Cosmology **NAME:**

Homework 4 All: The Geometry of the Universe

1. The Friedmann equation written in term of density parameter components with some specializations is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0\left(\Omega + \Omega_k + \Omega_\Lambda\right)$$

where H is the Hubble parameter, H_0 is the Hubble constant, Ω is the sum of all density components excluding the curvature and Λ components,

$$\Omega_k = -\frac{kc^2}{H_0^2 a^2}$$

is the curvature component, and

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2} = \frac{\Lambda/(8\pi G)}{3H_0^2/(8\pi G)} = \frac{\rho_\Lambda}{\rho_{\text{crit},0}}$$

is the Λ component (i.e., the cosmological constant component). At the fiducial cosmic present,

$$\Omega_{k,0} = -\frac{kc^2}{H_0^2 a_0^2}$$

and we are free to factorize k/a_0 as we like. In fact, the Robertson-Walker metric choice is to make $k = 0$ for flat space (i.e., Euclidean space), $k = 1$ for positive curvature space (i.e., hyperspherical space), and $k = -1$ for negative curvature space (i.e., hyperbolical space). For non-flat space, this implies a definite physical scale for a_0:

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_k|}} = \frac{(4.2827\ldots\,\text{Gpc})/h_{70}}{\sqrt{|\Omega_k|}} = \frac{(13.968\ldots\,\text{Gly})/h_{70}}{\sqrt{|\Omega_k|}}$$

(where $h_{70} = H_0/[70(\text{km/s})/\text{Mpc}]$) which can be called the curvature radius of the universe. Note for cosmic present, by construction $\Omega_0 + \Omega_{k,0} + \Omega_\Lambda = 1$, and so $\Omega_{k,0} = 1 - \Omega_0 - \Omega_\Lambda$, and so $\Omega_{k,0}$ follows if all other density parameters are known by assumption or a fit to data. Formally, the Gaussian curvature radius is defined

$$R_G = \frac{a_0}{\sqrt{k}}$$

which is imaginary for $k = -1$ (CL-12). Tristram et al. (2023) give $\Omega_k = -0.012(10)$ consistent with 0, and so consistent with flat space. Assuming $\Omega_k = -0.01$, what is the curvature radius and how does that compare with the radius of the observable universe according to the Λ-CDM model 14.25 Gpc which must be approximately true whatever the correct universe model is (Wikipedia: Observable universe).

   a) 43 Gpc; large. b) 430 Gpc; large. c) 43 Gpc; small. d) 430 Gpc; small.
   e) 0.043 Gpc; small.

2. For a positive curvature space (i.e., $k = 1$ space), the proper distance to the antipodes point according to the Robertson-Walker metric formulation at cosmic present is

   a) a_0. b) πa_0. c) $2\pi a_0$. d) $a_0/2$. e) $a_0/4$.

3. A geodesic is a ____________ between two points in some geometry. It is not in general a global minimum path or a global maximum ____________. However, a sufficiently small segment is always the shortest distance between points in that segment.

   a) non-stationary path b) straight line c) great circle d) stationary path
   e) small circle

4. The metric (which in relativity is usually called the spacetime interval) in general is

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\mu$$

where $g_{\mu\nu}$ is the ____________ or sometimes just the metric in another meaning of the term. Note Einstein summation on repeated indices is used.

a) Lorentz tensor b) geodesic c) metric tensor d) gravity tensor
e) stress-energy tensor

5. "Let's play *Jeopardy*! For $100, the answer is:

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] .$$

What is the ____________ metric, Alex?

a) Einstein-Hilbert b) de-Sitter-Schwarzschild c) Eddington-Lemaître
d) Milne-McCrea e) Robertson-Walker

6. The Robertson-Walker metric is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] ,$$

where $ds^2 = d\tau^2$ is the spacetime interval (and also the squared proper time differential in the convention adopted here) and dt is differential cosmic time. The $a(t)$ is the physical curvature radius and r is the conventional dimensionless comoving coordinate and t is cosmic time. The alternative conventional dimensionless comoving coordinate is χ though this symbol may just be the particular choice of CL-11. Note

$$r = \begin{cases} \sin\chi & \text{for } k = 1 \text{ (positive curvature);} \\ \chi & \text{for } k = 0 \text{ (flat space);} \\ \sinh\chi & \text{for } k = -1 \text{ (negative curvature)} \end{cases}$$

and

$$dr = \begin{cases} \cos\chi\,d\chi & \text{for } k = 1 \text{ (positive curvature);} \\ d\chi & \text{for } k = 0 \text{ (flat space);} \\ \cosh\chi\,d\chi & \text{for } k = -1 \text{ (negative curvature)} \end{cases}$$

implying

$$d\chi = \frac{dr}{\sqrt{1-kr^2}}$$

where we have used the hyperbolic identity $\cosh^2 - \sinh^2 = 1$ (Wikipedia: Hyperbolic functions: Useful relations).

The differential radial proper distance is

$$dD_{\text{proper,radial}} = a(t)\left(\frac{dr}{\sqrt{1-kr^2}}\right) = a(t)\,d\chi .$$

The differential transverse proper distance $dD_{\text{proper,transverse}}$ is:

a) $4\pi[a(t)r]^2$. b) $a(t)r$. c) $a(t)r\sqrt{d\theta^2 + \sin^2\theta\,d\phi^2}$. d) $\pi a(t)$. e) $2\pi a(t)$.

7. The Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right] ,$$

where ds is the differential spacetime interval (also equal to $d\tau$ the proper time in the present convention), dt is the differential cosmic time interval, the coordinates are for an arbitrary origin in the homogeneous and isotropic spacetime of the Robertson-Walker metric, θ and ϕ are the ordinary polar coordinates, r a dimensionless (i.e., unitless) comoving coordinate, t is cosmic time, $a(t)$ is the cosmic scale factor with dimensions of length, and $k = 0$ for Euclidean space (i.e., flat space), $k = 1$ for hyperspherical space (i.e., positive curvature space with the geometry of the surface of a 3-sphere which is sphere in 4-dimensional Euclidean space: see Wikipedia: n-sphere) and $k = -1$ for hyperbolical space (i.e., negative curvature space). Note an ordinary sphere is a 2-sphere in math jargon. For $ds^2 > 0$ / $ds^2 = 0$ / $ds^2 < 0$, the interval is timelike / lightlike (or null) / spacelike (CL-10; Carroll-9).

For non-flat space, the Robertson implies a definite physical scale for a_0:

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_k|}} = \frac{(4.2827\ldots\mathrm{Gpc})/h_{70}}{\sqrt{|\Omega_k|}} = \frac{(13.968\ldots\mathrm{Gly})/h_{70}}{\sqrt{|\Omega_k|}}$$

(where $h_{70} = H_0/[70(\mathrm{km/s})/\mathrm{Mpc}])$ which can be called the curvature radius of the universe. Note for cosmic present, by construction $\Omega_0 + \Omega_{k,0} + \Omega_\Lambda = 1$, and so $\Omega_{k,0} = 1 - \Omega_0 - \Omega_\Lambda$, and so $\Omega_{k,0}$ follows if all other density parameters are known by assumption or a fit to data. The quantity $R_\mathrm{G} = a_0/\sqrt{k}$ is called the Gaussian curvature radius (CL-12). It is imaginary for $k = -1$. For $k = 0$, there is no physically determined a_0 value and one can set it for convenience: e.g., $a_0 = 1\,\mathrm{Gpc}$ or $a_0 = c/H_0 = [4.2827\ldots)/h_{70}]\,\mathrm{Gpc}$.

The radial proper distance D_P to radial comoving distance r is given by

$$D_\mathrm{P} = a(t)\begin{cases}\sin(\chi) & k = 1 \text{ with } \chi \in [0, \pi];\\ \chi & k = 0 \text{ with } \chi \in [0, \infty];\\ \sinh(\chi) & k = -1 \text{ with } \chi \in [0, \infty],\end{cases}$$

where r has been parameterized by χ the alternative comoving coordinate:

$$r = \begin{cases}\sin(\chi) & k = 1 \text{ with } \chi \in [0, \pi];\\ \chi & k = 0 \text{ with } \chi \in [0, \infty];\\ \sinh(\chi) & k = -1 \text{ with } \chi \in [0, \infty],\end{cases}$$

$$dr = \begin{cases}\cos(\chi)\,d\chi = \sqrt{1-r^2},d\chi & k = 1 \text{ with } \chi \in [0, \pi];\\ d\chi & k = 0 \text{ with } \chi \in [0, \infty];\\ \cosh(\chi)\,d\chi = \sqrt{1+r^2}\,d\chi & k = -1 \text{ with } \chi \in [0, \infty],\end{cases}$$

where we have used the hyperbolic function identity $\cosh^2(\chi) - \sinh^2(\chi) = 1$. The transverse proper distance $D_\mathrm{p,transverse}$ at radial comoving distance r is given by

$$D_\mathrm{p,transverse} = a(t)r\sqrt{d\theta^2 + \sin^2\theta\,d\phi^2}\ .$$

Let's just consider the spatial geometry for the hyperspherical case ($k = 1$). Now we have the proper distance D_P formula

$$dD_\mathrm{P}^2 = a(t)^2\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right] = a(t)^2\left[d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2\theta\,d\phi^2)\right]\ .$$

NOTE: There are parts a,b. This question has **MULTIPLE PAGES** on an exam.

a) What is the general formula for circumference of a circle C at r in terms of r and χ? Sketch plot of C as a function of χ for all cases of k.

b) Mentally integrate over all solid angle to find the proper surface area A of the curved-space 2-sphere surrounding the origin at comoving coordinate r. This area is analogous to the circumference of a small circle on a ordinary sphere at polar angle θ. Sketch plot of A as a function of χ for all cases of k. Hint: $d\theta^2 + \sin^2\theta\,d\phi^2)$ is a differential path distance creating using the differential Pythagorean theorem and not a differential piece of solid angle.

c) The differential volume for the sphere is $dV = A(\chi)a\,d\chi$. For all k, determine $V(\chi)$ small χ and then for general χ. What is the maximum value of $V(\chi)$ for $k = 1$? **Hint:** You will need the identities $\sin^2(x) = (1/2)[1 - \cos(2x)]$ and $\sinh^2(x) = (1/2)[\cos(2x) - 1]$.

d) For the $k = 1$ case, what angles from the origin do radial geodesics lead to the antipodal point (i.e., the antipode)? How far in proper distance is it from the origin to the antipodal point along a radial geodesic? How far in proper distance to make the geodesic round trip from origin to origin?

8. The Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]\ .$$

Note that r is the radial comoving coordinate chosen so that r is proportional to proper distance in the transverse direction (i.e., perpendicular to the radial direction).

Prove Hubble's law in general form from the Robertson-Walker metric: i.e., prove

$$v_{\mathrm{R}} = HD_{\mathrm{P}} \ ,$$

where $v_{\mathrm{R}} = \dot{D}_{\mathrm{P}}$ is the recession velocity, $H = \dot{a}/a$ is the Hubble parameter, and D_{P} is proper (radial) distance. Note proper distance is distance that can be measured at one instant in cosmic time using a ruler: i.e., with $dt = 0$, it is

$$D_{\mathrm{P}} = \int \sqrt{-ds^2} \ .$$

The general form of Hubble's law is an exact result, but alas containing two quantities that are not direct observables, v_{R} and D_{P}, except asymptotically as $z \to 0$ or, in other words, in the limit where the 1st-order-in-small-z formulae can be treated as exact. The observational Hubble's law is

$$v_{\mathrm{red}} = H_0 D_{\mathrm{P,1st}} \ ,$$

where $v_{\mathrm{red}} = zc$ is redshift velocity (a direct observable) and $D_{\mathrm{P,1st}}$ is proper distance to 1st order in small z as measured from luminosity distance or angular diameter distance (which are direct observables). The observational Hubble's law is very plausible a priori, but a formal proof is left to a later problem.

9. The Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right] \ .$$

Note that r is the radial comoving coordinate chosen so that r is proportional to proper distance in the transverse direction (i.e., perpendicular to the radial direction).

NOTE: There are parts a,b,c. This question has **MULTIPLE PAGES** on an exam.

a) For light signals coming radially from remote source prove with few words the cosmological time-dilation effect (CL-16,19):

$$\frac{dt}{a(t)} = \frac{dt_0}{a_0} \qquad \text{or} \qquad \frac{dt_0}{dt} = \frac{a_0}{a(t)} \ ,$$

where t is the cosmic time of emission, t_0 is the cosmic time of observation (i.e., the cosmic present), and $a_0 = a(t_0)$.

b) Prove without words the cosmological redshift formula $1 + z = a_0/a(t)$.

c) The cosmological redshift formula is a very useful connecting the direct observable cosmological redshit z and the scaling up of the universe to since a light signal was emitted $a_0/a(t)$. Why can't it be used to directly determing $a(t)$?

10. The basic Λ-CDM model has its cosmic scale factor $a(t)$ fully specified via the Friedmann equation (FE) by the Hubble constant H_0 and three density parameters: i.e., $\Omega_{\mathrm{R},0}$ ("radiation"), $\Omega_{\mathrm{m},0}$ ("matter"), and ω_Λ (cosmological constant or constant dark energy). The obtaining the parameters is a major observational goal. In principle, only 3 are independent, but observational uncertainties make obtaining all 4 somewhat independently useful goal.

If the FE model is not flat, the Friedmann equation (in its derivation from general relativity) plus Robertson-Walker metric tells us that the physical scale of the of FE models at cosmic present t_0 is given by

$$a_0 = \frac{c/H_0}{\sqrt{|\Omega_0 - 1|}} = \frac{c/H_0}{\sqrt{|\Omega_{k,0}|}} = \frac{(4.2827\ldots\mathrm{Gpc})/h_{70}}{\sqrt{|\Omega_{k,0}|}} = \frac{(13.968\ldots\mathrm{Gly})/h_{70}}{\sqrt{|\Omega_k|}} \ ,$$

where Ω_0 is the sum of all density parameters, except $\Omega_{k,0}$, and $h_{70} = H_0/[70\,(\mathrm{km/s})/\mathrm{Mpc}]$ is the reduced Hubble constant which must be 1 to within a few percent. If the FE model is flat, there is no physical scale for the model and a_0 can be chosen arbitrarily or set to dimensionless 1 in which case the comoving distances r have length units and are equal to the proper distance of the cosmic present. In all cases, the proper distance to an object at comoving distance r is

$$D_{\mathrm{P}} = a_0 \int_0^r \frac{dr}{\sqrt{1-kr^2}} = a_0 f(r) \ ,$$

where r is comoving coordinate independent of time and $k = 1$ for hyperspherical space, $k = 0$ for Euclidean space (i.e., flat space in which case $f(r) = r$), and $k = -1$ for hyperbolical space. The variable k is called the curvature.

One way to test a FE model or fit it to observations is to plot some observable cosmic distance measure $D_{\rm C}$ for objects versus their cosmological redshifts z (which are the only easily obtained direct observables) and then compare to the theoretical cosmic distance measure $D_{\rm C}$ plotted as a function of z. The two best known observable cosmic distance measures (other than cosmological redshift z) are the luminosity distance $D_{\rm L}$ and the angular diameter distance $D_{\rm A}$ both of which have explicit dependence on z, but also depend on z via the comoving coordinate $r(z)$ whose z dependence is an observational constraint, not an intrinsic dependence.

NOTE: There are parts a,b,c,d. This question has **MULTIPLE PAGES** on an exam.

a) Recall the Robertson-Walker metric in standard form is

$$ds^2 = c^2\,dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right] .$$

For a light signal traveling from a source at comoving coordinate r, time t, and cosmological redshift z to the origin (i.e., us) at time t_0 along a radial path, derive an equation from the Robertson-Walker metric relating spatial integral $f(r)$ to time integral $\chi(t)$ (which is actually an alternative comoving coordinate though the symbol χ is probably not a standard for it). The left-hand side should depend only on parameters r and k and the right-hand side only on t and t_0. Do **NOT** use any words: just the expressions.

b) Formal expressions for r, t, and lookback time $t_{\rm LB}$ for a light signal are, respectively,

$$r = f^{-1}\left[\chi(z)\right] = f^{-1}\left\{\chi\left[t(z)\right]\right\} = f^{-1}\left\{\chi\left[t\left(\frac{a_0}{1+z}\right)\right]\right\} , \qquad t = t(z) = t\left(\frac{a_0}{1+z}\right) ,$$

and

$$t_{\rm LB} = -\Delta t = -[t(a) - t_0] ,$$

where we have used the cosmological redshift formula

$$1 + z = \frac{a_0}{a(t)} .$$

Note that $f(r) = r$ and $f^{-1}(r) = r$ if the curvature $k = 0$.

In order to obtain the proper distance $D_{\rm P} = a_0 f(r) = a_0\chi(z)$ explicitly, from the foregoing formulae, we need to specify an FE model. In general, only numerical results can be obtained. However, the de-Sitter universe (with k general) allows explicit simple formulae for some cosmological distance measures. For the de-Sitter universe,

$$a(t) = a_0 e^{H_0 \Delta t} ,$$

where in this case the Hubble constant $H_0 = \sqrt{\Lambda/3}$ is time-independent.

Determine in order the explicit formulae for $\Delta t(z)$, $t_{\rm LB}(z)$, $\chi(z)$, radial proper distance $D_{\rm P}$, and recession velocity $v_{\rm R}(z)$ for the de-Sitter universe.

What is odd about $t_{\rm LB}$ relative to the case of a cosmological model with a point origin (AKA Big Bang singularity)?

c) What is the explicit expression for the deceleration parameter $q_0 = -\ddot{a}_0 a_0/\dot{a}_0^2$ for the de Sitter universe?

d) The formal expressions for the standard cosmological distance measures (expressed in observational form if it exists and is distinct from theoretical forms and then in the theoretical forms) are as follows:

Cosmological redshift: $\quad z = \dfrac{\lambda_0 - \lambda}{\lambda} = \dfrac{a_0}{a(t)} - 1 \qquad 1 + z = \dfrac{a_0}{a(t)}$

Lookback time: $\quad t_{\rm LB} = t_0 - t(a) = -\Delta t$

Comoving coordinate r: $\quad r = f^{-1}[\chi(z)] = f^{-1}[\chi(t)]$

Comoving coordinate χ: $\quad \chi(z) = \chi(t) = \displaystyle\int_t^{t_0} \frac{c\,dt}{a(t)}$

Proper distance: $\quad D_{\rm P} = a_0\chi(z) = a_0\chi(t) = a_0 f(r)$

Recessional velocity: $\quad v_{\rm R} = H_0 D_{\rm P}$

Redshift velocity: $\quad v_{\rm red} = zc$

Luminosity distance: $\quad D_{\rm L} = \sqrt{\dfrac{L}{4\pi f}} = a_0 r(1+z)$

Angular diameter distance: $\quad D_{\rm A} = \dfrac{D_{\rm ruler}}{\theta} = \dfrac{a_0 r}{(1+z)}$

Distance-duality relation: $\quad D_{\rm L} = D_{\rm A}(1+z)^2$,

where the distance-duality relation is also called the Etherington reciprocity relation. Determine special case expressions for the cosmological distance measures above as a functions of z for the de Sitter universe. Note that some were already determined in part (b) and some already functions of z. What is odd about $D_{\rm A}$ as z goes to infinity in the case of $k = 0$?

11. The alternative comoving coordinate

$$\chi = \int_t^{t_0} \frac{c\,dt}{a(t)}$$

is also what is called conformal time.

NOTE: There are parts a,b,c,d,f. On an exam, this question has **MULTIPLE PAGES**.

a) Starting from the scaled Friedmann equation form

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left(\sum_p \Omega_{p,0} x^{-p}\right)$$

(where $x = a/a_0$) derive without words an integral formula for $\chi(x)$.

b) Now change the integral formula so that we have $\chi(z)$.

c) In what limit would $\chi(z)$ have an analytic formula?

d) Assuming there is only a single density component with $p > 0$, derive the exact solution for $\chi(z)$.

e) Assuming there is only a single density component with $p = 0$, derive the exact solution for $\chi(z)$.

f) Give the formula for radial proper distance $D_{\rm P}$ with $\chi(z)$ expanded into the integral form. Does $D_{\rm P}$ depend on a_0? Give the formula for $a_0 r$ for all cases of k with $\chi(z)$ unexpanded. Does $a_0 r$ depend on a_0?

12. The theoretical cosmological distance measures to 2nd order in small cosmological redshift z are conventionally written in terms of the Hubble constant $H_0 = \dot a_0/a_0$ and the deceleration parameter $q_0 = -\ddot a_0 a_0/\dot a_0^2$ (which is unitless or rather has natural units). In fact in the 1970s, cosmology was sometimes comically oversimplified as a search for two numbers: H_0 and q_0 (see A.R. Sandage, 1970, Physics Today, 23, 34, *Cosmology: A search for two numbers*). Nowadays, q_0 has lost some of its glamor. It is now not regarded as a basic parameter of cosmological models, but just one of the derived parameters and its peculiar definition just a historical convention. The fact that the universal expansion is accelerating makes the deceleration parameter negative which is an incongruity.

There are parts a,b.

NOTE: This question has **MULTIPLE PAGES** on an exam.

a) Taylor expand $a(t)$ in small $\Delta t = t - t_0$ to 2nd order and rewrite the coefficients in terms of H_0 and q_0. The rewritten expansion should begin $a(t) = a_0[1 + \ldots$.

b) Recalling the cosmological redshift formaula $1 + z = a_0/a$, rewrite the formula from the part (a) answer as an expansion for z to 2nd order small Δt. Hint: You will need the geometric series:

$$\frac{1}{1-x} = \sum_{\ell=0}^{\infty} x^\ell ,$$

which converges for $|x| < 0$ (Ar-279).

c) Now we need to invert the power series for z to find lookback time $t_{\rm LB} = t_0 - t = -\Delta t$ to 2nd order in small z. We will need the power series inversion cofficients. Given

$$\Delta y = \sum_{\ell=1}^{\infty} a_\ell \Delta x^\ell \qquad \text{and} \qquad \Delta x = \sum_{\ell=1}^{\infty} b_\ell \Delta y^\ell \ ,$$

where the inversion coefficients b_i run $b_1 = 1/a_1$, $b_2 = -a_2/a_1^3$, ... (Ar-316–317).

d) The Friedmann acceleration equation can be used to get a useful expression for the deceleration parameter q_0. Behold:

$$\begin{aligned}
\frac{\ddot a}{a} &= -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \frac{\Lambda}{3} \\
\frac{\ddot a a}{\dot a^2} H^2 &= -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2} + \rho_\Lambda + 3\frac{p_\Lambda}{c^2}\right) \\
-qH^2 &= -\frac{4\pi G}{3}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{4\pi G}{3H^2}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{1}{2}\frac{1}{\rho_{\rm critical}}\left[\rho(1+3w) + \rho_\Lambda(1+3w_\Lambda)\right) \\
q &= \frac{1}{2}\left[\Omega_{\rm M}(1+3w) + \Omega_\Lambda(1+3w_\Lambda)\right] \\
q &= \frac{1}{2}\left[\Omega_{\rm M} - 2\Omega_\Lambda\right] = \frac{\Omega_{\rm M}}{2} - \Omega_\Lambda \qquad \text{with } w = 0 \text{ and } w_\Lambda = -1 \text{ as per usual} \\
q &= \frac{1}{2}\left[0.3\alpha_{\rm M} - 2\times(0.7\alpha_\Lambda)\right] = \frac{1}{2}\left[0.3\alpha_{\rm M} - 1.4\alpha_\Lambda\right] = 0.15\alpha_{\rm M} - 0.7\alpha_\Lambda \ ,
\end{aligned}$$

where $\alpha_{\rm M} = \Omega_{\rm M}/0.3$ (0.3 being a modern fiducial value) and $\alpha_\Lambda = \Omega_\Lambda/0.7$ (0.7 being a modern fiducial value). Wit the modern fiducial values, one obtains a fidicial modern value $q_0 = -0.55$. Before 1998, people mostly thought $\Omega_\Lambda = 0$ which with $\Omega_{\rm M} = 0.3$ (which was what it seemed then as well as now) gives $q_0 = 0.15$. However, some people then hoped that $\Omega_{\rm M} = 1$ which would give $q_0 = 1/2$ which many thought was the great good value. Why?

13. To get the small cosmological redshift z formulae for cosmological distance measures one expands $a(t)$ around current time t_0 to 2nd order in $\Delta t = t - t_0$, parameterizes the first expansion coefficients with the Hubble constant $H_0 = \dot a_0/a_0$ and the deceleration parameter $q_0 = -\ddot a_0 a_0/\dot a_0^2$, substitutes for $a(t)$ with z (and thereby assuming t is the start time for a light signal coming from z), and inverts the power series to get lookback time $t_{\rm LB}$ to 2nd order in small z:

$$t_{\rm LB} = \frac{z}{H_0}\left[1 - \left(1 + \frac{1}{2}q_0\right)z + \ldots\right] \ .$$

One then uses the $t_{\rm LB}$ formula with the Robertson-Walker metric applied to the light signal to get the comoving coordinate r to 2nd order in z:

$$r = \frac{zc}{a_0 H_0}\left[1 - \frac{1}{2}(1+q_0)z + \ldots\right] \ .$$

There are parts a,b,c,d. The parts can be done be at least semi-independently, so don't stop necessarily if you can't do a part.

NOTE: This question has **MULTIPLE PAGES** on an exam.

a) Use the 2nd-order-in-z formulae given in the preamble to get the **2nd-order-in-z** formulae (simplified so that there is only one second order term appearing) and **1st-order-in-z** formulae (expressed just one term appearing) for the following standard cosmological distance measures (expressed in observational form if it exists and then theoretical form), except for expression for z

itself included for completeness:

Cosmological redshift: $z = \dfrac{\lambda_0 - \lambda_{\rm e}}{\lambda_{\rm e}} = \dfrac{a_0}{a(t)} - 1 \qquad 1 + z = \dfrac{a_0}{a(t)}$

Lookback time: $t_{\rm LB} = t_0 - t(a)$

Comoving coordinate r: $r = f^{-1}\left\{A\left[t_0, t\left(\dfrac{a_0}{1+z}\right)\right]\right\}$

Proper distance: $D_{\rm P} = a_0 f(r)$

Recessional velocity: $v_{\rm R} = H_0 D_{\rm P}$

Redshift velocity: $v_{\rm red} = zc$

Luminosity distance: $D_{\rm L} = \sqrt{\dfrac{L}{4\pi f}} = a_0 r(1+z)$

Angular diameter distance: $D_{\rm A} = \dfrac{D_{\rm ruler}}{\theta} = \dfrac{a_0 r}{(1+z)}$.

b) Under what conditions are the cosmological distances measures direct observables to 1st and 2nd order given that one can measure z?

c) Prove that all the standard cosmological distance measures are the same to 1st order in small z aside from constants. Show what they are in terms of quantity zc/H_0, where $c/H_0 = (13.968\ldots\,\text{Gly})/h_{70} = (4.2827\ldots\,\text{Gpc})/h_{70}$ is the Hubble length with $h_{70} = H_0/[70\,(\text{km/s})/\text{Mpc}]$.

d) Prove the observational Hubble's law:

$$v_{\rm red} = H_0 D_{\text{P-1st}} ,$$

where $D_{\text{P-1st}}$ is proper distance to 1st order in small z as measured from luminosity distance or angular diameter distance.

e) Given that $|q_0| \lesssim 1$, at what z values would one expect the standard cosmological distance measures (with constants applied as needed to make them all all equal to 1st order in z) to diverge by of order or less than 1 %, 10 %, 30 %, 50 %, and 100 %.