Cosmology **NAME:**

Homework 4: The Geometry of the Universe

---

004 qmult 00120 1 4 1 easy deducto-memory: factoring the curvature term

1. The Friedmann equation written in terms of density parameter components with some specializations is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\,(\Omega + \Omega_k + \Omega_\Lambda)$$

where H is the Hubble parameter, H_0 is the Hubble constant, Ω is the sum of all density parameter components (excluding the curvature and Λ components),

$$\Omega_k = -\frac{kc^2}{H_0^2 a^2}$$

is the curvature density parameter component, and

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2} = \frac{\Lambda/(8\pi G)}{3H_0^2/(8\pi G)} = \frac{\rho_\Lambda}{\rho_{\text{crit},0}}$$

is the Λ density parameter component (i.e., the cosmological constant component). At the fiducial cosmic present,

$$\Omega_{k,0} = -\frac{kc^2}{H_0^2 a_0^2}$$

and we are free to factorize k/a_0^2 as we like. In fact, the Robertson-Walker metric choice is to make $k = 0$ for flat space (i.e., Euclidean space), $k = 1$ for positive curvature space (i.e., hyperspherical space with $\Omega_{k,0} < 0$), and $k = -1$ for negative curvature space (i.e., hyperbolical space with $\Omega_{k,0} > 0$). For non-flat space, this implies a definite physical scale for a_0:

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_k|}} = \frac{(4.2827\ldots\,\text{Gpc})/h_{70}}{\sqrt{|\Omega_k|}} = \frac{(13.968\ldots\,\text{Gly})/h_{70}}{\sqrt{|\Omega_k|}}$$

(where $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$) which can be called the curvature radius of the universe. Note formally the Gaussian curvature radius is defined

$$R_G = \frac{a_0}{\sqrt{k}}$$

which is imaginary for $k = -1$ (CL-12).

The Friedmann equation as written has 3 free parameters for cosmic present which we can choose to be H_0, Ω_0, and Ω_Λ. This means we have the constraint $\Omega_0 + \Omega_{k,0} + \Omega_\Lambda = 1$, and so $\Omega_{k,0} = 1 - \Omega_0 - \Omega_\Lambda$, and so $\Omega_{k,0}$ follows if all other density parameters are known by assumption or a fit to data. Tristram et al. (2023) give $\Omega_{k,0} = -0.012(10)$ consistent with 0, and so consistent with flat space.

Assuming $\Omega_k = -0.01$, what is the approximate curvature radius and how does that compare with the radius of the observable universe according to the Λ-CDM model 14.25 Gpc which must be approximately true whatever the correct universe model is (Wikipedia: Observable universe).

a) 43 Gpc; large. b) 430 Gpc; large. c) 43 Gpc; small. d) 430 Gpc; small.
e) 0.043 Gpc; small.

SUGGESTED ANSWER: (a) Behold:

$$a_0 = \frac{(4.2827\ldots\,\text{Gpc})h_{70}}{\sqrt{|\Omega_k|}} = \frac{(4.2827\ldots\,\text{Gpc})h_{70}}{0.1} \approx 43\text{Gpc}\ .$$

Wrong answers:

b) You've divided by 0.01.

Redaction: Jeffery, 2008jan01

---

004 qmult 00150 1 1 2 easy memory: proper distance to the antipodal point

2. For a positive curvature space (i.e., $k = 1$ space), the proper distance to the antipodal point according to the Robertson-Walker metric formulation at cosmic present is

   a) a_0. b) πa_0. c) $2\pi a_0$. d) $a_0/2$. e) $a_0/4$.

   SUGGESTED ANSWER: (b) The Robertson-Walker metric is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] ,$$

where $ds^2 = d\tau^2$ is the spacetime interval (CL-10) and also the squared proper time differential in the convention adopted here (CL-10). The $a(t)$ is the physical curvature radius and r is the conventional dimensionless comoving coordinate and t is cosmic time. The r coordinate is proportional to tangential proper distance at any time. The alternative conventional dimensionless comoving coordinate is χ though this symbol may just be the particular choice of CL-11. The χ is proportional to the radial proper distance at any time. Note

$$r = \begin{cases} \sin(\chi) & \text{for } k = 1 \text{ (positive curvature);} \\ \chi & \text{for } k = 0 \text{ (flat space);} \\ \sinh(\chi) & \text{for } k = -1 \text{ (negative curvature)} \end{cases}$$

and

$$dr = \begin{cases} \cos(\chi)\,d\chi = \sqrt{1-\sin^2(\chi)}\,d\chi = \sqrt{1-r^2}\,d\chi & \text{for } k = 1 \text{ (positive curvature);} \\ d\chi & \text{for } k = 0 \text{ (flat space);} \\ \cosh(\chi)\,d\chi = \sqrt{1+\sinh^2(\chi)}\,d\chi = \sqrt{1+r^2}\,d\chi & \text{for } k = -1 \text{ (negative curvature),} \end{cases}$$

where we have used the hyperbolic identity $\cosh^2 - \sinh^2 = 1$ (Wikipedia: Hyperbolic functions: Useful relations). We now find

$$d\chi = \frac{dr}{\sqrt{1-kr^2}} .$$

There are two ways of seeing for positive curvature space (i.e., $k = 1$ space) that the antipodal point is at proper distance πa_0 at cosmic present. First, any circumference about the origin is perpendicular to a radius from the origin, and thus can be calculated from the tangential proper distance for $d\phi = 0$. One obtains $2\pi r a_0 = 2\pi a_0 \sin(\chi)$. At the antipodal point all the radii converge, and so the circumference there is 0. Thus $\chi = \pi$.

Second, the surface area of a 2-sphere (which is just an ordinary sphere: see Wikipedia: n-sphere) surrounding the origin $\pi a_0^2 r^2 = \pi a_0^2 \sin^2(\chi)$ goes to zero when $\sin(\pi) = 0$. Thus, $\chi = \pi$ must give the antipodal point from the origin.

By either way, πa_0 is the proper distance to the origin at cosmic present.

Wrong answers:

a) A nonsense answer.

Redaction: Jeffery, 2008jan01

---

004 qmult 00180 1 1 4 easy memory: geodesic is a stationary path

3. A geodesic is a ____________ between two points in a general geometry. It is not in general a global minimum path nor a global maximum ____________. However, a sufficiently small segment is always the shortest distance between points in that segment.

   a) non-stationary path b) straight line c) great circle d) stationary path
   e) small circle

   SUGGESTED ANSWER: (d)

Wrong answers:

a) A nonsense answer.

Redaction: Jeffery, 2008jan01

---

004 qmult 00200 1 1 3 easy memory: general metric

4. The spacetime interval (which in relativity is also called the metric) in general is

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\mu$$

where $g_{\mu\nu}$ is the ____________ or sometimes just the metric in another meaning of the term. Note Einstein summation on repeated indices is used.

a) Minkowski tensor b) geodesic c) metric tensor d) gravity tensor
e) stress-energy tensor

SUGGESTED ANSWER: (c) Wikipedia makes it clear that in pure differential geometry the metric is the metric tensor (Wikipedia: Metric tensor), but in relativity the metric can also be the (differential) spacetime interval (Wikipedia: Friedmann-Lemaire-Robertson-Walker metric: Concept).

Wrong answers:
a) This is a special case which is usually called the Minkowski metric.

Redaction: Jeffery, 2008jan01

---

004 qmult 00210 1 1 3 easy memory: Minkowski metric tensor tests

5. The ____________ is

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(CL-24). Some authors define ____________ with an overall negative sign compared to the definition above.

a) Robertson Walker metric tensor b) geodesic tensor c) Minkowski metric tensor
d) gravity tensor e) stress-energy tensor

SUGGESTED ANSWER: (c)

Wrong answers:
a) As Lurch would say AAAaargh.

Redaction: Jeffery, 2008jan01

---

004 qmult 00220 1 4 5 easy deducto-memory: Robertson-Walker metric identified

6. "Let's play *Jeopardy*! For \$100, the answer is:

$$ds^2 = c^2\, dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right] .$$

What is the ____________ metric, Alex?

a) Einstein-Hilbert b) de-Sitter-Schwarzschild c) Eddington-Lemaître
d) Milne-McCrea e) Robertson-Walker

SUGGESTED ANSWER: (e)

Wrong answers:
a) As Lurch would say AAAARGH.
c) Alexander Friedmann and Georges Lemaître independently derived the Robertson-Walker metric in the 1920s and it is sometimes called the Friedmann-Lemaître-Robertson-Walker metric (FLRM metric), but that is too longwinded to say. Robertson and Walker in the 1930s generalized the derivation.

Redaction: Jeffery, 2008jan01

---

001 qmult 00240 1 1 3 easy memory: radial and transverse proper distances

7. The Robertson-Walker metric is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] \ ,$$

where $ds^2 = d\tau^2$ is the spacetime interval (CL-10) and also the squared proper time differential in the convention adopted here (CL-10). The $a(t)$ is the physical curvature radius and r is the conventional dimensionless comoving coordinate and t is cosmic time. The r coordinate is proportional to tangential proper distance at any time. The alternative conventional dimensionless comoving coordinate is χ though this symbol may just be the particular choice of CL-11. The χ is proportional to the radial proper distance at any time. Note

$$r = \begin{cases} \sin(\chi) & \text{for } k=1 \text{ (positive curvature);} \\ \chi & \text{for } k=0 \text{ (flat space);} \\ \sinh(\chi) & \text{for } k=-1 \text{ (negative curvature)} \end{cases}$$

and

$$dr = \begin{cases} \cos(\chi)\,d\chi = \sqrt{1-\sin^2(\chi)}\,d\chi = \sqrt{1-r^2}\,d\chi & \text{for } k=1 \text{ (positive curvature);} \\ d\chi & \text{for } k=0 \text{ (flat space);} \\ \cosh(\chi)\,d\chi = \sqrt{1+\sinh^2(\chi)}\,d\chi = \sqrt{1+r^2}\,d\chi & \text{for } k=-1 \text{ (negative curvature),} \end{cases}$$

where we have used the hyperbolic identity $\cosh^2 - \sinh^2 = 1$ (Wikipedia: Hyperbolic functions: Useful relations). We now find

$$d\chi = \frac{dr}{\sqrt{1-kr^2}} \ .$$

The differential radial proper distance is

$$dD_{\text{proper,radial}} = a(t)\left(\frac{dr}{\sqrt{1-kr^2}}\right) = a(t)\,d\chi \ .$$

The differential transverse proper distance $dD_{\text{proper,transverse}}$ is:

a) $4\pi[a(t)r]^2$. b) $a(t)r$. c) $a(t)r\sqrt{d\theta^2 + \sin^2\theta\,d\phi^2}$. d) $\pi a(t)$. e) $2\pi a(t)$.

SUGGESTED ANSWER: (c)

Wrong answers:
a) A nonsense answer.

Redaction: Jeffery, 2008jan01

---

004 qfull 00350 1 3 0 easy math: some of the geometry of Robertson-Walker metric

8. The Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] \ ,$$

where ds is the (differential) spacetime interval (also equal to $d\tau$ the proper time in the present convention: CL-10), dt is the differential cosmic time interval, the coordinates are for an arbitrary origin in the homogeneous and isotropic spacetime of the Robertson-Walker metric, θ and ϕ are the ordinary polar coordinates, r a dimensionless (i.e., unitless) comoving coordinate for the tangential direction, t is cosmic time, $a(t)$ is the cosmic scale factor with dimensions of length, and $k = 0$ for Euclidean space (i.e., flat space), $k = 1$ for hyperspherical space (i.e., positive curvature space with the geometry of the surface of a 3-sphere which is sphere in 4-dimensional Euclidean space: see Wikipedia: n-sphere) and $k = -1$ for hyperbolical space (i.e., negative curvature space). Note an ordinary sphere is a 2-sphere in math jargon. For $ds^2 > 0$ / $ds^2 = 0$ / $ds^2 < 0$, the interval is timelike / lightlike (or null) / spacelike (CL-10; Carroll-9).

For non-flat space, the Robertson implies a definite physical scale for a_0:

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_k|}} = \frac{(4.2827\ldots\,\mathrm{Gpc})/h_{70}}{\sqrt{|\Omega_k|}} = \frac{(13.968\ldots\,\mathrm{Gly})/h_{70}}{\sqrt{|\Omega_k|}}$$

(where $h_{70} = H_0/[70\,(\mathrm{km/s})/\mathrm{Mpc}]$) which can be called the curvature radius of the universe. Note formally the Gaussian curvature radius is defined

$$R_G = \frac{a_0}{\sqrt{k}}$$

which is imaginary for $k = -1$ (CL-12).

The Friedmann equation as written has 3 free parameters for cosmic present which we can choose to be H_0, Ω_0, and Ω_Λ. This means we have the constraint $\Omega_0 + \Omega_{k,0} + \Omega_\Lambda = 1$, and so $\Omega_{k,0} = 1 - \Omega_0 - \Omega_\Lambda$, and so $\Omega_{k,0}$ follows if all other density parameters are known by assumption or a fit to data. Tristram et al. (2023) give $\Omega_{k,0} = -0.012(10)$ consistent with 0, and so consistent with flat space. For $k = 0$, there is no physically determined a_0 value and one can set it for convenience: e.g., $a_0 = 1\,\mathrm{Gpc}$ or $a_0 = c/H_0 = [4.2827\ldots)/h_{70}]\,\mathrm{Gpc}$ which is the Hubble length. However, for flat universe models, one usually makes $a(t)$ dimensionless and sets $a_0 = 1$. In these models, the comoving coordinates are dimensioned and given units (e.g., Gpc).

The r coordinate is the tangential comoving coordinate since it is proportional to tangential proper distance at any time. The alternative conventional dimensionless comoving coordinate is χ though this symbol may just be the particular choice of CL-11. The χ is proportional to the radial proper distance at any time.

The radial proper distance $D_{\mathrm{P,radial}}$ is given by

$$D_{\mathrm{P,radial}} = a(t)\begin{cases} \chi & \text{for } k = 1 \text{ with } \chi \in [0,\pi]; \\ \chi & \text{for } k = 0 \text{ with } \chi \in [0,\infty]; \\ \chi & \text{for } k = -1 \text{ with } \chi \in [0,\infty], \end{cases}$$

The r coordinate is related to the χ by

$$r = \begin{cases} \sin(\chi) & \text{for } k = 1 \text{ (positive curvature);} \\ \chi & \text{for } k = 0 \text{ (flat space);} \\ \sinh(\chi) & \text{for } k = -1 \text{ (negative curvature)} \end{cases}$$

and

$$dr = \begin{cases} \cos(\chi)\,d\chi = \sqrt{1-\sin^2(\chi)}\,d\chi = \sqrt{1-r^2}\,d\chi & \text{for } k = 1 \text{ (positive curvature);} \\ d\chi & \text{for } k = 0 \text{ (flat space);} \\ \cosh(\chi)\,d\chi = \sqrt{1+\sinh^2(\chi)}\,d\chi = \sqrt{1+r^2}\,d\chi & \text{for } k = -1 \text{ (negative curvature),} \end{cases}$$

where we have used the hyperbolic identity $\cosh^2 - \sinh^2 = 1$ (Wikipedia: Hyperbolic functions: Useful relations). We now find

$$d\chi = \frac{dr}{\sqrt{1-kr^2}} \ .$$

The transverse proper distance $D_{\mathrm{P,transverse}}$ is given by

$$D_{\mathrm{P,transverse}} = a(t)r\sqrt{d\theta^2 + \sin^2\theta\,d\phi^2} \ .$$

The general differential the proper distance D_{P} formula is

$$\begin{aligned} dD_{\mathrm{P}}^2 &= a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right] \\ &= a(t)^2\left[d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2\theta\,d\phi^2)\right] \ . \end{aligned}$$

NOTE: There are parts a,b,c,d. On exams, do **ONLY** parts a,b,c. The parts can be done independently, and so do not stop if you cannot do a part.

a) For the $k = 1$ case, what directions from the origin do radial geodesics lead to the antipodal point (i.e., the antipode)? How far in proper distance is it from the origin to the antipodal point along a radial geodesic? How far in proper distance to make the geodesic round trip from origin to origin?

b) What is the general formula for circumference C in proper distance for a circle at r in terms of r and χ? Sketch a plot of C as a function of χ for all cases of k.

c) Integrate over all solid angle to find the proper surface area A of the curved-space 2-sphere surrounding the origin at comoving coordinate r. This area is analogous to the circumference of a small circle on a ordinary sphere at polar angle θ. Sketch a plot of A as a function of χ for all cases of k. **HINT:** The integration is really easy and $d\theta^2 + \sin^2\theta\, d\phi^2$ is a differential path distance created using the differential Pythagorean theorem and not a differential piece of solid angle.

d) The differential volume for the sphere is $dV = A(\chi)a\, d\chi$. For all k, determine explicit formulae for $V(\chi)$ small χ and then for general χ. What is the maximum value of $V(\chi)$ for $k = 1$? **HINT:** You will need the identities $\sin^2(x) = (1/2)[1 - \cos(2x)]$ and $\sinh^2(x) = (1/2)[\cosh(2x) - 1]$.

SUGGESTED ANSWER:

a) Radial geodesics from the origin lead to the antipodal point for all directions: all roads lead to Rome. This behavior is analogous to following meridians from the pole of an ordinary sphere (i.e., a 2-sphere: see Wikipedia: n-sphere) to the antipodal pole. The proper distance along a geodesic from the origin is

$$D_{\rm P} = \begin{cases} a\chi & \text{in general for } \chi \in [0,\pi]; \\ \pi a & \text{for } \chi = \pi; \\ 2\pi a & \text{for a round trip from the origin to the origin.} \end{cases}$$

So the proper distance to the antipodal point is πa and the proper distance for the round trip is $2\pi a$. These results are analogous to the distances on a ordinary sphere (i.e., a 2-sphere).

b) Behold:

$$C = 2\pi a(t)r = 2\pi a(t) \begin{cases} \sin(\chi) & k = 1 \text{ with } \chi \in [0,\pi]; \\ \chi & k = 0 \text{ with } \chi \in [0,\infty]; \\ \sinh(\chi) & k = -1 \text{ with } \chi \in [0,\infty]. \end{cases}$$

You will have imagine the plot. However, all three curves will grow linearly for small χ with χ starting from $\chi = 0$ with slope $2\pi a(t)$. The $k = 1$ curve will become sinusoidal and reach a maximum a $\chi = \pi/2$ and then fall sinusoidally to zero at the antipodal point where $\chi = \pi$. The $k = 0$ curve just continues as a straight line with χ to $\chi = \infty$. The $k = -1$ curve steepens into an exponential with exponential factor $e^\chi/2$.

c) The differential piece of solid angle is $d\theta \sin\theta\, d\phi$ which integrates immediately to 4π just as in ordinary space. The differential piece of proper area is $(ar)^2 d\theta \sin\theta\, d\phi$. Therefore, the surface area of a sphere surrounding the origin is

$$A(r) = A(\chi) = 4\pi(ar)^2 = 4\pi a^2 \begin{cases} \sin^2(\chi) & k = 1 \text{ with } \chi \in [0,\pi]; \\ \chi^2 & k = 0 \text{ with } \chi \in [0,\infty]; \\ \sinh^2(\chi) & k = -1 \text{ with } \chi \in [0,\infty]. \end{cases}$$

You will have imagine the plot. However, all three curves will initially grow quadratically with χ (i.e., as parabola χ^2) starting from $\chi = 0$. The $k = 1$ curve will become sinusoidal squared and reach a maximum a $\chi = \pi/2$ and then fall squared sinusoidally to zero at the antipodal point where $\chi = \pi$. Near the two endpoints, its shape is parabolic. The $k = 0$ curve just continues as parabola χ^2 to $\chi = \infty$. The $k = -1$ curve steepens into an exponential with exponential factor $e^{2\chi}/4$.

d) For small χ,

$$V(\chi << 1) = \int_0^\chi A(\chi')a\, d\chi' = 4\pi a^3 \int_0^\chi \chi'^2\, d\chi' = \frac{4\pi}{3}(a\chi)^3$$

which is just what you would get for flat space for all χ. For general χ,

$$V(\chi) = \int_0^\chi A(\chi')a\,d\chi' = 4\pi a^3 \int_0^\chi d\chi' \begin{cases} \sin^2(\chi) & k=1 \text{ with } \chi\in[0,\pi]; \\ \chi^2 & k=0 \text{ with } \chi\in[0,\infty]; \\ \sinh^2(\chi) & k=-1 \text{ with } \chi\in[0,\infty]; \end{cases}$$

$$= 4\pi a^3 \int_0^\chi d\chi' \begin{cases} \frac{1}{2}[1-\cos(2\chi')] & k=1 \text{ with } \chi\in[0,\pi]; \\ \chi^2 & k=0 \text{ with } \chi\in[0,\infty]; \\ \frac{1}{2}[\cosh(2\chi')-1] & k=-1 \text{ with } \chi\in[0,\infty]; \end{cases}$$

$$= 4\pi a^3 \begin{cases} \frac{1}{2}\left[\chi-\frac{1}{2}\sin(2\chi')\right] & k=1 \text{ with } \chi\in[0,\pi]; \\ \frac{1}{3}\chi^3 & k=0 \text{ with } \chi\in[0,\infty]; \\ \frac{1}{2}\left[\frac{1}{2}\sinh(2\chi')-\chi\right] & k=-1 \text{ with } \chi\in[0,\infty]; \end{cases}$$

$$= 4\pi a^3 \begin{cases} \frac{\pi}{2} = 2\pi^2 a^3 = (19.7392\ldots)a^3 & k=1 \text{ with } \chi=\pi; \\ \frac{\pi^3}{3} = \frac{4\pi^4}{3}a^3 = (129.878788\ldots)a^3 & k=0 \text{ with } \chi=\pi; \\ \frac{1}{2}\left[\frac{1}{2}\sinh(2\pi)-\pi\right] = (821.406\ldots)a^3 & k=-1 \text{ with } \chi=\pi. \end{cases}$$

So for the hyperspherical space ($k=1$) the total volume is $2\pi^2a^3$. The Fortran-95 code for the numerical calculation is:

```
      pi=acos(-1.0_np)
      pi=3.14159265358979323846264338327950288419716939937510
           23456789a123456789b123456789c123456789d123456789e1
      v1=2.0_np*pi**2
      v0=(4.0_np/3.0_np)*pi**4
      vn=4.0_np*pi*0.5_np*(0.5_np*sinh(2.0_np*pi)-pi)
      print*,'v1,v0,vn'
      print*,v1,v0,vn
! 19.739208802178717239 129.87878804533658300 821.40618335325637295
```

Redaction: Jeffery, 2018jan01

---

004 qfull 00400 1 3 0 easy math: prove Hubble's law from the RW metric

9. The Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] .$$

Note that r is the radial comoving coordinate chosen so that r is proportional to proper distance in the transverse direction (i.e., perpendicular to the radial direction).

Prove Hubble's law in general form from the Robertson-Walker metric: i.e., prove

$$v_{\rm R} = HD_{\rm P} ,$$

where $v_{\rm R} = \dot{D}_{\rm P}$ is the recession velocity, $H = \dot{a}/a$ is the Hubble parameter, and $D_{\rm P}$ is proper (radial) distance. Note proper distance is distance that can be measured at one instant in cosmic time using a ruler: i.e., with $dt=0$, it is

$$D_{\rm P} = \int \sqrt{-ds^2} .$$

The general form of Hubble's law is an exact result, but alas containing two quantities that are not direct observables, $v_{\rm R}$ and $D_{\rm P}$, except asymptotically as $z\to 0$ or, in other words, in the limit where the 1st-order-in-small-z formulae can be treated as exact. The observational Hubble's law is

$$v_{\rm red} = H_0 D_{\rm P,1st} ,$$

where $v_{\rm red} = zc$ is redshift velocity (a direct observable) and $D_{\rm P,1st}$ is proper distance to 1st order in small z as measured from luminosity distance or angular diameter distance (which are direct observables). The observational Hubble's law is very plausible a priori, but a formal proof is left to a later problem.

SUGGESTED ANSWER:

For a proper distance along a radial direction we have

$$D_{\rm P} = a(t)\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = a(t)f(r) \ ,$$

where $f(r)$ is just the displayed integral which is, in fact, time independent. Thus

$$v_{\rm R} = \dot D_{\rm P} = \dot a f(r) \ .$$

Dividing the second by the first expression and rearranging, we get

$$v_{\rm R} = \frac{\dot a}{a} D_{\rm P} = HD_{\rm P} \ , \qquad \text{or, compactly,} \qquad v_{\rm R} = HD_{\rm P} \qquad \text{QED.}$$

Redaction: Jeffery, 2018jan01

---

004 qfull 00500 1 3 0 easy math: cosmological time dilation and cosmological redshift

10. The Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right] \ .$$

Note that r is the radial comoving coordinate chosen so that r is proportional to proper distance in the transverse direction (i.e., perpendicular to the radial direction).

NOTE: There are parts a,b,c,d. The parts can be done independently, so don't stop if you can't do one.

a) For a lightlike interval $ds^2 = 0$ for a light source at comoving coordinate r distant from an observer, prove that

$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = f(r) = F(t_0) - F(t) = \int_t^{t_0} \frac{c\,dt'}{a(t')} \ ,$$

where $f(r)$ is just r integral and $F(t)$ is the antiderivative (or indefinite integral) of $c/a(t)$. The right-hand side integral is the conformal time for light to travel from the light source at comoving coordinate r to the observer. What does the proven result imply about the conformal time in this case?

b) For light signals coming from comoving coordinate r to the observer prove with few words the cosmological time-dilation effect (CL-16,19):

$$\frac{dt_0}{a_0} = \frac{dt}{a(t)} \qquad \text{or} \qquad \frac{dt_0}{dt} = \frac{a_0}{a(t)} \ ,$$

where t is the cosmic time of emission, t_0 is the cosmic time of observation (i.e., the cosmic present), $a_0 = a(t_0)$, dt is the differential time between two emitted light signals, and dt_0 is the differential time between the corresponding two observed signals.

c) Prove without words the cosmological redshift formula $1 + z = a_0/a(t)$. **HINT:** You will have to use the part (b) answer to relate frequency/wavelength of emission to frequency/wavelength of reception.

d) The cosmological redshift formula is a very useful connecting the direct observable cosmological redshift z and the scaling up of the universe since a light signal was emitted $a_0/a(t)$. Why can't it be used to directly determining the function $a(t)$?

SUGGESTED ANSWER:

a) Behold:

$$\frac{dr^2}{1-kr^2} = \frac{c^2\,dt^2}{a(t)^2}$$

$$\pm\frac{dr}{\sqrt{1-kr^2}} = \frac{c\,dt}{a(t)}$$

$$-\int_r^0 \frac{dr'}{\sqrt{1-kr'^2}} = \int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = f(r) = F(t_0) - F(t) = \int_t^{t_0} \frac{c\,dt'}{a(t')} \ .$$

The proven result implies that the conformal time is time independent. No matter when a light signal starts out from the light source it always takes the same conformal time to reach the observer.

b) Behold:

$$f(r) = F(t_0) - F(t) = F(t_0+dt_0) - F(t+dt)$$

$$F(t_0) - F(t) = F(t_0+dt_0) - F(t+dt)$$

$$F(t_0) - F(t) = F(t_0) + dt_0\frac{dF}{dt}\bigg|_{t_0} - \left[F(t) + dt\frac{dF}{dt}\bigg|_t\right]$$

$$dt_0\frac{dF}{dt}\bigg|_{t_0} = dt\frac{dF}{dt}\bigg|_t$$

$$\frac{c\,dt_0}{a(t_0)} = \frac{c\,dt}{a(t)}$$

$$\frac{dt_0}{a(t_0)} = \frac{dt}{a(t)} \qquad \text{or} \qquad \frac{dt_0}{dt} = \frac{a_0}{a(t)} \ ,$$

where we have used the fact that $f(r)$ is independent of cosmic time and have used 1st order Taylor expansions.

c) Behold:

$$\frac{dt_0}{a_0} = \frac{dt}{a(t)} \qquad \frac{1}{\nu_0 a_0} = \frac{1}{\nu a(t)} \qquad \frac{\lambda_0}{a_0} = \frac{\lambda}{a(t)} \qquad \frac{\lambda_0}{\lambda} = \frac{a_0}{a(t)}$$

$$z = \frac{\lambda_0 - \lambda}{\lambda} = \frac{a_0}{a(t)} - 1 \qquad 1+z = \frac{a_0}{a(t)} \ . \qquad \text{QED}$$

d) The cosmic time of emission t is not a direct observable. It would be great if galaxies had clock faces showing cosmic time, but they don't.

Redaction: Jeffery, 2018jan01

---

004 qfull 00610 1 3 0 easy math: Robertson-Walker metric and observables

11. The basic Λ-CDM model has its cosmic scale factor $a(t)$ fully specified via the Friedmann equation (FE) by the Hubble constant H_0 and three density parameters: i.e., $\Omega_{\mathrm{R},0}$ ("radiation"), $\Omega_{\mathrm{m},0}$ ("matter"), and Ω_Λ (cosmological constant or constant dark energy). Obtaining the parameters is a major observational goal. In principle, only 3 are independent, but observational uncertainties make obtaining all 4 somewhat independently useful goal.

If the FE model is not flat, the Friedmann equation (in its derivation from general relativity) plus Robertson-Walker metric tells us that the physical scale of the FE models at cosmic present t_0 is given by

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_0 - 1|}} = \frac{c/H_0}{\sqrt{|\Omega_{k,0}|}} = \frac{(4.2827\ldots\,\mathrm{Gpc})/h_{70}}{\sqrt{|\Omega_{k,0}|}} = \frac{(13.968\ldots\,\mathrm{Gly})/h_{70}}{\sqrt{|\Omega_{k,0}|}} \ ,$$

where Ω_0 is the sum of all density parameters, except $\Omega_{k,0}$, and $h_{70} = H_0/[70\,(\mathrm{km/s})/\mathrm{Mpc}]$ is the reduced Hubble constant which must be 1 to within a few percent. If the FE model is flat, there is no physical scale for the model and a_0 can be chosen arbitrarily or set to dimensionless 1 in which case the comoving distances r have length units and are equal to the proper distances of the cosmic present.

In all cases with a_0 set to a dimensioned physical scale, the proper distance to an object at comoving distance r is

$$D_{\rm P} = a_0 \int_0^r \frac{dr}{\sqrt{1-kr^2}} = a_0 f(r) \ ,$$

where r is comoving coordinate independent of time and $k = 1$ for hyperspherical space, $k = 0$ for Euclidean space (i.e., flat space in which case $f(r) = r$), and $k = -1$ for hyperbolical space. The variable k is called the curvature (Li-24).

One way to test a FE model or fit it to observations is to plot some observable cosmic distance measures $D_{\rm C}$ for objects versus their cosmological redshifts z (which are the only easily obtained direct observables) and then compare to the theoretical cosmic distance measure $D_{\rm C}$ plotted as a function of z. The two best known observable cosmic distance measures (other than cosmological redshift z itself) are the luminosity distance $D_{\rm L}$ and the angular diameter distance $D_{\rm A}$ both of which have explicit dependence on z, but also depend on z via the comoving coordinate $r(z)$ whose z dependence is an observational constraint, not an intrinsic dependence.

NOTE: There are parts a,b,c,d. On exams, omit part d. Use minimal words. Some of the parts can be done independently, and so not stop if you cannot do one.

a) Recall the Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] \ .$$

For a light signal traveling from a source at comoving coordinate r, time t, and cosmological redshift z to the origin (i.e., us) at time t_0 along a radial path, derive an equation from the Robertson-Walker metric relating spatial integral $f(r)$ to time integral $\chi(t)$ (which is actually an alternative comoving coordinate though the symbol χ used by CL-11 may not a standard for it). The left-hand side should depend only on parameters r and k and the right-hand side only on t and t_0. Do **NOT** use any words: just the expressions. **HINT:** The interval is lightlike for a light signal: i.e., $ds = 0$.

b) Formal expressions for r, t, and lookback time $t_{\rm LB}$ for a light signal are, respectively,

$$r = f^{-1}[\chi(t)] = f^{-1}[\chi(a)] = f^{-1}\left[\chi\left(\frac{a_0}{1+z}\right)\right] = f^{-1}[\chi(z)] \ , \qquad t = t(a) = t\left(\frac{a_0}{1+z}\right) = t(z) \ ,$$

and

$$t_{\rm LB} = -\Delta t = -[t(a) - t_0] \ ,$$

where we have used the cosmological redshift formula

$$1+z = \frac{a_0}{a(t)} \ .$$

Note that $f(r) = r$ and $f^{-1}(r) = r$ if curvature $k = 0$.

In order to obtain the proper distance $D_{\rm P} = a_0 f(r) = a_0\chi(z)$ explicitly, from the foregoing formulae, we need to specify an FE model. In general, only numerical results can be obtained. However, the de-Sitter universe (with k general) allows explicit simple formulae for some cosmological distance measures. For the de-Sitter universe,

$$a(t) = a_0 e^{H_0\Delta t} \ ,$$

where in this case the Hubble constant $H_0 = \sqrt{\Lambda/3}$ is time-independent and Δt is the time relative to cosmic present.

For the de-Sitter universe, determine in order the explicit formulae for $\Delta t(z)$, $t_{\rm LB}(z)$, $\chi(z)$, radial proper distance $D_{\rm P}(z)$, and recession velocity $v_{\rm R}(z)$.

What is odd about lookback time $t_{\rm LB}$ as $z \to \infty$ relative to the case of a cosmological model with a point origin (AKA Big Bang singularity)?

c) What is the explicit expression for the deceleration parameter $q_0 = -\ddot{a}_0 a_0/\dot{a}_0^2$ for the de Sitter universe?

d) The formal expressions for the standard cosmological distance measures (expressed in observational form if it exists and is distinct from theoretical forms and then in the theoretical forms) are as follows:

$$\text{Cosmological redshift:} \qquad z = \frac{\lambda_0 - \lambda}{\lambda} = \frac{a_0}{a(t)} - 1$$

$$\text{Lookback time:} \qquad t_{\rm LB} = t_0 - t(a) = -\Delta t$$

$$\text{Comoving coordinate } r\text{:} \qquad r = f^{-1}[\chi(z)] = f^{-1}[\chi(t)]$$

$$\text{Comoving coordinate } \chi\text{:} \qquad \chi(z) = \chi(t) = \int_t^{t_0} \frac{c\,dt'}{a(t')}$$

$$\text{Radial proper distance:} \qquad D_{\rm P} = a_0\chi(z) = a_0\chi(t) = a_0 f(r)$$

$$\text{Recessional velocity:} \qquad v_{\rm R} = H_0 D_{\rm P}$$

$$\text{Redshift velocity:} \qquad v_{\rm red} = zc$$

$$\text{Luminosity distance:} \qquad D_{\rm L} = \sqrt{\frac{L}{4\pi f}} = a_0 r(1+z)$$

$$\text{Angular diameter distance:} \qquad D_{\rm A} = \frac{D_{\rm ruler}}{\theta} = \frac{a_0 r}{(1+z)}$$

$$\text{Distance-duality relation:} \qquad \frac{D_{\rm L}}{D_{\rm A}} = (1+z)^2 \ ,$$

where the distance-duality relation is also called the Etherington reciprocity relation.

Determine special case expressions (if they exist) for the cosmological distance measures above as a functions of z for the de Sitter universe. Note that some were already determined in part (b) and some already functions of z. What is odd about $D_{\rm A}$ as z goes to infinity in the case of $k = 0$?

SUGGESTED ANSWER:

a) Behold:

$$ds^2 = c^2\,dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right]$$

$$\begin{aligned} \pm a(t) \left(\frac{dr}{\sqrt{1-kr^2}} \right) &= c\,dt \\ -\int_r^0 \frac{dr}{\sqrt{1-kr^2}} &= \int_t^{t_0} \frac{c\,dt'}{a(t')} \\ \int_0^r \frac{dr}{\sqrt{1-kr^2}} &= \int_t^{t_0} \frac{c\,dt'}{a(t')} \\ f(r) &= \chi(t) \ . \end{aligned}$$

b) Clearly,

$$\Delta t = \frac{1}{H_0} \ln\left(\frac{a}{a_0}\right) = -\frac{1}{H_0}\ln(1+z) \ , \qquad \text{and so} \qquad t_{\rm LB} = -\Delta t = \frac{1}{H_0}\ln(1+z) \ .$$

There is more than one way to find χ. The smartest way is probably

$$\chi = \int_{\Delta t}^0 \frac{c\,d\Delta t'}{a} = \int_a^{a_0} \frac{c\,da'}{a'\dot{a}'} = \frac{c}{H_0}\int_a^{a_0} \frac{da'}{a'^2} = -\frac{c}{H_0}\left(\frac{1}{a_0} - \frac{1}{a}\right) = -\frac{c}{a_0 H_0}\left(1 - \frac{a_0}{a}\right) = \frac{zc}{a_0 H_0} \ .$$

Another way using $\Delta t = t - t_0$, $\Delta t_0 = 0$, and $d\Delta t = dt$ is

$$\chi = \frac{c}{a_0}\int_{\Delta t}^0 e^{-H_0\Delta t'}\,d\Delta t' = -\frac{c}{a_0 H_0}\left(1 - e^{-H_0\Delta t}\right) = -\frac{c}{a_0 H_0}\left(1 - \frac{a_0}{a}\right) = \frac{zc}{a_0 H_0} \ .$$

Either way, we find

$$\chi = \frac{zc}{a_0 H_0} .$$

Thus,

$$D_{\rm P} = a_0 \chi = \frac{zc}{H_0} \qquad \text{and} \qquad v_{\rm R} = H_0 D_{\rm P} = zc = v_{\rm red} .$$

In this special case, the recession velocity equals the redshift velocity defined by $v_{\rm red} = zc$.

For cosmological models with at point origin, $z \to \infty$ at a finite lookback time $t_{\rm LB}$. But for the de Sitter universe, $a(t) \to 0$ only as $t \to (-\infty)$, and so as $z \to \infty$, $t_{\rm LB} \to \infty$.

c) Behold:

$$q_0 = -\frac{\ddot{a}_0 a_0}{\dot{a}_0^2} = -\frac{H_0^2}{H_0^2} = -1 \qquad \text{or, compactly,} \qquad q_0 = -1 .$$

The deceleration parameter is negative because the exponential universe expansion is positively accelerating.

d) Behold:

Cosmological redshift: $z = \dfrac{\lambda_0 - \lambda}{\lambda} = \dfrac{a_0}{a(t)} - 1$

Lookback time: $t_{\rm LB} = t_0 - t(a) = -\Delta t = \dfrac{1}{H_0} \ln(1+z)$

Comoving coordinate r: $r = f^{-1}[\chi(z)] = f^{-1}\left(\dfrac{zc}{a_0 H_0}\right)$

Comoving coordinate χ: $\chi(z) = \dfrac{zc}{H_0 a_0}$

Radial proper distance: $D_{\rm P} = a_0 f(r) = a_0 \chi(z) = \dfrac{zc}{H_0}$

Recessional velocity: $v_{\rm R} = H_0 D_{\rm P} = zc$

Redshift velocity: $v_{\rm red} = zc$

Luminosity distance: $D_{\rm L} = \sqrt{\dfrac{L}{4\pi f}} = a_0 r(1+z) = a_0 f^{-1}\left(\dfrac{zc}{a_0 H_0}\right)(1+z)$

Angular diameter distance: $D_{\rm A} = \dfrac{D_{\rm ruler}}{\theta} = \dfrac{a_0 r}{(1+z)} = a_0 f^{-1}\left(\dfrac{zc}{a_0 H_0}\right)\dfrac{1}{(1+z)}$

Distance-duality equation: $\dfrac{D_{\rm L}}{D_{\rm A}} = (1+z)^2$.

The odd thing about $D_{\rm A}$ as z goes to infinity for $k = 0$ (where $f(r) = r$, and so $r = f^{-1}(r)$) is that it goes to a constant c/H_0 which is, in fact, the Hubble length. The proof is

$$\lim_{z\to\infty} D_{\rm A} = \lim_{z\to\infty} a_0 f^{-1}\left(\frac{zc}{a_0 H_0}\right)\frac{1}{(1+z)} = \lim_{z\to\infty}\left(\frac{zc}{H_0}\right)\frac{1}{(1+z)} = \frac{c}{H_0} .$$

This means the standard ruler goes to a constant angular diameter as z goes to infinity. The constancy I think is mostly because you are seeing the ruler sort of where it was in the past. But note that the luminosity distance continues to increase, and so that the ruler keeps getting fainter if it is also a standard candle. Note also that the angular diameter distance is based on the small angle approximation and that might fail in some way if the angular diameter distance starts getting smaller (meaning the ruler is bigger on the sky) with z as, in fact, it does for the Λ-CDM model.

Redaction: Jeffery, 2018jan01

---

004 qfull 00650 1 3 0 easy math: conformal time and cosmological redshift

12. The alternative comoving coordinate

$$\chi = \int_t^{t_0} \frac{c\,dt}{a(t)}$$

is also what is called conformal time.

NOTE: There are parts a,b,c,d,f.

a) Starting from the scaled Friedmann equation form

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\sum_p \Omega_{p,0} x^{-p}\right)$$

(where $x = a/a_0$) derive without words an integral formula for $\chi(x)$.

b) Now change the integral formula so that we have $\chi(z)$.

c) In what limit would $\chi(z)$ have an analytic formula?

d) Assuming there is only a single density component with $p > 0$, derive the exact solution for $\chi(z)$.

e) Assuming there is only a single density component with $p = 0$, derive the exact solution for $\chi(z)$.

f) Give the formula for radial proper distance $D_{\rm P}$ with $\chi(z)$ expanded into the integral form. Does $D_{\rm P}$ depend on a_0? Give the formula for $a_0 r$ for all cases of k with $\chi(z)$ unexpanded. Does $a_0 r$ depend on a_0?

SUGGESTED ANSWER:

a) Behold:

$$1) \quad H_0\,dt = \frac{da}{a\sqrt{\sum_p \Omega_{p,0} x^{-p}}} \qquad 2) \quad \frac{H_0}{c}\frac{c\,dt}{a} = \frac{da}{a^2\sqrt{\sum_p \Omega_{p,0} x^{-p}}}$$

$$3) \quad \frac{H_0 a_0}{c}\frac{c\,dt}{a} = \frac{dx}{x^2\sqrt{\sum_p \Omega_{p,0} x^{-p}}} \qquad 4) \quad \chi(x) = \frac{c}{H_0 a_0}\int_x^1 \frac{d\tilde{x}}{\sqrt{\sum_p \Omega_{p,0}\tilde{x}^{-p+4}}} \ .$$

Note for the set of p consisting of $\{4, 3, 2\}$ (i.e., set of $4 - p$ consisting of $\{0, 1, 2\}$), an exact solution exists for the integral. Unfortunately, this exact solution is not for an especially interesting case.

b) Note

$$1) \quad \frac{a_0}{a} = 1 + z \qquad 2) \quad x = \frac{1}{1+z} \qquad 3) \quad dx = -\frac{dz}{(1+z)^2} = -x^2\,dz \ .$$

Thus,

$$\int_x^1 \frac{d\tilde{x}}{\tilde{x}^2\sqrt{\sum_p \Omega_{p,0}\tilde{x}^{-p}}} = \int_0^z \frac{d\tilde{z}}{\sqrt{\sum_p \Omega_{p,0}(1+\tilde{z})^p}} \ ,$$

and so

$$\chi(z) = \frac{c}{H_0 a_0}\int_0^z \frac{d\tilde{z}}{\sqrt{\sum_p \Omega_{p,0}(1+\tilde{z})^p}} \ .$$

Note for $y = 1 + z$, we get

$$\chi(y) = \frac{c}{H_0 a_0}\int_0^y \frac{d\tilde{y}}{\sqrt{\sum_p \Omega_{p,0}\tilde{y}^p}}$$

which for the set of p of $\{0, 1, 2\}$ has an exact solution for the integral. Unfortunately, this exact solution is not for an especially interesting case.

c) In the small z limit where integral for $\chi(z)$ could be expanded in small z series. However, the series probably only converges for the $z < 1$.

d) Behold:

$$\chi(z) = \frac{c}{H_0 a_0} \int_0^z \frac{d\tilde{z}}{\sqrt{(1+\tilde{z})^p}} = \frac{c}{H_0 a_0} \frac{(1+\tilde{z})^{-p/2+1}}{(-p/2+1)}\bigg|_0^z = \frac{c}{H_0 a_0} \frac{1}{(p/2-1)} \left[1 - \frac{1}{(1+z)^{p/2-1}}\right] ,$$

where for interesting cases $p > 2$.

e) Behold:

$$\chi(z) = \frac{c}{H_0 a_0} \int_0^z d\tilde{z} = \frac{zc}{H_0 a_0}$$

which is the de Sitter universe case.

f) Behold:

$$D_{\rm P} = a_0 \chi(z) = \frac{c}{H_0} \int_0^z \frac{d\tilde{z}}{\sqrt{\sum_p \Omega_{p,0}(1+\tilde{z})^p}} .$$

The radial proper distance has no dependence on a_0. Behold:

$$a_0 r = \begin{cases} a_0 \sin[\chi(z)] & \text{for } k = 1; \\ a_0 \chi(z) & \text{for } k = 0; \\ a_0 \sinh[\chi(z)] & \text{for } k = -1. \end{cases}$$

For $k \neq 0$, the $a_0 r$ does depend on a_0 except in the limit of z small. For $k = 0$, the a_0 cancels out just as for $D_{\rm P}$ and in this case $D_{\rm P} = a_0 r = a_0 \chi(z)$.

Redaction: Jeffery, 2018jan01

---

004 qfull 00700 1 3 0 easy math: deceleration parameter

13. The theoretical cosmological distance measures to 2nd order in small cosmological redshift z are conventionally written in terms of the Hubble constant $H_0 = \dot{a}_0/a_0$ and the deceleration parameter $q_0 = -\ddot{a}_0 a_0/\dot{a}_0^2$ (which is unitless or rather has natural units). In fact in the 1970s, cosmology was sometimes comically oversimplified as a search for two numbers: H_0 and q_0 (see A.R. Sandage, 1970, Physics Today, 23, 34, *Cosmology: A search for two numbers*). Nowadays, q_0 has lost some of its glamor. It is now not regarded as a basic parameter of cosmological models, but just one of the derived parameters and its peculiar definition just a historical convention. The fact that the universal expansion is accelerating makes the deceleration parameter negative which is an incongruity.

There are parts a,b.

a) Taylor expand $a(t)$ in small $\Delta t = t - t_0$ to 2nd order and rewrite the coefficients in terms of H_0 and q_0. The rewritten expansion should begin $a(t) = a_0[1 + \ldots$.

b) Recalling the cosmological redshift formaula $1 + z = a_0/a$, rewrite the formula from the part (a) answer as an expansion for z to 2nd order small Δt. **HINT:** You will need the geometric series:

$$\frac{1}{1-x} = \sum_{\ell=0}^{\infty} x^\ell ,$$

which converges for $|x| < 0$ (Ar-279).

c) Now we need to invert the power series for z to find lookback time $t_{\rm LB} = t_0 - t = -\Delta t$ to 2nd order in small z. We will need the power series inversion cofficients. Given

$$\Delta y = \sum_{\ell=1}^{\infty} a_\ell \Delta x^\ell \qquad \text{and} \qquad \Delta x = \sum_{\ell=1}^{\infty} b_\ell \Delta y^\ell ,$$

where the inversion coefficients b_i run $b_1 = 1/a_1$, $b_2 = -a_2/a_1^3$, ... (Ar-316–317).

d) The Friedmann acceleration equation can be used to get a useful expression for the deceleration parameter q_0. Behold:

$$\begin{aligned}
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda}{3} \\
\frac{\ddot{a}a}{\dot{a}^2}H^2 &= -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2} + \rho_\Lambda + 3\frac{p_\Lambda}{c^2}\right) \\
-qH^2 &= -\frac{4\pi G}{3}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{4\pi G}{3H^2}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{1}{2}\frac{1}{\rho_{\text{critical}}}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{1}{2}\left[\Omega_{\text{M}}(1+3w) + \Omega_\Lambda(1+3w_\Lambda)\right] \\
q &= \frac{1}{2}\left[\Omega_{\text{M}} - 2\Omega_\Lambda\right] = \frac{\Omega_{\text{M}}}{2} - \Omega_\Lambda \qquad \text{with } w = 0 \text{ and } w_\Lambda = -1 \text{ as per usual} \\
q &= \frac{1}{2}\left[0.3\alpha_{\text{M}} - 2\times(0.7\alpha_\Lambda)\right] = \frac{1}{2}\left[0.3\alpha_{\text{M}} - 1.4\alpha_\Lambda\right] = 0.15\alpha_{\text{M}} - 0.7\alpha_\Lambda\ ,
\end{aligned}$$

where $\alpha_{\text{M}} = \Omega_{\text{M}}/0.3$ (0.3 being a modern fiducial value) and $\alpha_\Lambda = \Omega_\Lambda/0.7$ (0.7 being a modern fiducial value). Wit the modern fiducial values, one obtains a fidicial modern value $q_0 = -0.55$. Before 1998, people mostly thought $\Omega_\Lambda = 0$ which with $\Omega_{\text{M}} = 0.3$ (which was what it seemed then as well as now) gives $q_0 = 0.15$. However, some people then hoped that $\Omega_{\text{M}} = 1$ which would give $q_0 = 1/2$ which many thought was the great good value. Why?

SUGGESTED ANSWER:

a) Behold:

$$\begin{aligned}
a(t) &= a_0 + \Delta t\dot{a}_0 + \frac{1}{2}\Delta t^2\ddot{a}_0 + \ldots = a_0\left[1 + \Delta t H_0 + \frac{1}{2}\Delta t^2\frac{\ddot{a}_0}{a_0} + \ldots\right] \\
&= a_0\left[1 + \Delta t H_0 - \frac{1}{2}\Delta t^2 q_0 H_0^2 + \ldots\right]
\end{aligned}$$

b) Behold:

$$\begin{aligned}
z &= -1 + a_0/a(t) = -1 + \left[1 - \Delta t H_0 + \frac{1}{2}\Delta t^2 q_0 H_0^2 + \Delta t^2 H_0^2 + \ldots\right] \\
&= -H_0\Delta t + \left(1 + \frac{1}{2}q_0\right)H_0^2\Delta t^2
\end{aligned}$$

c) Behold:

$$t_{\text{LB}} = t_0 - t = -\Delta t = \frac{z}{H_0}\left[1 - \left(1 + \frac{1}{2}q_0\right)z + \ldots\right]\ .$$

d) It made the universe geometry flat (which makes it simpler to understand) and didn't need a cosmological constant. It is also true that nearly exact flatness was a prediction of inflation which was thought of as a promising theory since circa 1980. However, the fact that Ω_{M} kept turning out to be ~ 0.3 suggested to some even before the discovery of the acceleration of the universal expansion that maybe we needed a cosmological constant if inflation was going to be maintained.

Redaction: Jeffery, 2018jan01

---

004 qfull 00710 1 3 0 easy math: small z expressions for the cosmological distance measures

14. To get the small cosmological redshift z formulae for cosmological distance measures one expands $a(t)$ around current time t_0 to 2nd order in $\Delta t = t - t_0$, parameterizes the first expansion coefficients with the Hubble constant $H_0 = \dot{a}_0/a_0$ and the deceleration parameter $q_0 = -\ddot{a}_0 a_0/\dot{a}_0^2$, substitutes for $a(t)$

with z (and thereby assuming t is the start time for a light signal coming from z), and inverts the power series to get lookback time $t_{\rm LB}$ to 2nd order in small z:

$$t_{\rm LB} = \frac{z}{H_0}\left[1 - \left(1 + \frac{1}{2}q_0\right)z + \ldots\right] .$$

One then uses the $t_{\rm LB}$ formula with the Robertson-Walker metric applied to the light signal to get the comoving coordinate r to 2nd order in z:

$$r = \frac{zc}{a_0 H_0}\left[1 - \frac{1}{2}(1 + q_0)z + \ldots\right] .$$

There are parts a,b,c,d. The parts can be done be at least semi-independently, so don't stop necessarily if you can't do a part.

a) Use the 2nd-order-in-z formulae given in the preamble to get the **2nd-order-in-z** formulae (simplified so that there is only one second order term appearing) and **1st-order-in-z** formulae (expressed just one term appearing) for the following standard cosmological distance measures (expressed in observational form if it exists and then theoretical form), except for expression for z itself included for completeness:

Cosmological redshift: $z = \dfrac{\lambda_0 - \lambda_{\rm e}}{\lambda_{\rm e}} = \dfrac{a_0}{a(t)} - 1 \qquad 1 + z = \dfrac{a_0}{a(t)}$

Lookback time: $t_{\rm LB} = t_0 - t(a)$

Comoving coordinate r: $r = f^{-1}\left\{A\left[t_0, t\left(\dfrac{a_0}{1+z}\right)\right]\right\}$

Proper distance: $D_{\rm P} = a_0 f(r)$

Recessional velocity: $v_{\rm R} = H_0 D_{\rm P}$

Redshift velocity: $v_{\rm red} = zc$

Luminosity distance: $D_{\rm L} = \sqrt{\dfrac{L}{4\pi f}} = a_0 r(1+z)$

Angular diameter distance: $D_{\rm A} = \dfrac{D_{\rm ruler}}{\theta} = \dfrac{a_0 r}{(1+z)}$.

b) Under what conditions are the cosmological distances measures direct observables to 1st and 2nd order given that one can measure z?

c) Prove that all the standard cosmological distance measures are the same to 1st order in small z aside from constants. Show what they are in terms of quantity zc/H_0, where $c/H_0 = (13.968\ldots\,{\rm Gly})/h_{70} = (4.2827\ldots\,{\rm Gpc})/h_{70}$ is the Hubble length with $h_{70} = H_0/[70\,({\rm km/s})/{\rm Mpc}]$.

d) Prove the observational Hubble's law:

$$v_{\rm red} = H_0 D_{\text{P-1st}} ,$$

where $D_{\text{P-1st}}$ is proper distance to 1st order in small z as measured from luminosity distance or angular diameter distance.

e) Given that $|q_0| \lesssim 1$, at what z values would one expect the standard cosmological distance measures (with constants applied as needed to make them all all equal to 1st order in z) to diverge by of order or less than 1 %, 10 %, 30 %, 50 %, and 100 %.

SUGGESTED ANSWER:

a) Behold:

Cosmological redshift: $z = \dfrac{\lambda_0 - \lambda_{\rm e}}{\lambda_{\rm e}} = \dfrac{a_0}{a(t)} - 1 \approx \dfrac{a_0}{a(t)}$ for $z >> 1$ $\quad 1 + z = \dfrac{a_0}{a(t)}$

Lookback time: $t_{\rm LB} = t_0 - t(a) = \dfrac{z}{H_0}\left[1 - (1 + \dfrac{1}{2}q_0)z + \ldots\right] = \dfrac{z}{H_0} + \ldots$

Comoving coordinate r: $r = f^{-1}\left\{A\left[t_0, t\left(\dfrac{a_0}{1+z}\right)\right]\right\} = \dfrac{zc}{a_0 H_0}\left[1 - \dfrac{1}{2}(1 + q_0)z + \ldots\right] = \dfrac{zc}{a_0 H_0} + \ldots$

Proper distance: $D_{\rm P} = a_0 f(r) = \dfrac{zc}{H_0}\left[1 - \dfrac{1}{2}(1 + q_0)z + \ldots\right] = \dfrac{zc}{H_0} + \ldots$

Recessional velocity: $v_{\rm R} = H_0 D_{\rm P} = zc\left[1 - \dfrac{1}{2}(1 + q_0)z + \ldots\right] = zc + \ldots$

Redshift velocity: $v_{\rm red} = zc$

Luminosity distance: $D_{\rm L} = \sqrt{\dfrac{L}{4\pi f}} = a_0 r(1 + z) = \dfrac{zc}{H_0}\left[1 + \dfrac{1}{2}(1 - q_0)z + \ldots\right] = \dfrac{zc}{H_0} + \ldots$

Angular diameter distance: $D_{\rm A} = \dfrac{D_{\rm ruler}}{\theta} = \dfrac{a_0 r}{(1 + z)} = \dfrac{zc}{H_0}\left[1 - \left(\dfrac{3}{2} + \dfrac{1}{2}q_0\right)z + \ldots\right] = \dfrac{zc}{H_0} + \ldots$.

b) All the standard cosmological distance measures are direct observables to 1st order in small z if H_0 is known and to 2nd order in small z if H_0 and q_0 are known.

c) By inspection from part (a) to 1st order in small z:

$$\begin{aligned} ct_{\rm LB} = a_0 r = D_{\rm P} = \frac{v_{\rm R}}{H_0} = \frac{v_{\rm red}}{H_0} = D_{\rm L} = D_{\rm A} = \frac{zc}{H_0} \\ = z\left(\frac{13.968\ldots \text{ Gly}}{h_{70}}\right) = z\left(\frac{4.2827\ldots \text{ Gpc}}{h_{70}}\right) \ . \end{aligned}$$

d) By inspection from part (a), we find the observational Hubble's law

$$v_{\rm red} = H_0 D_{\text{P-1st}} \ ,$$

where $D_{\rm P,1st}$ is proper distance to 1st order in small z as measured from luminosity distance or angular diameter distance.

e) By z equal to 0.01, 0.1, 0.3, 0.5 and 1.

Redaction: Jeffery, 2018jan01