Cosmology & Galaxies Name:

Homework 6: The Cosmic Background Radiation, the Cosmic Temperature, and Recombination

1. Specific intensity and related quantities (e.g., energy density per unit wavelength) are conventionally given in three representations: photon energy representation I_E, frequency representation I_ν, and wavelength representation I_λ. These representations are related by differential expression

$$I_E\,dE = I_\nu\,d\nu = I_\lambda\,(-d\lambda) \ ,$$

where the minus sign is occasionally omitted if one knows what one means—which is that a differential increase in photon energy/frequency corresponds to a differential decrease in wavelength.

There are parts a,b,c,d,e. On exams, omit parts d,e and use mimimal words. Parts a,b,c can done independently, and so do not stop if you can't do a part.

a) As well as the three conventional representations, there is a logarithmic representation

$$EI_E = \nu I_\nu = \lambda I_\lambda$$

which has the same value whichever of E, ν, or λ is used as the independent variable. Prove the logarithmic representation equality. **HINT:** You will have to use differentials of the logarithm of the independent variables (e.g., $d[\ln(E)]$) and make use of the de Broglie relations $E = h\nu = hc/\lambda$.

b) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency representation is

$$B_\nu = \frac{2hv^3}{c^2}\frac{1}{e^x-1} \ , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda} \ .$$

Derive the explicit energy representation B_E, wavelength representation B_λ, and logarithmic representation $EB_E = \nu B_\nu = \lambda B_\lambda$ in all three of the E, ν and λ forms.

c) Write the Planck's law in the dimensionless frequency representation expression $B_x\,dx$ and derive for $B_x\,dx$ the Rayleigh-Jeans law form (small x) and the Wien approximation form (large x).

d) Suggest one or two reasons why people might want to use the logarithmic representation for plots.

e) Derive the Rayleigh-Jeans law (small x, small E, small ν, large λ approximation) and the Wien approximation (large x, large E, large ν, small λ approximation) for B_E, B_ν, and B_λ **HINT:** This pretty easy albeit tedious.

2. The total Debye function (i.e., the sum of the first and second Debye functions) is

$$D_z = \int_0^\infty \frac{x^z}{e^x-1}\,dx = z!\zeta(z+1) \ ,$$

(e.g., Wolfram Mathworld: Debye functions; Wikipedia: Debye function) where the factorial function

$$z! = \begin{cases} \int_0^\infty x^z e^{-x}\,dx = z(z-1)! & \text{for } z \text{ not a negative integer and also} \\ & \quad z \neq 0 \text{ for the second form (Ar-543);} \\ n! & \text{for integer } n \geq 0; \\ \sqrt{\pi} & \text{for } z = -1/2 \text{ (Ar-543,544);} \\ \dfrac{(2z)!!}{2^{(z+1/2)}}\sqrt{\pi} & \text{for half-integer } z \geq -1/2 \text{ with } (-1)!! = 1; \end{cases}$$

and Riemann zeta function (without anayltic continuation considered)

$$\zeta(s) = \begin{cases} \displaystyle\sum_{\ell=1}^{\infty} \frac{1}{\ell^s} & \text{in general;} \\ \zeta(1) = \displaystyle\sum_{\ell=1}^{\infty} \frac{1}{\ell} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots & \text{the divergent harmonic series (Ar-279);} \\ \zeta(2) = \dfrac{\pi^2}{6} = \dfrac{\pi^2}{2 \cdot 3} = 1.6449340668482264364724151666646\ldots & \\ \zeta(3) = 1.2020569031595942853997381615114\ldots & \\ \zeta(4) = \dfrac{\pi^4}{90} = \dfrac{\pi^4}{2 \cdot 3^2 \cdot 5} = 1.0823232337111381915160036965411\ldots & \\ \zeta(5) = 1.0369277551433699263313654864570\ldots & \\ \zeta(6) = \dfrac{\pi^6}{945} = \dfrac{\pi^6}{3^3 \cdot 5 \cdot 7} = 1.0173430619844491397145179297909\ldots & \\ \zeta(7) = 1.0083492773819228268397975498497\ldots & \\ \zeta(8) = \dfrac{\pi^8}{9450} = \dfrac{\pi^8}{2 \cdot 3^3 \cdot 5^2 \cdot 7} = 1.0040773561979443393786852385087\ldots & \\ \zeta(9) = 1.0020083928260822144178527692324\ldots & \\ \approx \displaystyle\sum_{\ell=1}^{k-1} \frac{1}{\ell^s} + \int_{k-1/2}^{\infty} \frac{1}{x^s}\,dx = \sum_{\ell=1}^{k-1} \frac{1}{\ell^s} + \frac{1/(k-1/2)^{s-1}}{s-1} & \text{integral approximation for } s > 1; \\ 1 + \dfrac{1}{2^s} & \text{2nd simplest asymptotic form as } s \to \infty; \\ 1 & \text{asymptotic form as } s \to \infty \end{cases}$$

(e.g., Wikipedia: Riemann zeta function; OEIS: Riemann zeta function).

There are parts a,b,c,d,e,f. On exams, do only parts a,b,c. Parts a,b,c can be done independently, so don't stop if you can't do one.

a) Prove $D_z = z!\zeta(z+1)$.

b) Determine the general moment formula M_n (where n is the moment power) for the distribution $f(x) = Ax^z/(e^x - 1)$, where A is the normalization constant which you must determine too. Specialize for $n = 0$ (the normalization), $n = 1$ (the mean), and $n = 2$. Determine the general formula for the variance σ^2.

c) From the Planck's law specific intensity,

$$B_\nu = \frac{2hv^3}{c^2} \frac{1}{e^x - 1} , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda} ,$$

show the total energy density of a blackbody radiation field is

$$\epsilon = a_{\mathrm{R}} T^4 ,$$

where the radiation constant

$$a_{\mathrm{R}} = \frac{8\pi^5 k^4}{15 h^3 c^3} = (7.56573325028000\ldots) \times 10^{-16}\,\mathrm{J/m^3/K^4} = 1\,\mathrm{J/m^3} \times \left(\frac{1}{6029.61649612301\ldots\ \ \mathrm{K}} \right)^4$$

and $T = 6029.61649612301$ is the temperature that gives $1\,\mathrm{J/m^3}$. The numerical values are **NOT** required for the answer. **HINT:** Remember to change an isotropic specific intensity into a density you must multiply by $4\pi/c$.

d) Show that the mean photon energy of blackbody radiation field is

$$E = \frac{\zeta(4)}{\zeta(3)}(3kT) = (2.70117803291906\ldots) \times kT$$
$$= 2.327695131004933 \times 10^{-4}\,\text{eV} \times T = 1\,\text{eV} \times \left(\frac{T}{4296.09525182222\ldots\,\text{K}}\right) ,$$

where $k = (0.8617333262\ldots) \times 10^4\,\text{eV/K}$. The numerical values are **NOT** required for the answer.

e) Prove by induction that

$$z! = \frac{(2z)!!}{2^{(z+1/2)}}\sqrt{\pi}$$

for half-integer $z \geq -1/2$ with $(-1)!! = 1$.

f) For $s > 1$ and $k \geq 2$,

$$\zeta(s) = \sum_{\ell=1}^{\infty}\frac{1}{\ell^s} \approx \sum_{\ell=1}^{k-1}\frac{1}{\ell^s} + \int_{k-1/2}^{\infty}\frac{1}{x^s}\,dx = \sum_{\ell=1}^{k-1}\frac{1}{\ell^s} + \frac{1/(k-1/2)^{s-1}}{s-1} ,$$

where the summation-to-integral approximation is just the reverse of the Riemann integral-to-midpoint-summation rule which remarkably is more accurate than the trapezoid rule (Wikipedia: Riemann sum: Midpoint rule). The series truncated at term k is always a lower limit on the Riemann zeta function since all the terms are positive. Prove that the integral approximation is always larger (except in the limit that $s \to \infty$) than the term k which means the integral approximation never underestimates the Riemann zeta function. **HINT:** You will need to use L'Hôpital's rule.

3. The primordial photon gas (which is conventionly called the cosmic microwave background (CMB) even before it redshifts into the microwave band) after recombination does not signficantly interact with itself, matter, or anything again and photon number in any box scaling with the expansion of the universe is conserved to excellent approximation.

There are parts a,b,c,d. On exams, do only parts a,b,c. Parts a,b,c can be done independently, so don't stop if you can't do one.

a) Prove that the energy density of the CMB obeys

$$\epsilon = \epsilon_0\left(\frac{a_0}{a}\right)^4 ,$$

where 0 refers to a fiducial cosmic time which could be cosmic present and a is the cosmic scale factor. Note we are not assuming the specific intensity has any particular distribution.

b) Planck's law (AKA the blackbody specific intensity spectrum) in the frequency representation is

$$B_\nu = \frac{2hv^3}{c^2}\frac{1}{e^x - 1} , \qquad \text{where} \qquad x = \frac{h\nu}{kT} = \frac{hc}{kT\lambda} .$$

Show that the CMB obeys this law at any general time t provided it obeys it at the fiducial time t_0 where $a = a_0$ and temperature is T_0. **HINT:** The photons in a frequency bin $d\nu = (a_0/a)\,d\nu_0$ stay in that frequency bin as the universe evolves, and so obey the same energy scaling as the overall CMB. Thus at general time t, we have

$$I_\nu\,d\nu = \left(\frac{a_0}{a}\right)^4 B_{\nu_0}\,d\nu_0 ,$$

where we have assumed the specific intensity at the fiducial time obeys Planck's law. The proof requires showing that $I_\nu\,d\nu = B_\nu\,d\nu$ using a temperature parameter T that obeys a simple formula depending on the cosmic scale factor a. Why is this temperature parameter T the actual temperature at general time t?

c) Given that the CMB specific intensity obeys Planck's law, its energy density is

$$\epsilon = a_{\text{R}}T^4 ,$$

where $a_{\rm R}$ is the radiation density constant (usually symbolized by a) and T is the temperature. Using the part (b) answer find the energy density at general time t in terms of the energy density ϵ_0 at fiducial time t_0. Is the result consistent with the part (a) answer?

d) It is quite possible to have a radiation field with a Planck's law shape, but not size. Say for example, say you have blackbody radiator sphere of radius R and you are a distance $r \geq R$ from the sphere center. The emitted specific intensity beams all have B_ν, and so the shape of the spectrum at r obeys Planck's law, but its size is smaller. The effect is called geometrical dilution. Determine the geometrical dilution factor $W(\mu)$ (where radial cosine $\mu = \cos(\theta)$) from the integral for mean specfic intensity J_ν at r

$$J_\nu = \frac{1}{4\pi}\int_0^\theta \int_0^{2\pi} B_\nu \sin(\theta')\, d\theta'\, d\phi = W B_\nu \ .$$

HINT: Transform the θ integral to a μ integral and draw a diagram.