Cosmology NAME:

Homework 5: Advanced Solutions of the Friedmann Equation

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005 qfull 00110 1 3 0 easy math: radiation-matter universe somewhat completely

1. The Friedmann equation for the radiation-matter universe (which applies to the observable universe up to of order 10 Gyr) in general scaled form is

$$\left(\frac{\dot{x}}{x}\right)^2 = \Omega_{4,0}x^{-4} + \Omega_{3,0}x^{-3}$$

where x is the cosmic scale factor with $x_0 = 1$ for cosmic present, $\tau = H_0 t$ is the scaled cosmic time with t being cosmic time in time units and H_0 being the Hubble constant, $\Omega_{4,0}$ is the radiation density parameter for cosmic present, and $\Omega_{3,0}$ is the matter density parameter for cosmic present.

NOTE: There are parts a,b,c,d,e,f,g. On exams, do **ONLY** parts a,b,c. The parts a,b,c can be done independently, and so don't stop if you can't do one.

a) Determine the radiation-matter equality scale factor x_{eq}: i.e., the x value that makes the radiation and matter mass-energy equal.

b) Defining $y = x/x_{\text{eq}}$, rewrite the Friedmann equation into a nice integrable form $dw = f(y)\,dy$ (i.e., a special case scaled form), where $w = \tau/\tau_{\text{sc}}$ is rescaled time and the form has no constants. What is τ_{sc} in terms of the density parameters?

c) Solve the Friedmann equation form found in part (b) for $w(y)$ with $w(y = 0) = 0$. You will need the table integral

$$\int \frac{y\,dy}{\sqrt{1+y}} = \frac{2}{3}(y-2)\sqrt{1+y}\ .$$

d) For $w(y)$, write out the special cases $w(y = 0)$ $w(y)$ to 2nd order in small y, $w(y = 1)$ (at the radiation-matter equality) $w(y = 2)$ (at 2 times the radition-matter equality) $w(y = 3)$ (at 3 times the radition-matter equality which is where the exact y(w) formula changes form), and $w(y >> 1)$ (the large y asymptotic limit).

d) Solve for the asymptotic limiting small w and large w forms of $y(w)$.

f) Transform the limiting forms found in part (d) into the general scaled forms: i.e., into $x(\tau)$ forms.

g) This a challenging part if you have some time. Yours truly has probably spent more time than it is worth trying to find good analytic approximate for solutions $x(\tau)$ for cases where no exact solution exists or the exact solution exists, but is too complex for easy understanding. In fact, the V model solutions (Jeffery 2025) provide understandable exact solutions which are analogues to the standard traditional, but non-exact, solutions for the Friedmann equation found by Alexander Alexandrovich Friedmann (1888–1925), Georges Lemaitre (1894–1966), Willem de Sitter (1872–1934), and others long ago. There may be no better way in general to understand those standard traditional, but non-exact, solutions than using those V model solution analogues. However, in special cases, there may be. One special case, is the radiaton-matter universe. In fact, an exact solution for $y(w)$ exists with two mathematically equivalent formulae that look rather different (Jeffery 2026). But both formulae are too complex for easy understanding. However, a fairly accurate, easy-to-understand interpolation formula does exist that agrees asymptotically with the symptotic limiting small w and large w forms of $y(w)$ found in part (d). See if you can find it. **HINT:** The formula uses $\arctan[y_{\text{small}}(w)]$ and it takes some playing around to find it.

SUGGESTED ANSWER:

a) Behold:

$$\Omega_{4,0}x^{-4} = \Omega_{3,0}x^{-3} \qquad \text{implies} \qquad x_{\text{eq}} = \frac{\Omega_{4,0}}{\Omega_{3,0}}\ .$$

b) Behold:

$$d\tau = \frac{dx}{x\sqrt{\Omega_{4,0}x^{-4} + \Omega_{3,0}x^{-3}}} = \frac{dy}{y\sqrt{\Omega_{4,0}x^{-4} + \Omega_{3,0}x^{-3}}} = \frac{y\,dy}{y^2\sqrt{\Omega_{4,0}x^{-4} + \Omega_{3,0}x^{-3}}}$$

$$= \frac{y\,dy}{y^2\sqrt{\Omega_{3,0}}\sqrt{x_{\rm eq}x^{-4}+x^{-3}}} = \frac{y\,dy}{y^2\sqrt{\Omega_{3,0}}\sqrt{x_{\rm eq}^{-3}y^{-4}+x_{\rm eq}^{-3}y^{-3}}}$$

$$\sqrt{\Omega_{3,0}x_{\rm eq}^{-3}}\,d\tau = \frac{y\,dy}{\sqrt{1+y}}$$

$$\frac{\Omega_{3,0}^2}{\Omega_{4,0}^{3/2}}\,d\tau = \frac{y\,dy}{\sqrt{1+y}}$$

$$dw = \frac{y\,dy}{\sqrt{1+y}}\ ,$$

where

$$\tau_{\rm sc} = \frac{\Omega_{4,0}^{3/2}}{\Omega_{3,0}^2}\ .$$

c) Behold:

$$w = \int_0^y \frac{y'\,dy'}{\sqrt{1+y'}} = \frac{2}{3}(y'-2)\sqrt{1+y'}\Big|_0^y = \frac{2}{3}(y-2)\sqrt{1+y}+\frac{4}{3}\ .$$

d) Behold:

$$w = \begin{cases} \frac{2}{3}(y-2)\sqrt{1+y}+\frac{4}{3} & \text{in general.} \\ 0 & \text{for } y=0. \\ \frac{2}{3}(y-2)\left(1+\frac{y}{2}-\frac{y^2}{8}\right)+\frac{4}{3} & \\ \quad = \frac{4}{3}\left[-\left(1-\frac{y}{2}\right)\left(1+\frac{y}{2}-\frac{y^2}{8}\right)+1\right] & \\ \quad = \frac{4}{3}\left(\frac{3y^2}{8}\right) = \frac{y^2}{2} & \text{to 2nd order in small } y. \\ -\frac{2}{3}\sqrt{2}+\frac{4}{3} = \frac{4}{3}\left(1-\frac{1}{\sqrt{2}}\right) & \text{for } y=1\text{: the radiation-matter} \\ \quad = 0.390524291751\ldots & \quad\text{equality.} \\ \frac{4}{3} = 1.333\ldots & \text{for } y=2\text{: 2 times} \\ & \quad\text{the radiation-matter equality.} \\ \frac{8}{3} = 2.666\ldots & \text{for } y=3\text{: 3 times} \\ & \quad\text{the radiation-matter} \\ & \quad\text{equality and where} \\ & \quad\text{the exact } y(w) \text{ formula} \\ & \quad\text{changes form.} \\ \frac{2}{3}y^{3/2} & \text{for } y>>1\text{: the large } y \text{ asymptotic} \\ & \quad\text{limit.} \end{cases}$$

e) Behold:

$$y_{\rm rad} = (2w)^{1/2} \qquad \text{and} \qquad y_{\rm mat} = \left(\frac{3}{2}w\right)^{2/3}\ ,$$

where the subscripts stand, respectively, for exact radiation universe solution and exact matter universe solution.

f) Behold:

$$x_{\rm rad} = \left[2x_{\rm eq}^2\left(\frac{\Omega_{3,0}^2}{\Omega_{4,0}^{3/2}}\right)\tau\right]^{1/2} = \left(2\sqrt{\Omega_{4,0}}\,\tau\right)^{1/2}$$

$$x_{\rm mat} = \left[\frac{3}{2} x_{\rm eq}^{3/2} \left(\frac{\Omega_{3,0}^2}{\Omega_{4,0}^{3/2}}\right) \tau\right]^{2/3} = \left(\frac{3}{2}\sqrt{\Omega_{3,0}}\,\tau\right)^{2/3} .$$

g) An approach to interpolation formulae for radiation-matter universe leads us to formulae that start as the pure radiation universe solution and then using a linear-saturation formula (speaking a bit loosely) transition to the matter universe solution gradually. All the interpolation formulae considered, of course, obey

$$y_{\rm interpol} = \begin{cases} y_{\rm rad} = (2w)^{1/2} & \text{for } w << 1. \\ y_{\rm mat} = \left(\frac{3}{2}w\right)^{2/3} & \text{for } w >> 1. \end{cases}$$

After some playing around, the following formulae were considered:

$$y_{\rm interpol} = \begin{cases} \left(\frac{y_{\rm rad}}{1+y_{\rm rad}}\right)(1+y_{\rm mat}) & \text{Making use of the simple linear-staturation formula to effect the transition.} \\ \tanh(y_{\rm rad})(1+y_{\rm mat}) & \text{Making use of the hyperoblic tangent's small argument linearity and large argument saturation property.} \\ \frac{\arctan[y_{\rm rad}/(\pi/2)]}{(\pi/2)}(1+y_{\rm mat}) & \text{Making use of the} \\ \quad = \left\{\frac{\arctan[(2w)^{1/2}/(\pi/2)]}{(\pi/2)}\right\} & \text{arctangent's small argument} \\ \quad \times \left[1+\left(\frac{3}{2}w\right)^{2/3}\right] & \text{linearity and large} \\ & \text{argument saturation property.} \\ S\left[1-e^{-[y_{\rm rad}+gw]/S}\right] & \text{Where the constants} \\ \quad +y_{\rm mat}\left(1-e^{-hw^{4/3}}\right) & S=1/C,\ g, \text{ and } h \text{ are chosen to make the formula agree with the exact radiation-matter universe solution to order } w^2 \text{ in small } w. \end{cases}$$

Of the three interpolation formulae using a linear-saturation formula, the simple linear-saturation formula is the poorest, the tangent linear-saturation formula is only a little better, and the arctangent linear-saturation formula is much better. The differences are just due to the peculiarities of these formulae.

To determine the relative error of our formulae, we specified y and calculate the exact w for that y from the exact radiation-matter universe solution $w(y)$ and then used the calculated w in the interpolation formulae to find the approximate y values for the calculated w value.

For arctangent linear-saturation formula, relative error goes to zero as $w^{1/2}$ as $w \to 0$, is $\sim -1\times 10^{-3}$ at $w = 4.9983\ldots\times 10^{-5}$ rises to ~ 0.045 at $w = 0.108588\ldots$ then falls to ~ -0.03 at $w = 8/3$, and then is falling asymptotically toward zero reaching ~ -0.044 as $w = 3853\ldots$. Over the range $w \in [4.9983\ldots, 8/3]$, the root mean-square error of representative sample of points is ~ 0.02. on the path to slowly going asymptotically to 0.

For the last formula, the constants can be determined from exact formulae which are too complex to be worth writing out explicitly for each of C, g, and h. The C constant is determined from a quadratic whose value feeds into a formula for g which and then C and

g feed into a formula for h. We have not done this calculation yet. However, the values are expected to be of order 1. In fact, using values of 1 for all the constants except $g = 4/3$ (which makes the formula agree to first order in w with the exact formula), the last formula gives a solution which has a greater maximum error than the arctangent linear-saturation formula by a factor of order 3, but is significantly more accurate as $w \to 0$ and asymptotically as w grows large.

Fortran-95 Code

```
      print*,'Radiation-matter universe for y=1.'
      w=(4.0_np/3.0_np)*(1.0_np-1.0_np/sqrt(2.0_np))
      print*,'1,w'
      print*,1,w
!     1  0.39052429175126996742
!           123456789a123456789b
```

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