}{2}\right)! \quad$ proven;

Step 1a $\quad \frac{1!!}{2^1}\sqrt{\pi} = \left(\frac{1}{2}\right)\sqrt{\pi} = \left(\frac{1}{2}\right)! \quad$ proven also;

Step 2 $\quad \frac{(2z-2)!!}{2^{(z-1/2)}}\sqrt{\pi} = (z-1)! \quad$ assumed;

Step 3 $\quad \frac{(2z)!!}{2^{(z+1/2)}}\sqrt{\pi} = \left(\frac{2z}{2}\right)\frac{(2z-2)!!}{2^{(z-1/2)}}\sqrt{\pi} = z(z-1)! = z! \quad$ proven

and that completes the proof: QED.

f) Behold:

$$R = \lim_{s\to\infty}\frac{1/(k-1/2)^{s-1}}{(s-1)/k^s} = \lim_{s\to\infty}\frac{(k-1/2)[k/(k-1/2)]^s}{(s-1)}$$

$$= \begin{cases} 0 & \text{for } [k/(k-1/2] \leq 1 \text{ which is impossible;} \\ \lim_{s\to\infty}\dfrac{(k-1/2)e^{s\ln[k/(k-1/2)]}}{(s-1)} & \\ \quad = \lim_{s\to\infty}(k-1/2)\ln[k/(k-1/2)]e^{s\ln[k/(k-1/2)]} & \text{using L'Hôpital's rule and note} \\ & \quad \ln[k/(k-1/2)] > 0 \text{ for finite } k; \\ \quad = \infty & \text{for } [k/(k-1/2] > 1 \text{ which is true.} \end{cases}$$

The upshot is that the integral approximation for $k \geq 2$ is always larger (except there is an equality in the limit that $s \to \infty$) than just truncating to at term k (which gives a lower limit on the Riemann zeta function value since all the series terms are positive), and so the integral approximation never underestimates the Riemann zeta function.

Fortran-95 Code

```
print*
print*,'Radiation constant etc.'
```

```
          pi=3.14159265358979323846264338327950288419716939937510_np
    !
!!23456789a123456789b123456789c123456789d123456789e123456789f123456789g12
    !           ! https://en.wikipedia.org/wiki/Pi#Approximate_value_and_digits 51
digits
    !      sigma=5.670374419e-8_np   ! ... exact MKS
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
          clight=2.99792458e8_np     ! exact
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
          bolt=1.380649e-23_np       ! exact
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
    !          1 23456789a1
          hplanck=6.62607015e-34_np  ! exact
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
          sigma=(2.0_np*pi**5/15.0_np)*bolt**4/(hplanck**3*clight**2) ! exact
          echarge=1.602176634 e-19_np  ! exact
https://physics.nist.gov/cuu/Constants/Table/allascii.txt
    !                                  ! which also echarge*(1 V) = ... J := 1 eV
                                       ! Thus factor of unity = 1 eV/(echarge*(1
V) J) .
          radcon=4.0_np*sigma/clight ! for 4/c
https://en.wikipedia.org/wiki/Stefan%E2%80%93Boltzmann_law#Energy_density
          tem_radcon=radcon**(-0.25_np)
          print*,'radcon,tem_radcon'
          print*,radcon,tem_radcon
    ! 7.5657332502800046477 3E-0016   6029.6164961230119483
    ! 1 23456789a123456             1234 56789a123456
          zeta3=1.2020569031595942853997381615114_np
          zeta4=1.0823232337111381915160036965411_np
          coef=3.0_np*zeta4/zeta3
          boltev=bolt/echarge
          coefev=coef*boltev
          tem_fid=1/coefev
          print*,'coef,coefev,tem_fid'
          print*,coef,coefev,tem_fid
      2.7011780329190638961          2.3276951310049330360 3E-0004
4296.0952518222229366
    ! 1 23456789a123456             1 23456789a123456             1234
56789a123456
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006 qfull 00320 1 3 0 easy math: The cosmic evolution of the primordial photon gas/CMB.

3. The primordial photon gas (which is conventionly called the cosmic microwave background (CMB) even before it redshifts into the microwave band) after recombination does not signficantly interact with itself, matter, or anything again and photon number in any box scaling with the expansion of the universe is conserved to excellent approximation.

   There are parts a,b,c,d. On exams, do only parts a,b,c. Parts a,b,c can be done independently, so don't stop if you can't do one.

   a) Prove that the energy density of the CMB obeys

   $$\epsilon = \epsilon_0 \left(\frac{a_0}{a}\right)^4 \ ,$$

   where 0 refers to a fiducial cosmic time which could be cosmic present and a is the cosmic scale factor. Note we are not assuming the specific intensity has any particular distribution.

   b) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency representation is

   $$B_\nu = \frac{2hv^3}{c^2}\frac{1}{e^x - 1} \ , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda} \ .$$

Show that the CMB obeys this law at any general time t provided it obeys it at the fiducial time t_0 where $a = a_0$ and temperature is T_0. **HINT:** The photons in a frequency bin $d\nu = (a_0/a)\,d\nu_0$ stay in that frequency bin as the universe evolves, and so obey the same energy scaling as the overall CMB. Thus at general time t, we have

$$I_\nu\,d\nu = \left(\frac{a_0}{a}\right)^4 B_{\nu_0}\,d\nu_0 \ ,$$

where we have assumed the specific intensity at the fiducial time obeys Planck's law. The proof requires showing that $I_\nu\,d\nu = B_\nu\,d\nu$ using a temperature parameter T that obeys a simple formula depending on the cosmic scale factor a. Why is this temperature parameter T the actual temperature at general time t?

c) Given that the CMB specific intensity obeys Planck's law, its energy density is

$$\epsilon = a_{\rm R}T^4 \ ,$$

where $a_{\rm R}$ is the radiation density constant (usually symbolized by a) and T is the temperature. Using the part (b) answer find the energy density at general time t in terms of the energy density ϵ_0 at fiducial time t_0. Is the result consistent with the part (a) answer?

d) It is quite possible to have a radiation field with a Planck's law shape, but not size. Say for example, say you have blackbody radiator sphere of radius R and you are a distance $r \geq R$ from the sphere center. The emitted specific intensity beams all have B_ν, and so the shape of the spectrum at r obeys Planck's law, but its size is smaller. The effect is called geometrical dilution. Determine the geometrical dilution factor $W(\mu)$ (where radial cosine $\mu = \cos(\theta)$) from the integral for mean specfic intensity J_ν at r

$$J_\nu = \frac{1}{4\pi}\int_0^\theta\int_0^{2\pi} B_\nu \sin(\theta')\,d\theta'\,d\phi = W B_\nu \ .$$

HINT: Transform the θ integral to a μ integral and draw a diagram.

SUGGESTED ANSWER:

a) Since photon number is conserved, photon density n goes as $1/a^3$. Now since individual photon energy E goes as $1/a$ due to the cosmological redshift, we must have

$$\epsilon \propto nE \propto \left(\frac{1}{a^3}\right)\left(\frac{1}{a}\right) = \frac{1}{a^4} \ , \qquad \text{and so} \qquad \epsilon = \epsilon_0\left(\frac{a_0}{a}\right)^4 \qquad \text{QED.}$$

b) Behold:

$$\begin{aligned} I_\nu\,d\nu &= \left(\frac{a_0}{a}\right)^4 B_{\nu_0}\,d\nu_0 = \left(\frac{a_0}{a}\right)^4 \frac{2h\nu_0^3}{c^2}\frac{1}{e^{x_0}-1}\,d\nu_0 \\ &= \frac{2h\nu^3}{c^2}\frac{1}{e^{x_0}-1}\,d\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{x}-1}\,d\nu \\ &= B_\nu\,d\nu \ , \end{aligned}$$

provided we define temperature parameter T by

$$\frac{h\nu}{kT} = x = x_0 = \frac{h\nu_0}{kT_0} \qquad \text{implying} \qquad \frac{T}{\nu} = \frac{T_0}{\nu_0} \ ,$$

and so

$$T = \left(\frac{\nu}{\nu_0}\right)T_0 = \left(\frac{a_0}{a}\right)T_0 \qquad \text{or compactly} \qquad T = \left(\frac{a_0}{a}\right)T_0 \ .$$

Since the specific intensity obeys Planck's law (both in shape and size) at general time t with with temperature parameter T acting as temperature, clearly T just is the temperature at general time t.

c) Using the part (b) answer, we have

$$\epsilon = a_{\rm R} T^4 = a_{\rm R} \left[T_0 \left(\frac{a_0}{a} \right) \right]^4 = \left(\frac{a_0}{a} \right)^4 a_{\rm R} T_0^4 = \left(\frac{a_0}{a} \right)^4 \epsilon_0 \ .$$

This result is exactly consistent with the part (a) answer.

d) Let $\mu = \cos\theta$, where θ is the angle measured from the radial direction from the sphere center. Note that

$$\sin(\theta')\, d\theta' = -d\mu'$$

and it is clear from a diagrm that the cosine of the angle from the radial direction to the limb of the sphere is

$$\mu = \cos\theta = \frac{\sqrt{r^2 - R^2}}{r} = \sqrt{1 - \left(\frac{R}{r} \right)^2} \ .$$

Behold:

$$W = \frac{1}{4\pi} \int_{\mu}^{1} \int_{0}^{2\pi} d\mu\, d\phi = \frac{1}{2}(1 - \mu) = \frac{1}{2} \left[1 - \sqrt{1 - \left(\frac{R}{r} \right)^2} \right]$$

or compactly

$$W = \begin{cases} \frac{1}{2} \left[1 - \sqrt{1 - \left(\frac{R}{r} \right)^2} \right] & \text{in general;} \\ \frac{1}{2} & \text{for } R/r = 1\text{: the sphere fills half the sky;} \\ \frac{1}{4} \left(\frac{R}{r} \right)^2 & \text{to 2nd order in small } R/r. \end{cases}$$

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